# Geometric theta-lifting for the dual pair $\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n}$

Sergey Lysenko

Abstract Let $X$ be a smooth projective curve over an algebraically closed field of characteristic $> 2$. Consider the dual pair $H = \mathbb{SO}_{2m}, G = \mathbb{S}\mathrm{p}_{2n}$ over $X$ with $H$ split. Write $\mathrm{Bun}_G$ and $\mathrm{Bun}_H$ for the stacks of $G$-torsors and $H$-torsors on $X$. The theta-kernel $\mathrm{Aut}_{G,H}$ on $\mathrm{Bun}_G \times \mathrm{Bun}_H$ yields the theta-lifting functors $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ between the corresponding derived categories. We describe the relation of these functors with Hecke operators.

In two particular cases it becomes the geometric Langlands functoriality for this pair (in the nonramified case). Namely, we show that for $n = m$ the functor $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ commutes with Hecke operators with respect to the inclusion of the Langlands dual groups $\check{H} \widetilde{\to} \mathbb{SO}_{2n} \hookrightarrow \mathbb{SO}_{2n+1} \widetilde{\to} \check{G}$. For $m = n + 1$ we show that the functor $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ commutes with Hecke operators with respect to the inclusion of the Langlands dual groups $\check{G} \widetilde{\to} \mathbb{SO}_{2n+1} \hookrightarrow \mathbb{SO}_{2n+2} \widetilde{\to} \check{H}$.

In other cases the relation is more complicated and involves the $\mathrm{SL}_2$ of Arthur. As a step of the proof, we establish the geometric theta-lifting for the dual pair $\mathrm{GL}_m, \mathrm{GL}_n$. Our global results are derived from the corresponding local ones, which provide a geometric analog of a theorem of Rallis.

## 1. Introduction

1.1 The classical Howe correspondence for dual reductive pairs is known to realize the Langlands functoriality in some particular cases (cf. [22], [1], [14]). In this paper, which is a continuation of [18], we depelop a similar geometric theory for dual reductive pairs $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2m})$ and also $(\mathrm{GL}_n, \mathrm{GL}_m)$. We consider only the everywhere unramified case.

Remind the classical construction of the theta-lifting operators. Let $X$ be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Let $F = \mathbb{F}_q(X)$, $\mathbb{A}$ be the adèles ring of $X$, $\mathcal{O}$ be the integer adèles. Let $G$, $H$ be split connected reductive groups over $\mathbb{F}_q$ that form a dual pair inside some symplectic group $\mathbb{S}\mathrm{p}_{2r}$. Assume that the metaplectic covering $\widetilde{\mathbb{S}\mathrm{p}}_{2r}(\mathbb{A}) \to \mathbb{S}\mathrm{p}_{2r}(\mathbb{A})$ splits over $G(\mathbb{A}) \times H(\mathbb{A})$. Let $S$ be the corresponding Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$. A choice of a theta-functional $\theta : S \to \bar{\mathbb{Q}}_\ell$ yields a morphism of modules over the global nonramified Hecke algebras $\mathcal{H}_G \otimes \mathcal{H}_H$

$$S^{(G\times H)(\mathcal{O})} \to \mathrm{Funct}((G \times H)(F)\backslash(G \times H)(\mathbb{A})/(G \times H)(\mathcal{O}))$$

sending $\phi$ to the function $(g, h) \mapsto \theta((g, h)\phi)$. The space $S^{(G\times H)(\mathcal{O})}$ has a distinguished nonramified vector, its image $\phi_0$ under the above map is the classical theta-function. Viewing $\phi_0$ as a kernel of integral operators, one gets the classical theta-lifting operators

$$F_G : \mathrm{Funct}(H(F)\backslash H(\mathbb{A})/H(\mathcal{O})) \to \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O}))$$

and

$$F_H : \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O})) \to \mathrm{Funct}(H(F)\backslash H(\mathbb{A})/H(\mathcal{O}))$$

For the dual pairs $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ and $(\mathrm{GL}_n, \mathrm{GL}_m)$ these operators realize the Langlands functoriality between the corresponding automorphic representations (as we will see below, its precise formulation involves the $\mathrm{SL}_2$ of Arthur). We establish a geometric analog of this phenomenon.

Remind that $S \widetilde{\to} \otimes'_{x\in X} S_x$ is the restricted tensor product of local Weil representations of $G(F_x) \times H(F_x)$. Here $F_x$ denotes the completion of $F$ at $x \in X$. The above functoriality in the classical case is a consequence of a local result describing the space of invariants $S_x^{G(\mathcal{O}_x)\times H(\mathcal{O}_x)}$ as a module over the tensor product ${}_x\mathcal{H}_G \otimes {}_x\mathcal{H}_H$ of local (nonramified) Hecke algebras. In the geomeric setting the main step is also to prove a local analog of this and then derive the global functoriality.

Let us underline the following phenomenon in the proof that we find striking. Let $G = \mathbb{S}\mathrm{p}_{2n}$, $H = \mathrm{S}\mathbb{O}_{2m}$. The Langlands dual groups are $\check{G} \widetilde{\to} \mathrm{S}\mathbb{O}_{2n+1}$ and $\check{H} \widetilde{\to} \mathrm{S}\mathbb{O}_{2m}$ over $\bar{\mathbb{Q}}_\ell$. Write $\mathrm{Rep}(\check{G})$ for the category of finite-dimensional representations of $\check{G}$ over $\bar{\mathbb{Q}}_\ell$, and similarly for $\check{H}$. Roughly speaking, there will be algebraic varieties $Y_H, Y_G$ over $k$ and fully faithful functors $f_H : \mathrm{Rep}(\check{H}) \to \mathrm{P}(Y_H)$ and $f_G : \mathrm{Rep}(\check{G}) \to P(Y_G)$ taking values in the categories of perverse sheaves (pure of weight zero) on $Y_H$ (resp., $Y_G$) with the following properties. Extend $f_H$ to a functor

$$f_H : \mathrm{Rep}(\check{H} \times \mathbb{G}_m) \to \oplus_{i\in\mathbb{Z}} P(Y_H)[i] \subset \mathrm{D}(Y_H)$$

naturally. That is, if $V$ is a representation of $\check{H}$ and $I$ is the standard representation of $\mathbb{G}_m$ then $f_H(V \boxtimes (I^{\otimes i})) \widetilde{\to} f_H(V)[i]$ is placed in perverse cohomological degree $-i$. For $n \geq m$ there will be a proper map $\pi : Y_G \to Y_H$ such that the following diagram is 2-commutative

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \stackrel{f_G}{\to} & P(Y_G) \\ \downarrow \mathrm{Res}^\kappa & & \downarrow \pi_! \\ \mathrm{Rep}(\check{H} \times \mathbb{G}_m) & \stackrel{f_H}{\to} & \oplus_{i\in\mathbb{Z}} P(Y_H)[i] \end{array}$$

for some homomoprhism $\kappa : \check{H} \times \mathbb{G}_m \to \check{G}$. For $n = m$ the restriction of $\kappa$ to $\mathbb{G}_m$ is trivial, so $\pi_! f_G$ takes values in the category of perverse sheaves in this case. Both $f_G$ and $f_H$ send an irreducible representation to an irreducible perverse sheaf. So, for $V \in \mathrm{Rep}(\check{G})$ the decomposition of $\mathrm{Res}^\kappa(V)$ into irreducible ones can be seen via the decomposition theorem of Beilinson, Bernstein and Deligne ([2]). To be more precise, $Y_H, Y_G$ are not algebraic varieties, but ind-proobjects in the category of algebraic varieties. There will also be an analog of the above result for $n < m$ (and also for the dual pair $\mathrm{GL}_n, \mathrm{GL}_m$).

The above phenomenon is a part of our main local results (Proposition 4 in Section 5.1, Theorem 7 in Section 6.2). They provide a geometric analog of the local theta correspondence for these dual pairs. The key technical tools in the proof are *the weak geometric analogs of the Jacquet functors* (cf. Section 4.7).

1.2 In the global setting let $\Omega$ denote the canonical line bundle on $X$. Let $G$ be the group scheme over $X$ of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$.

Let $H = \mathbb{SO}_{2m}$. Write $\mathrm{Bun}_H$ for the stack of $H$-torsors on $X$, similarly for $G$. Using the construction from [17], we introduce a geometric analog $\mathrm{Aut}_{G,H}$ of the above function $\phi_0$, this is an object of the derived category of $\ell$-adic sheaves on $\mathrm{Bun}_G \times \mathrm{Bun}_H$. It yields the theta-lifting functors

$$F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$$

and

$$F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$$

between the corresponding derived categories. Our mains global results for the pair $(G, H)$ are Theorems 3 and 4 describing the relation between the theta-lifting functors and the Hecke functors on $\mathrm{Bun}_G$ and $\mathrm{Bun}_H$. They agree with the conjectures of Adams ([1]). One of the advantages of the geometric setting compared to the classical one is that the $\mathrm{SL}_2$ of Arthur appears naturally (one can not even formulate a correct result in general without incorporating this $\mathrm{SL}_2$).

An essential difficulty in the proof was the fact that the complex $\mathrm{Aut}_{G,H}$ is not perverse (it has infinitely many perverse cohomologies), it is not even a direct sum of its perverse cohomologies (cf. Section 8.3.7).

We also establish the global theta-lifting for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$ (cf. Theorem 5).

1.3 Let us briefly duscuss how the paper is organized. Our main results are collected in Section 2. In Section 3 we remind some classical constructions at the level of functions, which we have in mind for geometrization.

In Section 4 we develop a geometric theory for the following classical objects. Let $K = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$. Given a reductive group $G$ over $\mathbb{F}_q$ and its finite dimensional representation $M$, the space of invariants in the Schwarz space $\mathcal{S}(M(K))^{G(\mathcal{O})}$ is a module over the nonramified Hecke algebra $\mathcal{H}_G$. We introduce the geometric analogs of the Fourier transform on this space and (some weak analogs) of the Jacquet functors. A way to relate this with the global case is proposed in Section 4.6.

In Section 5 we develop the local theta correspondence for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$. The key ingredients here are decomposition theorem from [2], the dimension estimates from [19] and hyperbolic localization results from [3].

In Section 6 we develop the local theta correspondence for the dual pair $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2m})$. In addition to the above tools, we use the classical result (Proposition 2) in the proof of our Propositions 7 and 8.

In Section 7 we derive the global theta-lifting results for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$.

In Section 8 we prove our main global results (Theorems 3 and 4) about theta-lifting for the dual pair $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2m})$. The relation between the local theory and the theta-kernel $\mathrm{Aut}_{G,H}$ comes from the results of [18]. In that paper we have introduced a scheme $\mathcal{L}_d(M(F_x))$ of discrete lagrangian lattices in a symplectic Tate space $M(F_x)$ and a certain $\mu_2$-gerb over it $\widetilde{\mathcal{L}}_d(M(F_x))$. The complex $\mathrm{Aut}_{G,H}$ on $\mathrm{Bun}_{G,H}$ comes from the stack $\widetilde{\mathcal{L}}_d(M(F_x))$ simply as the inverse image. The key observation is that it is much easier to prove the Hecke property of $\mathrm{Aut}_{G,H}$ on $\widetilde{\mathcal{L}}_d(M(F_x))$, because over the latter stack it is perverse.

In Section 8.3 we give another proof of a somewhat weaker statement than Theorem 4 using the ideas from [17]. The key step here is Proposition 12, which is partially inspired by the proof of the functional equation for geometric Eisenstein series ([4]). There seems no obvious way to derive Theorem 4 directly from Proposition 12, as $\mathrm{Aut}_{G,H}$ is not pure in general. However, at the level of functions or at the level of Grothendieck groups, Proposition 12 easily implies the classical analog of Theorem 4.

**Acknowledgements.** I am very grateful to Vincent Lafforgue for stimulating discussions which we have regularly for about last two years. They have contributed to this paper. He has also read the first version of the manuscript and indicated several mistakes. I also thank Alexander Braverman for nice discussions. Most of this work was realized in the University Paris 6 and the final part in the Institute for Advanced Study (Princeton), where the author was supported by NSF grant No. DMS-0111298.

## 2. Main results

2.1 Notation From now on $k$ denotes an algebraically closed field of characteristic $p > 2$ (except in Section 3, where $k = \mathbb{F}_q$). All the schemes (or stacks) we consider are defined over $k$.

Let $X$ be a smooth projective connected curve. Set $F = k(X)$. For a closed point $x \in X$ write $F_x$ for the completion of $F$ at $x$, let $\mathcal{O}_x \subset F_x$ be the ring of integers. Let $D_x = \operatorname{Spec} \mathcal{O}_x$ denotes the disc around $x$. Write $\Omega$ for the canonical line bundle on $X$.

Fix a prime $\ell \neq p$. For a scheme (or stack) $S$ write $\mathrm{D}(S)$ for the bounded derived category of $\ell$-adic étale sheaves on $S$, and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves. Set $\mathrm{DP}(S) = \oplus_{i \in \mathbb{Z}} \mathrm{P}(S)[i] \subset \mathrm{D}(S)$. By definition, we let for $K, K' \in \mathrm{P}(S), i, j \in \mathbb{Z}$

$$\mathrm{Hom}_{\mathrm{DP}(S)}(K[i], K'[j]) = \begin{cases} \mathrm{Hom}_{\mathrm{P}(S)}(K, K'), & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

Write $\mathrm{P}^{ss}(S) \subset \mathrm{P}(S)$ for the full subcategory of semi-simple perverse sheaves. Let $\mathrm{DP}^{ss}(S) \subset \mathrm{DP}(S)$ be the full subcategory of objects of the form $\oplus_{i \in \mathbb{Z}} K_i[i]$ with $K_i \in \mathrm{P}^{ss}$ for all $i$.

Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$. Since we are working over an algebraically closed field, we systematically ignore Tate twists. For a morphism of stacks $f : Y \to Z$ we denote by $\operatorname{dim.rel}(f)$ the function of a connected component $C$ of $Y$ given by $\dim C - \dim C'$, where $C'$ is the connected component of $Z$ containing $f(C)$.

If $V \to S$ and $V^* \to S$ are dual rank $r$ vector bundles over a stack $S$, we normalize the Fourier transform $\mathrm{Four}_\psi : \mathrm{D}(V) \to \mathrm{D}(V^*)$ by $\mathrm{Four}_\psi(K) = (p_{V^*})_!(\xi^* \mathcal{L}_\psi \otimes p_V^* K)[r]$, where $p_V, p_{V^*}$ are the projections, and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

Write $\mathrm{Bun}_k$ for the stack of rank $k$ vector bundles on $X$. For $k = 1$ we also write $\operatorname{Pic} X$ for the Picard stack $\mathrm{Bun}_1$ of $X$. We have a line bundle $\mathcal{A}_k$ on $\mathrm{Bun}_k$ with fibre $\det \mathrm{R}\Gamma(X, V)$ at $V \in \mathrm{Bun}_k$. View it as a $\mathbb{Z}/2\mathbb{Z}$-graded placed in degree $\chi(V) \mod 2$. Our conventions about $\mathbb{Z}/2\mathbb{Z}$-grading are those of ([17], 3.1).

For a sheaf of groups $G$ on a scheme $S$, $\mathcal{F}^0_G$ denotes the trivial $G$-torsor on $S$. For a representation $V$ of $G$ and a $G$-torsor $\mathcal{F}_G$ on $S$ we write $V_{\mathcal{F}_G} = V \times^G \mathcal{F}_G$ for the induced vector bundle on $S$.

In Section 8.2 we assume that the reader is familiar with the results of the first paper [18] of this series.

2.2.1 HECKE OPERATORS For a connected reductive group $G$ over $k$, let $\mathcal{H}_G$ be the Hecke stack classifying $(x, \mathcal{F}_G, \mathcal{F}'_G, \beta)$, where $\mathcal{F}_G, \mathcal{F}'_G$ are $G$-torsors on $X$, $x \in X$ and $\beta : \mathcal{F}_G \mid_{X-x} \widetilde{\to} \mathcal{F}'_G \mid_{X-x}$ is an isomorphism. We have a diagram of projections

$$X \times \operatorname{Bun}_G \stackrel{\operatorname{supp} \times h^{\leftarrow}_G}{\leftarrow} \mathcal{H}_G \stackrel{h^{\rightarrow}_G}{\to} \operatorname{Bun}_G,$$

where $h^{\leftarrow}_G$ (resp., $h^{\rightarrow}_G$, supp) sends the above collection to $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$, $x$). Write ${}_x\mathcal{H}_G$ for the fibre of $\mathcal{H}_G$ over $x \in X$.

Let $T \subset B \subset G$ be a maximal torus and Borel subgroup, we write $\Lambda_G$ (resp., $\check{\Lambda}_G$) for the coweights (resp., weights) lattice of $G$. Let $\Lambda^+_G$ (resp., $\check{\Lambda}^+_G$) denote the set of dominant coweights (resp., dominant weights) of $G$. Write $\check{\rho}_G$ (resp., $\rho_G$) for the half sum of the positive roots (resp., coroots) of $G$, $w_0$ for the longest element of the Weyl group of $G$. For $\check{\lambda} \in \check{\Lambda}^+_G$ we write $\mathcal{V}^{\check{\lambda}}$ for the corresponding Weyl $G$-module.

For $x \in X$ we write $\operatorname{Gr}_{G,x}$ for the affine grassmanian $G(F_x)/G(\mathcal{O}_x)$ (cf. [4], Section 3.2 for a detailed discussion). It can be seen as an ind-scheme classifying a $G$-torsor $\mathcal{F}_G$ on $X$ together with a trivialization $\beta : \mathcal{F}_G \mid_{X-x} \widetilde{\to} \mathcal{F}^0_G \mid_{X-x}$ over $X - x$. For $\lambda \in \Lambda^+_G$ let $\overline{\operatorname{Gr}}^{\lambda}_{G,x} \subset \operatorname{Gr}_{G,x}$ be the closed subscheme classifying $(\mathcal{F}_G, \beta)$ for which $V_{\mathcal{F}^0_G}(-\langle \lambda, \check{\lambda} \rangle x) \subset V_{\mathcal{F}_G}$ for every $G$-module $V$ whose weights are $\le \check{\lambda}$. The unique dense open $G(\mathcal{O}_x)$-orbit in $\overline{\operatorname{Gr}}^{\lambda}_{G,x}$ is denoted $\operatorname{Gr}^{\lambda}_{G,x}$.

Let $\mathcal{A}^{\lambda}_G$ denotes the intersetion cohomology sheaf of $\overline{\operatorname{Gr}}^{\lambda}_G$. Let $\check{G}$ denote the Langlands dual group to $G$. We write $\operatorname{Sph}_G$ for the category of $G(\mathcal{O}_x)$-equivariant perverse sheaves on $\operatorname{Gr}_{G,x}$. By ([19]), this is a tensor category, and there is a canonical equivalence of tensor categories $\operatorname{Loc} : \operatorname{Rep}(\check{G}) \widetilde{\to} \operatorname{Sph}_G$, where $\operatorname{Rep}(\check{G})$ is the category of $\check{G}$-representations over $\bar{\mathbb{Q}}_\ell$. Under this equivalence $\mathcal{A}^{\lambda}_G$ corresponds to the irreducible representation of $\check{G}$ with h.w. $\lambda$.

Write $\operatorname{Bun}_{G,x}$ for the stack classifying $\mathcal{F}_G \in \operatorname{Bun}_G$ together with a trivialization $\mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{D_x}$. Following ([4], Section 3.2.4), write $\mathrm{id}^l, \mathrm{id}^r$ for the isomorphisms

$${}_x\mathcal{H}_G \widetilde{\to} \operatorname{Bun}_{G,x} \times^{G(\mathcal{O}_x)} \operatorname{Gr}_{G,x}$$

such that the projection to the first factor corresponds to $h^{\leftarrow}_G, h^{\rightarrow}_G$ respectively. Let ${}_x\overline{\mathcal{H}}^{\lambda}_G \subset {}_x\mathcal{H}_G$ be the closed substack that identifies with $\operatorname{Bun}_{G,x} \times^{G(\mathcal{O}_x)} \overline{\operatorname{Gr}}^{\lambda}_{G,x}$ via $\mathrm{id}^l$.

To $\mathcal{S} \in \operatorname{Sph}_G, K \in \mathrm{D}(\operatorname{Bun}_G)$ one attaches their twisted external products $(K \tilde{\boxtimes} \mathcal{S})^l$ and $(K \tilde{\boxtimes} \mathcal{S})^r$ on ${}_x\mathcal{H}_G$, they are normalized to be perverse for $K, \mathcal{S}$ perverse. The Hecke functors

$${}_x\mathrm{H}^{\leftarrow}_G, {}_x\mathrm{H}^{\rightarrow}_G : \operatorname{Sph}_G \times \mathrm{D}(\operatorname{Bun}_G) \to \mathrm{D}(\operatorname{Bun}_G)$$

are given by

$${}_x\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K) = (h^{\leftarrow}_G)_!(*\mathcal{S} \tilde{\boxtimes} K)^r \quad \text{and} \quad {}_x\mathrm{H}^{\rightarrow}_G(\mathcal{S}, K) = (h^{\rightarrow}_G)_!(\mathcal{S} \tilde{\boxtimes} K)^l$$

We have denoted by $* : \mathrm{Sph}_G \widetilde{\to} \mathrm{Sph}_G$ the covariant equivalence of categories induced by the map $G(F_x) \to G(F_x)$, $g \mapsto g^{-1}$. Write also $* : \mathrm{Rep}(\check{G}) \widetilde{\to} \mathrm{Rep}(\check{G})$ for the corresponding functor (in view of Loc), it sends an irreducible $\check{G}$-module with h.w. $\lambda$ to the irreducible $\check{G}$-module with h.w. $-w_0(\lambda)$.

By ([6], Proposition 5.3.9), we have canonically ${}_x\mathrm{H}^{\leftarrow}_G(*\mathcal{S}, K) \widetilde{\to} {}_x\mathrm{H}^{\to}_G(\mathcal{S}, K)$. Besides, the functors $K \mapsto {}_x\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K)$ and $K \mapsto {}_x\mathrm{H}^{\to}_G(\mathbb{D}(\mathcal{S}), K)$ are mutually (both left and right) adjoint.

Letting $x$ move around $X$, one similarly defines Hecke functors

$$\mathrm{H}^{\leftarrow}_G, \mathrm{H}^{\to}_G : \mathrm{Sph}_G \times \mathrm{D}(S \times \mathrm{Bun}_G) \to \mathrm{D}(X \times S \times \mathrm{Bun}_G),$$

where $S$ is a scheme. The Hecke functors are compatible with the tensor structure on $\mathrm{Sph}_G$ and commute with Verdier duality (cf. *loc.cit*). Sometimes we write $\mathrm{Rep}(\check{G})$ instead of $\mathrm{Sph}_G$ in the definition of Hecke functors in view of Loc.

2.2.2 We introduce the category

$$\mathrm{D}\,\mathrm{Sph}_G := \oplus_{r\in\mathbb{Z}}\ \mathrm{Sph}_G[r] \subset \mathrm{D}(\mathrm{Gr}_G)$$

It is equipped with a tensor structure, associativity and commutativity constraints so that the following holds. There is a canonical equivalence of tensor categories $\mathrm{Loc}^{\mathfrak{r}} : \mathrm{Rep}(\check{G} \times \mathbb{G}_m) \widetilde{\to} \mathrm{D}\,\mathrm{Sph}_G$ such that $\mathbb{G}_m$ acts on $\mathrm{Sph}_G[r]$ by the character $x \mapsto x^{-r}$. So, the grading by cohomological degrees in $\mathrm{D}\,\mathrm{Sph}_G$ corresponds to grading by the characters of $\mathbb{G}_m$ in $\mathrm{Rep}(\check{G}\times\mathbb{G}_m)$. In cohomological degree zero the equivalence $\mathrm{Loc}^{\mathfrak{r}}$ specializes to Loc.

The action of $\mathrm{Sph}_G$ on $\mathrm{D}(\mathrm{Bun}_G)$ extends to an action of $\mathrm{D}\,\mathrm{Sph}_G$. Namely, we still denote by

$${}_x\mathrm{H}^{\leftarrow}_G, {}_x\mathrm{H}^{\to}_G : \mathrm{D}\,\mathrm{Sph}_G \times \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)$$

the functors given by ${}_x\mathrm{H}^{\leftarrow}_G(\mathcal{S}[r], K) = {}_x\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K)[r]$ and ${}_x\mathrm{H}^{\to}_G(\mathcal{S}[r], K) = {}_x\mathrm{H}^{\to}_G(\mathcal{S}, K)[r]$ for $\mathcal{S} \in \mathrm{Sph}_G$ and $K \in \mathrm{D}(\mathrm{Bun}_G)$.

We still denote by $* : \mathrm{D}\,\mathrm{Sph}_G \to \mathrm{D}\,\mathrm{Sph}_G$ the functor given by $*(\mathcal{S}[i]) = (*\mathcal{S})[i]$ for $\mathcal{S} \in \mathrm{Sph}_G$. Write $\mathrm{Loc}_X$ for the tensor category of local systems on $X$. Set

$$\mathrm{D}\,\mathrm{Loc}_X = \oplus_{i\in\mathbb{Z}}\ \mathrm{Loc}_X[i] \subset \mathrm{D}(X)$$

We also equip it with a tensor structure so that a choice of $x \in X$ yields an equivalence of tensor categories $\mathrm{Rep}(\pi_1(X, x) \times \mathbb{G}_m) \widetilde{\to} \mathrm{D}\,\mathrm{Loc}_X$. The cohomological grading in $\mathrm{D}\,\mathrm{Loc}_X$ corresponds to grading by the characters of $\mathbb{G}_m$.

For the standard definition of a Hecke eigen-sheaf we refer the reader to ([10], Section 2.7). Since we need to take into account the maximal torus of $\mathrm{SL}_2$ of Arthur, we modify this standard definition as follows.

**Definition 1.** Given a tensor functor $E : \mathrm{Sph}_G \to \mathrm{D}\,\mathrm{Loc}_X$, a $E$-Hecke eigensheaf is a complex $K \in \mathrm{D}(\mathrm{Bun}_G)$ equipped with an isomorphism

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K) \widetilde{\to} E(\mathcal{S}) \boxtimes K[1]$$

functorial in $\mathcal{S} \in \mathrm{Sph}_G$ and satisfying the compatibility conditions (as in *loc.cit.*). Note that once $x \in X$ is chosen, a datum of $E$ becomes equivalent to a datum of a homomorphism $\sigma : \pi_1(X, x) \times \mathbb{G}_m \to \check{G}$. In other words, we are given a homomorphism $\mathbb{G}_m \to \check{G}$ of algebraic groups over $\bar{\mathbb{Q}}_\ell$, and a continuous homomorphism $\pi_1(X, x) \to Z_{\check{G}}(\mathbb{G}_m)$, where $Z_{\check{G}}(\mathbb{G}_m)$ is the centralizer of $\mathbb{G}_m$ in $\check{G}$.

Given $\sigma : \pi_1(X, x) \times \mathbb{G}_m \to \check{G}$ as above we write $\sigma^{ex} : \pi_1(X, x) \times \mathbb{G}_m \to \check{G} \times \mathbb{G}_m$ for the homomorphism, whose first component is $\sigma$, and the second component $\pi_1(X, x) \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection.

*Example* 1. The constant perverse sheaf $\bar{\mathbb{Q}}_\ell[\dim \mathrm{Bun}_G]$ on $\mathrm{Bun}_G$ is a $\sigma$-Hecke eigensheaf for the homomoprhism $\sigma : \pi_1(X, x) \times \mathbb{G}_m \to \check{G}$ given by $2\rho : \mathbb{G}_m \to \check{G}$ and trivial on $\pi_1(X, x)$.

2.3 THETA-SHEAF Let $G_r$ denote the sheaf of automorphisms of $\mathcal{O}_X^r \oplus \Omega^r$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^r \oplus \Omega^r) \to \Omega$. The stack $\mathrm{Bun}_{G_r}$ of $G_r$-torsors on $X$ classifies $M \in \mathrm{Bun}_{2r}$ equipped with a symplectic form $\wedge^2 M \to \Omega$.

Remind the following objects introduced in [17]. Write $\mathcal{A}_{G_r}$ for the line bundle on $\mathrm{Bun}_{G_r}$ with fibre $\det \mathrm{R}\Gamma(X, M)$ at $M$. We view it as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle purely of degree zero. Denote by $\widetilde{\mathrm{Bun}}_{G_r} \to \mathrm{Bun}_{G_r}$ the $\mu_2$-gerbe of square roots of $\mathcal{A}_{G_r}$. The theta-sheaf $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$ is a perverse sheaf on $\widetilde{\mathrm{Bun}}_{G_r}$ (cf. [17] for details).

2.4. DUAL PAIR $\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n}$

2.4.1 Let $n, m \geq 1$, $G = G_n$ and $\mathcal{A}_G = \mathcal{A}_{G_n}$. Let $H = \mathbb{SO}_{2m}$ be the split orthogonal group of rank $m$ over $k$. The stack $\mathrm{Bun}_H$ of $H$-torsors on $X$ classifies: $V \in \mathrm{Bun}_{2m}$, a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}_X$, and a compatible trivialization $\gamma : \det V \widetilde{\to} \mathcal{O}_X$. Let $\mathcal{A}_H$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundle on $\mathrm{Bun}_H$ with fibre $\det \mathrm{R}\Gamma(X, V)$ at $V$.

Write $\mathrm{Bun}_{G,H} = \mathrm{Bun}_G \times \mathrm{Bun}_H$. Let

$$\tau : \mathrm{Bun}_{G,H} \to \mathrm{Bun}_{G_{2nm}}$$

be the map sending $(M, V)$ to $M \otimes V$ with the induced symplectic form $\wedge^2(M \otimes V) \to \Omega$. The following is proved in ([16], Proposition 2).

**Lemma 1.** *There is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism of line bundles on* $\mathrm{Bun}_{G,H}$

$$\tau^* \mathcal{A}_{G_{2nm}} \widetilde{\to} \mathcal{A}_H^{2n} \otimes \mathcal{A}_G^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-4nm} \tag{1}$$

Let $\tilde{\tau} : \mathrm{Bun}_{G,H} \to \widetilde{\mathrm{Bun}}_{G_{2nm}}$ be the map sending $(\wedge^2 M \to \Omega, \mathrm{Sym}^2 V \to \mathcal{O})$ to $(\wedge^2(M \otimes V) \to \Omega, \mathcal{B})$, where

$$\mathcal{B} = \frac{\det \mathrm{R}\Gamma(X, V)^n \otimes \det \mathrm{R}\Gamma(X, M)^m}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm}},$$

and $\mathcal{B}^2$ is identified with $\det \mathrm{R}\Gamma(X, M \otimes V)$ via (1).

**Definition 2.** Set $\mathrm{Aut}_{G,H} = \tilde{\tau}^* \mathrm{Aut}[\dim. \mathrm{rel}(\tau)]$. As in ([16], Section 3.2) for the diagram of projections

$$\mathrm{Bun}_H \stackrel{\mathfrak{q}}{\leftarrow} \mathrm{Bun}_{G,H} \stackrel{\mathfrak{p}}{\rightarrow} \mathrm{Bun}_G$$

define $F_G : \mathrm{D}^b(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}^b(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ by

$$F_G(K) = \mathfrak{p}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{q}^* K)[-\dim \mathrm{Bun}_H]$$

$$F_H(H) = \mathfrak{q}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{p}^* K)[-\dim \mathrm{Bun}_G]$$

Since $\mathfrak{p}$ and $\mathfrak{q}$ are not representable, a priori $F_G$ and $F_H$ may send a bounded complex to a complex, which is unbounded even over some open substack of finite type. We don't know if this really happens.

The Langlands dual groups are $\check{G} \widetilde{\to} \mathbb{SO}_{2n+1}$ and $\check{H} \widetilde{\to} \mathbb{SO}_{2m}$ over $\bar{\mathbb{Q}}_\ell$. For convenience of the reader, we first formulate our main result in particular cases that yield Langlands functoriality.

**Theorem 1.** *1) Case $n = m$. There is an inclusion $\check{H} \hookrightarrow \check{G}$ such that there exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_G(V, F_G(K)) \widetilde{\to} (\mathrm{id} \boxtimes F_G)(\mathrm{H}^{\leftarrow}_H(\mathrm{Res}^{\check{G}}_{\check{H}}(V), K)) \quad (2)$$

*over $X \times \mathrm{Bun}_G$ functorial in $V \in \mathrm{Rep}(\check{G})$ and $K \in \mathrm{D}(\mathrm{Bun}_H)$. Here $\mathrm{id} \boxtimes F_G : \mathrm{D}(X \times \mathrm{Bun}_H) \to \mathrm{D}(X \times \mathrm{Bun}_G)$ is the corresponding theta-lifting functor.*
*2) Case $m = n+1$. There is an inclusion $\check{G} \hookrightarrow \check{H}$ such that there exists an isomorphism*

$$\mathrm{H}^{\rightarrow}_H(V, F_H(K)) \widetilde{\to} (\mathrm{id} \boxtimes F_H)(\mathrm{H}^{\rightarrow}_G(\mathrm{Res}^{\check{H}}_{\check{G}}(V), K)) \quad (3)$$

*over $X \times \mathrm{Bun}_H$ functorial in $V \in \mathrm{Rep}(\check{H})$ and $K \in \mathrm{D}(\mathrm{Bun}_G)$. Here $\mathrm{id} \boxtimes F_H : \mathrm{D}(X \times \mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_H)$ is the corresponding theta-lifting functor.*

We will derive Theorem 1 from the following Hecke property of $\mathrm{Aut}_{G,H}$.

**Theorem 2.** *1) Case $n = m$. There is an inclusion $\check{H} \to \check{G}$ such that there exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_G(V, \mathrm{Aut}_{G,H}) \widetilde{\to} \mathrm{H}^{\rightarrow}_H(\mathrm{Res}^{\check{G}}_{\check{H}}(V), \mathrm{Aut}_{G,H}) \quad (4)$$

*over $X \times \mathrm{Bun}_{G,H}$ functorial in $V \in \mathrm{Rep}(\check{G})$.*
*2) Case $m = n+1$. There is an inclusion $\check{G} \hookrightarrow \check{H}$ such that there exists an isomorphism*

$$\mathrm{H}^{\rightarrow}_H(V, \mathrm{Aut}_{G,H}) \widetilde{\to} \mathrm{H}^{\leftarrow}_G(\mathrm{Res}^{\check{H}}_{\check{G}}(V), \mathrm{Aut}_{G,H}) \quad (5)$$

*over $X \times \mathrm{Bun}_{G,H}$ functorial in $V \in \mathrm{Rep}(\check{H})$.*

2.4.2 In the case $m \le n$ define the map $\kappa : \check{H} \times \mathbb{G}_m \to \check{G}$ as follows.

Set $W_0 = \bar{\mathbb{Q}}_\ell^n$, write $W_0 = W_1 \oplus W_2$, where $W_1$ (resp., $W_2$) is the subspace generated by the first $m$ (resp., last $n-m$) base vectors. Equip $W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell$ with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_n \in \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ is the unity. Realize $\check{G}$ as $\mathbb{SO}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$. Equip the subspace $W_1 \oplus W_1^* \subset W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell$ with the induced symmetric form, and realize $\check{H}$ as $\mathbb{SO}(W_1 \oplus W_1^*)$. This fixes the inclusion $i_\kappa : \check{H} \hookrightarrow \check{G}$. The centralizer of $\check{H}$ in $\check{G}$ contains the group $\mathbb{O}(W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell)$. Let $\check{T}_{\mathrm{GL}(W_2)}$ be the maximal torus of diagonal matrices in $\mathrm{GL}(W_2)$. We have $\mathrm{Hom}(\mathbb{G}_m, \check{T}_{\mathrm{GL}(W_2)}) = \mathbb{Z}^{n-m}$ canonically, and we let $\alpha_\kappa = (2, 4, \dots, 2n-2m) \in \mathrm{Hom}(\mathbb{G}_m, \check{T}_{\mathrm{GL}(W_2)})$. View $\alpha_\kappa$ as a map $\mathbb{G}_m \to \check{G}$. Finally, set $\kappa = (i_\kappa, \alpha_\kappa) : \check{H} \times \mathbb{G}_m \to \check{G}$.

Another way to think of $\alpha_\kappa$ is to say that $W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell$ can be thought of as an irreducible representation of *the* $\mathrm{SL}_2$ *of Arthur*, and $\alpha_\kappa$ is its restriction to the standard maximal torus

$$\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\sigma} \mathbb{SO}(W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell)$$

As predicted by Adams ([1]), the representation $\sigma$ corresponds to the principal unipotent orbit in $\mathbb{SO}(W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell)$, so $\alpha_\kappa = 2\rho_{\mathbb{SO}(W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell)}$ for a suitable choice of positive roots of $\mathbb{SO}(W_2 \oplus W_2^* \oplus \bar{\mathbb{Q}}_\ell)$.

Write $\mathrm{gRes}^\kappa : \mathrm{Sph}_G \to \mathrm{D}\,\mathrm{Sph}_H$ for the geometric restriction functor corresponding to $\kappa$.

In the case $m > n$ define $\kappa : \check{G} \times \mathbb{G}_m \to \check{H}$ as follows. Set in this case $W_0 = \bar{\mathbb{Q}}_\ell^m$, let $W_1$ (resp., $W_2$) be the subspace of $W_0$ generated by the first $n$ (resp., last $m-n$) base vectors. Equip $W_0 \oplus W_0^*$ with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where $E_m \in \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$ is the unity. Realize $\check{H}$ as $\mathbb{SO}(W_0 \oplus W_0^*)$.

Write $\{e_i\}$ for the standard base of $W_0$, and $\{e_i^*\}$ for the dual base in $W_0^*$. Write $W_2 = W_3 \oplus W_4$, where $W_3$ (resp., $W_4$) is spanned by $e_{n+1}$ (resp., by $e_{n+2}, \dots, e_m$). Let $\bar{W} \subset W_3 \oplus W_3^*$ be any nondegenerate one-dimensional subspace. Equip $W_1 \oplus W_1^* \oplus \bar{W}$ with the induced symmetric form and set $\check{G} = \mathbb{SO}(W_1 \oplus W_1^* \oplus \bar{W})$. This fixes the inclusion $i_\kappa : \check{G} \hookrightarrow \check{H}$.

Let $\bar{W}^\perp$ denote the orthogonal complement of $\bar{W}$ in $W_2 \oplus W_2^*$. The centralizer of $\check{G}$ in $\check{H}$ contains $\mathbb{O}(\bar{W}^\perp)$. Realize $\mathrm{GL}(W_4)$ as the Levi subgroup of $\mathbb{SO}(\bar{W}^\perp)$ using the standard inclusion $W_4 \oplus W_4^* \subset \bar{W}^\perp$. Let $\check{T}_{\mathrm{GL}(W_4)}$ be the maximal torus of diagonal matrices in $\mathrm{GL}(W_4)$. Set

$$\alpha_\kappa = (-2, -4, \dots, 2-2m+2n) \in \mathbb{Z}^{m-n-1} = \mathrm{Hom}(\mathbb{G}_m, \check{T}_{\mathrm{GL}(W_4)})$$

View $\alpha_\kappa$ as a map $\mathbb{G}_m \to \check{H}$, set $\kappa = (i_\kappa, \alpha_\kappa) : \check{G} \times \mathbb{G}_m \to \check{H}$.

Another way to think of $\alpha_\kappa$ is to say that $\bar{W}^\perp$ can be thought of as an irreducible representation of *the* $\mathrm{SL}_2$ *of Arthur*, and $\alpha_\kappa$ is the restriction to the standard maximal torus

$$\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\sigma} \mathbb{SO}(\bar{W}^\perp)$$

As predicted by Adams ([1]), the representation $\sigma$ corresponds to the principal unipotent orbit in $\mathrm{S}\mathbb{O}(\bar{W}^{\perp})$, so $\alpha_\kappa = 2\rho_{\mathrm{S}\mathbb{O}(\bar{W}^{\perp})}$ for a suitable choice of positive roots of $\mathrm{S}\mathbb{O}(\bar{W}^{\perp})$. As above, the geometric restriction functor corresponding to $\kappa$ is denoted $\mathrm{gRes}^{\kappa} : \mathrm{Sph}_H \to \mathrm{D}\,\mathrm{Sph}_G$.

Here is our main global result.

**Theorem 3.** *1) Case $m \le n$. There exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, F_G(K)) \widetilde{\to} (\mathrm{id} \boxtimes F_G)(\mathrm{H}^{\leftarrow}_H(\mathrm{Res}^{\kappa}(\mathcal{S}), K)) \tag{6}$$

*in $\mathrm{D}(X \times \mathrm{Bun}_G)$ functorial in $\mathcal{S} \in \mathrm{Sph}_G$ and $K \in \mathrm{D}(\mathrm{Bun}_H)$. Here $\mathrm{id} \boxtimes F_G : \mathrm{D}(X \times \mathrm{Bun}_H) \to \mathrm{D}(X \times \mathrm{Bun}_G)$ is the corresponding theta-lifting functor.*
*2) Case $m > n$. There exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_H(\mathcal{S}, F_H(K)) \widetilde{\to} (\mathrm{id} \boxtimes F_H)(\mathrm{H}^{\rightarrow}_G(\mathrm{Res}^{\kappa}(*\mathcal{S}), K)) \tag{7}$$

*in $\mathrm{D}(X \times \mathrm{Bun}_H)$ functorial in $\mathcal{S} \in \mathrm{Sph}_H$ and $K \in \mathrm{D}(\mathrm{Bun}_G)$. Here $\mathrm{id} \boxtimes F_H : \mathrm{D}(X \times \mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_H)$ is the corresponding theta-lifting functor.*

We will derive Theorem 3 from the following Hecke property of $\mathrm{Aut}_{G,H}$.

**Theorem 4.** *1) Case $m \le n$. There exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, \mathrm{Aut}_{G,H}) \widetilde{\to} \mathrm{H}^{\leftarrow}_H(*\,\mathrm{gRes}^{\kappa}(\mathcal{S}), \mathrm{Aut}_{G,H}) \tag{8}$$

*in $\mathrm{D}(X \times \mathrm{Bun}_{G,H})$ functorial in $\mathcal{S} \in \mathrm{Sph}_G$.*
*2) Case $m > n$. There exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_H(\mathcal{S}, \mathrm{Aut}_{G,H}) \widetilde{\to} \mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(*\mathcal{S}), \mathrm{Aut}_{G,H}) \tag{9}$$

*in $\mathrm{D}(X \times \mathrm{Bun}_{G,H})$ functorial in $\mathcal{S} \in \mathrm{Sph}_H$.*

2.4.3 There is an automorphism $\sigma_H : \check{H} \widetilde{\to} \check{H}$ inducing the functor $* : \mathrm{Rep}(\check{H}) \widetilde{\to} \mathrm{Rep}(\check{H})$ defined in Section 2.2.2. For $m > n$ write $\tilde{\kappa} = \sigma_H \circ \kappa$. Note also that the functor $* : \mathrm{Rep}(\check{G}) \widetilde{\to} \mathrm{Rep}(\check{G})$ is isomorphic to the identity functor. From Theorem 3 one derives the following.

**Corollary 1.** *1) For $m \le n$ let $K \in \mathrm{Bun}_H$ be a $\sigma$-Hecke eigensheaf for some $\sigma : \pi_1(X, x) \times \mathbb{G}_m \to \check{H}$. Let $\tau$ be the composition*

$$\pi_1(X, x) \times \mathbb{G}_m \stackrel{\sigma^{ex}}{\to} \check{H} \times \mathbb{G}_m \stackrel{\kappa}{\to} \check{G},$$

*where $\sigma^{ex}$ is as in Definition 1. Then $F_G(K)$ is equipped with a structure of a $\tau$-Hecke eigensheaf.*
*2) For $m > n$ let $K \in \mathrm{Bun}_G$ be a $\sigma$-Hecke eigensheaf for some $\sigma : \pi_1(X, x) \times \mathbb{G}_m \to \check{G}$. Let $\tau$ be the composition*

$$\pi_1(X, x) \times \mathbb{G}_m \stackrel{\sigma^{ex}}{\to} \check{G} \times \mathbb{G}_m \stackrel{\tilde{\kappa}}{\to} \check{H},$$

*where $\sigma^{ex}$ is as in Definition 1. Then $F_H(K)$ is equipped with a structure of a $\tau$-Hecke engeinsheaf.*

### 2.5 Dual pair $\mathrm{GL}_m, \mathrm{GL}_n$

Let $n, m \geq 0$. Remind that $\mathrm{Bun}_n$ denotes the stack of rank $n$ vector bundles on $X$. Our convention is that $\mathrm{GL}_0 = \{1\}$ and $\mathrm{Bun}_0 = \operatorname{Spec} k$.

Let $\mathcal{W}_{n,m}$ denote the stack classifying $L \in \mathrm{Bun}_n, U \in \mathrm{Bun}_m$ and a section $s : \mathcal{O}_X \to L \otimes U$. We have a diagram

$$\mathrm{Bun}_n \stackrel{h_n}{\leftarrow} \mathcal{W}_{n,m} \stackrel{h_m}{\to} \mathrm{Bun}_m,$$

where $h_m$ (resp., $h_n$) sends $(L, U, t)$ to $U$ (resp., to $L$). Let $\mathcal{W}'_{n,m}$ be the stack classifying $L \in \mathrm{Bun}_n, U \in \mathrm{Bun}_m$ and a section $s' : L \otimes U \to \Omega$. We have a diagram

$$\mathrm{Bun}_n \stackrel{h'_n}{\leftarrow} \mathcal{W}'_{n,m} \stackrel{h'_m}{\to} \mathrm{Bun}_m,$$

where $h'_m$ (resp., $h'_n$) sends $(L, U, s')$ to $U$ (resp., to $L$).

**Definition 3.** The theta-lifting functors $F_{n,m}, F'_{n,m} : \mathrm{D}^b(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_m)$ are given by

$$F_{n,m}(K) = (h_m)_! h_n^* K[\dim \mathrm{Bun}_m + a_{n,m}] \quad \text{and} \quad F'_{n,m}(K) = (h'_m)_! (h'_n)^* K[\dim \mathrm{Bun}_m - a_{n,m}]$$

Here $a_{n,m}$ is a function of a connected component of $\mathrm{Bun}_n \times \mathrm{Bun}_m$ given by $a_{n,m} = \chi(L \otimes U)$ for $L \in \mathrm{Bun}_n, U \in \mathrm{Bun}_m$. By restriction under $h_n \times h_m$ (resp., under $h'_n \times h'_m$), we view $a_{n,m}$ in the above formulas as a function on $\mathcal{W}_{n,m}$ (resp., on $\mathcal{W}'_{n,m}$).

Since $h_m$ and $h'_m$ are not representable, a priori $F_{n,m}$ and $F'_{n,m}$ may send a bounded complex to an unbounded one. The following result can be thought of as a functional equation for the theta-lifting functors.

**Lemma 2.** *There is a canonical isomorphism of functors* $F'_{n,m} \widetilde{\to} F_{n,m}$. $\square$

*Proof* Write $\phi, \phi'$ for the projections from $\mathcal{W}_{n,m}$ and from $\mathcal{W}'_{n,m}$ to $\mathrm{Bun}_n \times \mathrm{Bun}_m$. As in ([4], Lemma 7.3.6) one shows that $\phi_i \bar{\mathbb{Q}}_\ell[a_{n,m}] \widetilde{\to} \phi'_! \bar{\mathbb{Q}}_\ell[-a_{n,m}]$ canonically. The assertion follows. $\square$

For the rest of Section 2.5 assume $m \geq n$ and set $G = \mathrm{GL}(L_0)$ and $H = \mathrm{GL}(U_0)$ for $U_0 = k^m$, $L_0 = k^n$. Write $U_0 = U_1 \oplus U_2$, where $U_1$ (resp., $U_2$) is the subspace generated by the first $n$ (resp., last $m - n$) base vectors. Let $M = \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \subset H$ be the corresponding Levi factor.

Define $\kappa : \check{G} \times \mathbb{G}_m \to \check{H}$ as the composition

$$\check{G} \times \mathbb{G}_m \stackrel{\mathrm{id} \times 2\check{\rho}_{\mathrm{GL}(U_2)}}{\to} \check{G} \times \check{\mathrm{GL}}(U_2) = \check{M} \hookrightarrow \check{H}$$

Write $\mathrm{gRes}^\kappa : \mathrm{Sph}_H \to \mathrm{D}\,\mathrm{Sph}_G$ for the corresponding geometric restriction functor.

The analog of Theorem 3 for the dual pair $(G, H)$ is as follows.

**Theorem 5.** *We assume $m \geq n$. There exists an isomorphism*

$$\mathrm{H}^{\leftarrow}_H(\mathcal{S}, F_{n,m}(K)) \widetilde{\to} (\mathrm{id} \boxtimes F_{n,m})(\mathrm{H}^{\to}_G(\mathrm{gRes}^\kappa(\mathcal{S}), K)) \tag{10}$$

*in* $\mathrm{D}(X \times \mathrm{Bun}_m)$ *functorial in* $\mathcal{S} \in \mathrm{Sph}_H$ *and* $K \in \mathrm{D}(\mathrm{Bun}_n)$. *Here* $\mathrm{id} \boxtimes F_{n,m} : \mathrm{D}(X \times \mathrm{Bun}_n) \to \mathrm{D}(X \times \mathrm{Bun}_m)$ *is the corresponding theta-lifting functor.*

If $n=m$ or $m=n+1$ then the restriction of $\kappa$ to $\mathbb{G}_m$ is trivial, so Theorem 5 in this case says that $F_{n,m}$ realizes the (nonramified) geometric Langlands functoriality with respect to an inclusion $\check{G}\hookrightarrow\check{H}$. For example, for $n=m$ one may show the following. For an irreducible rank $n$ local system $E$ on $X$ write $\mathrm{Aut}_E$ for the automorphic sheaf on $\mathrm{Bun}_n$ corresponding to $E$ (cf. [7]). Then $F_{n,n}(\mathrm{Aut}_E)$ is isomorphic to $\mathrm{Aut}_{E^*}$ tensored by some constant complex.

For a closed point $x\in X$ let ${}_{x,\infty}\mathcal{W}_{n,m}$ denote the stack classifying $L\in\mathrm{Bun}_n, U\in\mathrm{Bun}_m$ and a section $s:\mathcal{O}_X\to L\otimes U(\infty x)$, which is allowed to have an arbitrary pole at $x$. This is an ind-algebraic stack.

In Section 7 we will define Hecke functors

$$ {}_x\mathrm{H}^{\leftarrow}_H, {}_x\mathrm{H}^{\rightarrow}_H :\ \mathrm{Sph}_H\times\mathrm{D}({}_{x,\infty}\mathcal{W}_{n,m})\to\mathrm{D}({}_{x,\infty}\mathcal{W}_{n,m}) \tag{11}$$

$$ {}_x\mathrm{H}^{\leftarrow}_G, {}_x\mathrm{H}^{\rightarrow}_G :\ \mathrm{Sph}_G\times\mathrm{D}({}_{x,\infty}\mathcal{W}_{n,m})\to\mathrm{D}({}_{x,\infty}\mathcal{W}_{n,m}) \tag{12}$$

Set

$$\mathcal{I}=(\bar{\mathbb{Q}}_\ell)_{\mathcal{W}_{n,m}}[\dim\mathrm{Bun}_m+\dim\mathrm{Bun}_n+a_{n,m}], \tag{13}$$

where $a_{n,m}$ is a function of a connected component of $\mathcal{W}_{n,m}$ defined above. We view $\mathcal{I}$ as a complex on $\mathcal{W}_{n,m}$ extended by zero to ${}_{x,\infty}\mathcal{W}_{n,m}$. We will derive Theorem 5 from the following 'Hecke property' of $\mathcal{I}$.

**Theorem 6.** *The two functors* $\mathrm{Sph}_H\to\mathrm{D}({}_{x,\infty}\mathcal{W}_{m,n})$ *given by*

$$\mathcal{T}\mapsto {}_x\mathrm{H}^{\leftarrow}_H(\mathcal{T},\mathcal{I})\quad\text{and}\quad \mathcal{T}\mapsto {}_x\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^\kappa(\mathcal{T}),\mathcal{I})$$

*are isomorphic.*

## 3. Classical setting and motivations

In Section 3 we assume $k=\mathbb{F}_q$.

### 3.1 Weil representation of $\mathrm{GL}_m\times\mathrm{GL}_n$.

Let $U_0$ (resp., $L_0$) be a $k$-vector space of dimension $m$ (resp., $n$). For Section 3.1 set $G=\mathrm{GL}(L_0)$ and $H=\mathrm{GL}(U_0)$. Let $\Pi_0=U_0\otimes L_0$ and $\Pi=\Pi_0(\mathcal{O})$.

Let $x\in X$ be a closed point. Remind that the Weil representation of $G(F_x)\times H(F_x)$ can be realized in the Schwarz space $\mathcal{S}(\Pi(F_x))$ of locally constant compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $\Pi(F_x)$. The action of $G(F_x)\times H(F_x)$ on this space comes from its natural action on $\Pi(F_x)$.

Write $\mathcal{H}_x(G)$ for the Hecke algebra of the pair $(G(F_x),G(\mathcal{O}_x))$, and similarly for $\mathcal{H}_x(H)$. Remind that $\mathcal{H}_x(G)$ identifies canonically with the Grothendieck group $K(\mathrm{Rep}(\check{G}))$ of the category $\mathrm{Rep}(\check{G})$ of $\check{G}$-representations over $\bar{\mathbb{Q}}_\ell$.

The space of invariants $\mathcal{S}(\Pi(F_x))^{G(\mathcal{O}_x)\times H(\mathcal{O}_x)}$ is naturally a module over $\mathcal{H}_x(G)\otimes\mathcal{H}_x(H)$. Let $\phi_0\in\mathcal{S}(\Pi(F))$ be the characteristic function of $\Pi(\mathcal{O})$. The following result is well-known (cf. [20], [22]), in Section 5 we prove its geometric version.

**Lemma 3.** *Assume $m \geq n$. The map $\mathcal{H}_x(G) \to \mathcal{S}(\Pi(F))^{(G\times H)(\mathcal{O}_x)}$ sending $h$ to $h\phi_0$ is an isomorphism of $\mathcal{H}_x(G)$-modules. There is a homomorphism $\kappa : \mathcal{H}_x(H) \to \mathcal{H}_x(G)$ such that the $\mathcal{H}_x(H)$-action on $\mathcal{S}(\Pi(F_x))^{G(\mathcal{O}_x)\times H(\mathcal{O}_x)}$ factors through $\kappa$.*

For $n = m$ the homomoprhism $\kappa$ comes from the functor $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G})$ of restriction with respect to an isomorphism $\check{G} \widetilde{\to} \check{H}$. For $m \geq n$ we will see that $\kappa$ comes from the functor $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G}\times\mathbb{G}_m) \widetilde{\to} \mathrm{D}\,\mathrm{Sph}_G$ of restriction with respect to a homomorphism $\check{G}\times\mathbb{G}_m \to \check{H}$. For $m > n + 1$ the restriction of this homomoprhism to $\mathbb{G}_m$ is nontrivial.

3.2 Weil representation of $\mathbb{SO}_{2m} \times \mathbb{Sp}_{2n}$

3.2.1 Let us explain the idea of the proof of Theorem 3. Keep the notation of Section 2.4. Let $U_0 = \mathcal{O}_X^m$ and $V_0 = U_0 \oplus U_0^*$, we equip $V_0$ with the symmetric form $\mathrm{Sym}^2 V_0 \to \mathcal{O}_X$ given by the pairing between $U_0$ and $U_0^*$, so $U_0$ and $U_0^*$ are isotropic subbundles in $V_0$. Think of $V_0$ as the standard representation of $H$.

Let $P(H) \subset H$ be the parabolic subgroup preserving $U_0$, let $U(H) \subset P(H)$ be its unipotent radical, so $U(H) \widetilde{\to} \wedge^2 U_0$ canonically.

Let $L_0 = \mathcal{O}_X^n$ and $M_0 = L_0 \oplus L_0^* \otimes \Omega$. We equip $M_0$ the symplectic form $\wedge^2 M_0 \to \Omega$ given by the pairing $L_0$ with $L_0^* \otimes \Omega$. So, $L_0$ and $L_0^* \otimes \Omega$ are lagrangian subbundles in $M_0$. Remind that $G$ is the group scheme over $X$ of automorphisms of $M_0$ preserving the symplectic form.

Let $P(G) \subset G$ be the parabolic subgroup preserving $L_0$, write $U(G) \subset P(G)$ for its unipotent radical. We have $U(G) \widetilde{\to} \Omega^{-1} \otimes \mathrm{Sym}^2 L_0$ canonically.

Set $\mathcal{M}_0 = V_0 \otimes M_0$, it is equipped with a symplectic form, which is the tensor product of forms on $V_0$ and $M_0$.

Set $F = k(X)$. Let $\mathbb{A}$ be the adèles ring of $F$, $\mathcal{O} \subset \mathbb{A}$ be the entire adeles. Write $F_x$ for the completion of $F$ at $x \in X$. Let $\chi : \Omega(\mathbb{A})/\Omega(F) \to \bar{\mathbb{Q}}_\ell^*$ denote the character

$$\chi(\omega) = \psi(\sum_{x\in X} \mathrm{tr}_{k(x)/k} \mathrm{Res}\, \omega_x)$$

Let $\mathrm{Hs} = \mathcal{M}_0 \oplus \Omega$ be the Heisenberg group over $X$ constructed out of the symplectic bundle $\mathcal{M}_0$. The product in Hs is given by

$$(m_1, \omega_1)(m_2, \omega_2) = (m_1 + m_2, \omega_1 + \omega_2 + \frac{1}{2}\langle m_1, m_2\rangle)$$

For the generalities on the metaplectic extension $\widetilde{\mathbb{Sp}}(\mathcal{M}_0)$ of $\mathbb{Sp}(\mathcal{M}_0)$ and its Weil representation we refer the reader to [17]. The natural map $G(\mathbb{A}) \times H(\mathbb{A}) \to \mathbb{Sp}(\mathcal{M}_0)(\mathbb{A})$ lifts naturally to a homomorphism $G(\mathbb{A}) \times H(\mathbb{A}) \to \widetilde{\mathbb{Sp}}(\mathcal{M}_0)(\mathbb{A})$. We use two Schrödinger models of the corresponding Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$.

Set $\mathcal{L}_0 = V_0 \otimes L_0 \subset V_0 \otimes M_0$, this is a Lagrangian subbundle in $\mathcal{M}_0$. Let

$$\chi_{\mathcal{L}} : \mathcal{L}_0(\mathbb{A}) \oplus \Omega(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^*$$

denote the character $\chi_{\mathcal{L}}(u, \omega) = \chi(\omega)$. Let $\mathcal{S}_{\mathcal{L},\psi}$ denote the induced representation of $(\mathcal{L}_0(\mathbb{A}) \oplus \Omega(\mathbb{A}), \chi_{\mathcal{L}})$ to $\mathrm{Hs}(\mathbb{A})$. By definition, $\mathcal{S}_{\mathcal{L},\psi}$ is the space of functions $f : \mathrm{Hs}(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell$ satisfying:

- $f(ah) = \chi_{\mathcal{L}}(a) f(h)$ for $a \in \mathcal{L}_0(\mathbb{A}) \oplus \Omega(\mathbb{A})$, $h \in \mathrm{Hs}(\mathbb{A})$;
- there is an open subgroup $U \subset \mathcal{M}_0(\mathbb{A})$ such that $f(h(u,0)) = f(h)$ for $u \in U$, $h \in \mathrm{Hs}(\mathbb{A})$;
- $f$ is of compact support modulo $\mathcal{L}_0(\mathbb{A}) \oplus \Omega(\mathbb{A})$.

For a free $\mathbb{A}$-module (or free $F_x$-module) $R$ of finite type denote by $\mathcal{S}(R)$ the Schwarz space of locally constant compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $R$. We have an isomorphism $\mathcal{S}_{\mathcal{L},\psi} \widetilde{\to} \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$ sending $f$ to $\phi$ given by $\phi(v) = f(v,0)$, $v \in V_0 \otimes L_0^* \otimes \Omega(\mathbb{A})$.

The theta-functional

$$\Theta_{\mathcal{L}} : \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A})) \to \bar{\mathbb{Q}}_\ell$$

is given by

$$\Theta_{\mathcal{L}}(\phi) = \sum_{v \in V_0 \otimes L_0^* \otimes \Omega(F)} \phi(v) \quad \text{for } \phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$$

Set $\mathcal{U}_0 = U_0 \otimes M_0$, this is a Lagrangian subbundle in $\mathcal{M}_0$. Let

$$\chi_{\mathcal{U}} : \mathcal{U}_0(\mathbb{A}) \oplus \Omega(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^*$$

be the character $\chi_{\mathcal{U}}(u,\omega) = \chi(\omega)$. Let $\mathcal{S}_{\mathcal{U},\psi}$ denote the induced representation of $(\mathcal{U}_0(\mathbb{A}) \oplus \Omega(\mathbb{A}), \chi_{\mathcal{U}})$ to $\mathrm{Hs}(\mathbb{A})$. As above, we identify it with the Schwarz space $\mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$.

The theta-functional $\Theta_{\mathcal{U}} : \mathcal{S}(U_0^* \otimes M_0(\mathbb{A})) \to \bar{\mathbb{Q}}_\ell$ is given by

$$\Theta_{\mathcal{U}}(\phi) = \sum_{t \in U_0^* \otimes M_0(F)} \phi(t) \quad \text{for } \phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$$

For a locally free $\mathcal{O}_X$-module $\mathcal{Y}$ of finite type write $\chi(\mathcal{Y})$ for the Euler characteristic of $\mathcal{Y}$. Set $\epsilon = q^{\chi(U_0^* \otimes L_0)}$. Let us construct a diagram of $H(\mathbb{A}) \times G(\mathbb{A})$-representations

$$\begin{array}{ccc} \mathcal{S}(U_0^* \otimes M_0(\mathbb{A})) & \stackrel{\theta_{\mathcal{U}}}{\to} & \mathrm{Funct}((H \times G)(F) \backslash (H \times G)(\mathbb{A})) \\ \uparrow \zeta & \nearrow \epsilon\theta_{\mathcal{L}} & \\ \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A})), & & \end{array} \tag{14}$$

where $H(\mathbb{A}) \times G(\mathbb{A})$ acts on the space of functions $\mathrm{Funct}((H \times G)(F) \backslash (H \times G)(\mathbb{A}))$ by right translations. The map $\theta_{\mathcal{U}}$ sends $\phi$ to $\theta_{\mathcal{U},\phi}$ given by $\theta_{\mathcal{U},\phi}(h,g) = \Theta_{\mathcal{U}}((h,g)\phi)$. The map $\theta_{\mathcal{L}}$ sends $\phi$ to $\theta_{\mathcal{L},\phi}$ given by

$$\theta_{\mathcal{L},\phi}(h,g) = \Theta_{\mathcal{L}}((h,g)\phi)$$

For $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$ let $\zeta\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$ be given by

$$(\zeta\phi)(b) = \int_{U_0 \otimes L_0^* \otimes \Omega(\mathbb{A})} \chi(\langle a, b_1 \rangle) \phi(a + b_2) da \tag{15}$$

Here for $b \in U_0^* \otimes M_0(\mathbb{A})$ we write $b = b_1 + b_2$ with $b_1 \in U_0^* \otimes L_0(\mathbb{A})$ and $b_2 \in U_0^* \otimes L_0^* \otimes \Omega(\mathbb{A})$, and $da$ is the Haar measure on $U_0 \otimes L_0^* \otimes \Omega(\mathbb{A})$ normalized by requiring that the volume of $U_0 \otimes L_0^* \otimes \Omega(\mathcal{O})$ is one. It is known that $\zeta$ is an isomorphism of $G(\mathbb{A}) \times H(\mathbb{A})$-modules (cf. [20]).

Let $\phi_{0,\mathcal{U}}$ (resp., $\phi_{0,\mathcal{L}}$) be the characteristic function of $U_0^* \otimes M_0(\mathcal{O})$ (resp., of $V_0 \otimes L_0^* \otimes \Omega(\mathcal{O})$). An easy calculation shows that $\zeta\phi_{0,\mathcal{L}} = \phi_{0,\mathcal{U}}$.

**Lemma 4.** *The diagram (14) commutes.*

*Proof* We have $\Theta_{\mathcal{L}}(\phi_{0,\mathcal{L}}) = q^{\dim \mathrm{H}^0(X, V_0\otimes L_0^*\otimes\Omega)}$ and $\Theta_{\mathcal{U}}(\phi_{0,\mathcal{U}}) = q^{\dim \mathrm{H}^0(X, U_0^*\otimes M_0)}$. Since

$$\dim \mathrm{H}^0(X, U_0^* \otimes M_0) = \chi(U_0^* \otimes L_0) + \dim \mathrm{H}^0(X, V_0 \otimes L_0^* \otimes \Omega),$$

we get $\Theta_{\mathcal{U}} \circ \zeta = \epsilon\Theta_{\mathcal{L}}$. Since $\zeta$ is an isomorphism of $G(\mathbb{A}) \times H(\mathbb{A})$-modules, our assertion follows. $\square$

Write $\mathcal{H}(H)$ for the Hecke algebra of the pair $(H(\mathcal{O}), H(\mathbb{A}))$, and similarly for $G$. Passing to the $(G \times H)(\mathcal{O})$-invariants, one gets from (14) the commutative diagram

$$\begin{array}{ccc} \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H\times G)(\mathcal{O})} & \stackrel{\theta_{\mathcal{U}}}{\to} & \mathrm{Funct}(\mathrm{Bun}_{G,H}(k)) \\ \uparrow \zeta & \nearrow \epsilon\theta_{\mathcal{L}} & \\ \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})} & & \end{array} \qquad (16)$$

of $\mathcal{H}(H) \otimes \mathcal{H}(G)$-modules. The notation $\mathrm{Bun}_{G,H}$ is that of Section 2.4.1.

Let $\phi_0 \in \mathrm{Funct}(\mathrm{Bun}_{G,H}(k))$ be the function trace of Frobenius of $\mathrm{Aut}_{G,H}$. Then $\theta_{\mathcal{U}}\phi_{0,\mathcal{U}}$ equals $\phi_0$ up to a multiple.

For $x \in X$ let $\phi_{0,\mathcal{U},x} \in \mathcal{S}(U_0^* \otimes M_0(F_x))$ be the characteristic function of $U_0^* \otimes M_0(\mathcal{O}_x)$, let $\phi_{0,\mathcal{L},x} \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(F_x))$ be the characteristic function of $V_0 \otimes L_0^* \otimes \Omega(\mathcal{O}_x)$.

Denote by $\mathcal{H}_x(G)$ the Hecke algebra of the pair $(G(\mathcal{O}_x), G(F_x))$, and similarly for $H$. Remind the decomposition as a restricted tensor product

$$\mathcal{H}(G) \widetilde{\to} \bigotimes_{x\in X}{}' \mathcal{H}_x(G)$$

Similarly, we have

$$\mathcal{S}(U_0^* \otimes M_0(\mathbb{A})) \widetilde{\to} \bigotimes_{x\in X}{}' \mathcal{S}(U_0^* \otimes M_0(F_x))$$

In view of this isomorphism $\mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$ is generated as a $\bar{\mathbb{Q}}_\ell$-vector space by functions of the form $\otimes_x \phi_x$ with $\phi_x \in \mathcal{S}(U_0^* \otimes M_0(F_x))$, where $\phi_x = \phi_{0,\mathcal{U},x}$ for all but finite number of $x \in X$.

In particular, we have a canonical diagram

$$\begin{array}{ccc} \mathcal{S}(U_0^* \otimes M_0(F_x))^{(H\times G)(\mathcal{O}_x)} & \hookrightarrow & \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H\times G)(\mathcal{O})} \\ \uparrow \zeta_x & & \uparrow \zeta \\ \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(F_x))^{(H\times G)(\mathcal{O}_x)} & \hookrightarrow & \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})}, \end{array}$$

where $\zeta_x$ is given by (15) with $U_0 \otimes L_0^* \otimes \Omega(\mathbb{A})$ replaced by $U_0 \otimes L_0^* \otimes \Omega(F_x)$.

Set

$$\begin{aligned} \mathcal{W}\mathrm{eil}_{G,H}(k) = \{(f_1, f_2) \mid f_1 \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(F_x))^{(H\times G)(\mathcal{O}_x)}, \\ f_2 \in \mathcal{S}(U_0^* \otimes M_0(F_x))^{(H\times G)(\mathcal{O}_x)} \text{ such that } \zeta_x(f_1) = f_2\} \end{aligned}$$

The Hecke property of $\phi_0$ (a classical analogue of Theorem 4) is as follows.

**Proposition 1.** *1) Assume $m \le n$. There is a homomorphism $\kappa : \mathcal{H}_x(G) \to \mathcal{H}_x(H)$ such that for $h \in \mathcal{H}_x(G)$ we have*

$$ {}_x\mathrm{H}^{\leftarrow}_G(h, \phi_0) = {}_x\mathrm{H}^{\rightarrow}_H(\kappa(h), \phi_0) $$

*2) Assume $m > n$. There is a homomoprhism $\kappa : \mathcal{H}_x(H) \to \mathcal{H}_x(G)$ such that for $h \in \mathcal{H}_x(H)$ we have*

$$ {}_x\mathrm{H}^{\leftarrow}_H(h, \phi_0) = {}_x\mathrm{H}^{\rightarrow}_G(\kappa(h), \phi_0) $$

The above discussion reduces the proof of Proposition 1 to the following local result.

**Proposition 2.** *1) Assume $m \le n$. There is a homomorphism $\kappa : \mathcal{H}_x(G) \to \mathcal{H}_x(H)$ such that for $h \in \mathcal{H}_x(G)$ we have*

$$ {}_x\mathrm{H}^{\leftarrow}_G(h, \phi_{0,\mathcal{U},x}) = \zeta_x({}_x\mathrm{H}^{\rightarrow}_H(\kappa(h), \phi_{0,\mathcal{L},x})) $$

*Moreover, $\mathcal{W}\mathrm{eil}_{G,H}(k)$ is a free module of rank one over $\mathcal{H}_x(H)$ generated by $\phi_{0,\mathcal{L},x}$.*
*2) Assume $m > n$. There is a homomoprhism $\kappa : \mathcal{H}_x(H) \to \mathcal{H}_x(G)$ such that for $h \in \mathcal{H}_x(H)$ we have*

$$ \zeta_x({}_x\mathrm{H}^{\leftarrow}_H(h, \phi_{0,\mathcal{L},x})) = {}_x\mathrm{H}^{\rightarrow}_G(\kappa(h), \phi_{0,\mathcal{U},x}) $$

*Moreover, $\mathcal{W}\mathrm{eil}_{G,H}(k)$ is a free module of rank one over $\mathcal{H}_x(G)$ generated by $\phi_{0,\mathcal{U},x}$.*

To the author's best knowledge, there are three different prooofs of Proposition 2 available in the litterature. First proof due to Rallis ([22]) is by some explicit calculation based on the following desciption of the Jacquet module. By ([15], Lemma 5.1) we have an isomorphism of $\mathrm{GL}(L_0)(F_x) \times H(F_x)$-representations

$$ \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(F_x))_{U(G)(F_x)} \widetilde{\rightarrow} \mathcal{S}(\mathrm{Cr}(V_0 \otimes L_0^* \otimes \Omega(F_x))) \quad (17) $$

Here $\mathrm{Cr}(V_0 \otimes L_0^* \otimes \Omega(F_x)) \subset V_0 \otimes L_0^* \otimes \Omega(F_x)$ is the subset of maps $v : L_0(F_x) \to V_0 \otimes \Omega(F_x)$ such that $s_{\mathcal{L}}(v) = 0$, where $s_{\mathcal{L}}(v)$ denotes the composition

$$ s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L_0(F_x) \stackrel{\mathrm{Sym}^2 v}{\rightarrow} \mathrm{Sym}^2(V_0 \otimes \Omega(F_x)) \to \Omega^2(F_x) $$

A different proof due to Howe (of a somewhat weaker statement) is exposed in [20] (a revisited version is given in [12]). One more proof is given by Kudla in [14]. Namely, in [15] it was shown that the Howe correspondence is compatible with the parabolic induction, this allows one to describe explicitely the image of a principal series representation under the Howe correspondence (cf. [14], Proposition 3.2, p.96), hence, to derive the functoriality ([14], Theorem on p. 105).

3.2.2 In Section 6 we prove Theorem 7, which is a geometric analogue of Proposition 2. The main difficulty is that the existing proofs of proof Proposition 2 do not geometrize in an obvious way. Our approach, though inspired by [22], is somewhat different.

One more feature is that classical proofs of Proposition 2 do not reveal a relation with the $\mathrm{SL}_2$ of Arthur, though it is believed to be relevant here (cf. the conjectures of Adams in [1]). In our approach at least the maximal torus of $\mathrm{SL}_2$ of Athur appears naturally.

3.2.3 In Section 8 we derive Theorem 4 from Theorem 7. In the rest of Section 3 we explain (at the level of functions) some ideas that will be used in Section 8 in the geometric setting.

For $a \in \mathbb{A}^*$ write $\mid a \mid \in \bar{\mathbb{Q}}_\ell$ for the absolut value of $a$. For a vector bundle $\mathcal{W}$ on $X$ and $e \in \mathrm{GL}(\mathcal{W})(\mathbb{A})$ we write $|e|=|\det e|$.

3.2.4 Action on $\mathcal{S}_{\mathcal{U},\psi}$.

Remind the canonical isomorphism $U(H)\widetilde{\to} \wedge^2 U_0$. For $b \in (\wedge^2 U_0)(\mathbb{A})$ the element $(b,1) \in H(\mathbb{A}) \times G(\mathbb{A})$ acts on $\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$ by

$$((b,1)\phi)(v) = \chi(\langle b, s_{\mathcal{U}}(v)\rangle)\phi(v), \quad v \in U_0^* \otimes M_0(\mathbb{A})$$

Here $s_{\mathcal{U}}(v)$ is the composition

$$(\wedge^2 U_0)(\mathbb{A}) \stackrel{\wedge^2 v}{\to} (\wedge^2 M_0)(\mathbb{A}) \to \Omega(\mathbb{A})$$

For $a \in \mathrm{GL}(U_0)(\mathbb{A}), g \in G(\mathbb{A})$ the pair $(a,g)$ acts on the left on $U_0^* \otimes M_0(\mathbb{A})$ sending $v : U_0(\mathbb{A}) \to M_0(\mathbb{A})$ to $g \circ v \circ a^{-1}$. So, $(a,g)$ acts on $\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$ by

$$((a,g)\phi)(v) = \mid a \otimes g \mid^{\frac{1}{2}} \phi(g^{-1} \circ v \circ a)$$

So, for $p = \begin{pmatrix} a & b \\ 0 & a^{*-1} \end{pmatrix} \in P(H)(\mathbb{A}), g \in G(\mathbb{A})$ the element $(p,g)$ acts on $\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))$ by

$$((p,g)\phi)(v) = \mid a \otimes g \mid^{\frac{1}{2}} \chi(\langle a^{-1}b, s_{\mathcal{U}}(g^{-1} \circ v \circ a)\rangle)\phi(g^{-1} \circ v \circ a) \tag{18}$$

Here $a \in \mathrm{GL}(U_0)(\mathbb{A}), b \in (U_0 \otimes U_0)(\mathbb{A})$. We get

$$\theta_{\mathcal{U},\phi}(p,g) = \sum_{v \in U_0^* \otimes M_0(F)} ((p,g)\phi)(v)$$

Let ${}_X\mathcal{V}^{ex}_{m,G}$ be the stack classifying $U \in \mathrm{Bun}_m, M \in \mathrm{Bun}_G$ and a map between the generic fibres $v : U(F) \to M(F)$ (here 'ex' stands for 'extended'). Its set of $k$-points identifies with $(GL(U_0)(F) \times G(F))\backslash X_{m,G}$, where

$$X_{m,G} = (U_0^* \otimes M_0)(F) \times (GL(U_0)(\mathbb{A})/\mathrm{GL}(U_0)(\mathcal{O})) \times (G(\mathbb{A})/G(\mathcal{O}))$$

is equipped with the diagonal left action of $GL(U_0)(F) \times G(F)$. Remind that $G(\mathbb{A})/G(\mathcal{O})$ is naturally in bijection with the isomorphism classes of pairs $(\mathcal{F}_G, \beta)$, where $\mathcal{F}_G \in \mathrm{Bun}_G$, $\beta : \mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{\mathrm{Spec}\, F}$ is a trivialization at the generic point.

Given $\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H \times G)(\mathcal{O})}$, for $(v, a\,\mathrm{GL}(U_0)(\mathcal{O}), gG(\mathcal{O})) \in X_{m,G}$ the value of

$$\phi(g^{-1} \circ v \circ a)$$

depends only on the image of $(v, a\,\mathrm{GL}(U_0)(\mathcal{O}), gG(\mathcal{O}))$ in ${}_X\mathcal{V}^{ex}_{m,G}(k)$. This gives rise to a map

$$\xi_{\mathcal{U}} : \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H \times G)(\mathcal{O})} \to \mathrm{Funct}({}_X\mathcal{V}^{ex}_{m,G}(k)) \tag{19}$$

sending $\phi$ to the function $(v, a\,\mathrm{GL}(U_0)(\mathcal{O}), gG(\mathcal{O})) \mapsto \phi(g^{-1} \circ v \circ a)$.

Let ${}_X\mathcal{V}_{m,G} \subset {}_X\mathcal{V}^{ex}_{m,G}$ be the substack classifying $v \in {}_X\mathcal{V}^{ex}_{m,G}$ such that the map $s_{\mathcal{U}}(v) : \wedge^2 U \to \Omega$ is regular over $X$.

**Lemma 5.** *For any $\phi \in \mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H\times G)(\mathcal{O})}$ its image under (19) is the extension by zero from ${}_X\mathcal{V}_{m,G}(k)$.*

*Proof* Note that $\phi$ is invariant under the action of $U(H)(\mathcal{O})$. So, if $v \in U_0^* \otimes M_0(\mathbb{A})$ and $\phi(v) \neq 0$ then for each $b \in (\wedge^2 U_0)(\mathcal{O})$ we have $\chi(\langle b, s_{\mathcal{U}}(v)\rangle) = 1$. So, $\phi(v) = 0$ unless $s_{\mathcal{U}}(v) \in (\Omega \otimes \wedge^2 U_0^*)(\mathcal{O})$.

Let $\eta = (M, U, U(F) \stackrel{v}{\to} M(F))$ be a $k$-point of ${}_X\mathcal{V}^{ex}_{m,G}$. Pick a collection

$$(t_0, a\,\mathrm{GL}(U_0)(\mathcal{O}), gG(\mathcal{O})) \in X_{m,G}$$

representing $\eta$. Then the condition $s_{\mathcal{U}}(g^{-1}\circ v\circ a) \in (\Omega\otimes\wedge^2 U_0^*)(\mathcal{O})$ is easily seen to be equivalent to the fact that $\eta \in {}_X\mathcal{V}_{m,G}$. $\square$

Let $\mathcal{Y}_{P(H)}$ be the stack classifying $U \in \mathrm{Bun}_m$ and $s : \wedge^2 U \to \Omega$. The stack $\mathrm{Bun}_{P(H)}$ classifies $U \in \mathrm{Bun}_m$ and an exact sequence $0 \to \wedge^2 U \to ? \to \mathcal{O} \to 0$. Thus, $\mathrm{Bun}_{P(H)}$ and $\mathcal{Y}_{P(H)}$ are (generalized) dual vector bundles over $\mathrm{Bun}_m$. Let

$$\pi_{\mathcal{U}} : {}_X\mathcal{V}_{m,G} \to \mathcal{Y}_{P(H)} \times \mathrm{Bun}_G$$

be the map sending $(M, U, U(F) \stackrel{v}{\to} M(F))$ to $(M, U,\, s_{\mathcal{U}}(v) : \wedge^2 U \to \Omega)$.

Let $\mathcal{V}_{m,G} \subset {}_X\mathcal{V}_{m,G}$ be the substack given by the condition that $v : U \to M$ is regular over $X$. From (18) one derives the following.

**Lemma 6.** *The diagram of $\mathcal{H}(G)$-modules commutes*

$$\begin{array}{ccccc}
\mathcal{S}(U_0^* \otimes M_0(\mathbb{A}))^{(H\times G)(\mathcal{O})} & \stackrel{\theta_{\mathcal{U}}}{\to} & \mathrm{Funct}(\mathrm{Bun}_{H\times G}(k)) & \to & \mathrm{Funct}(\mathrm{Bun}_{P(H)\times G}(k)) \\
\downarrow \xi_{\mathcal{U}} & & & \nearrow \mathrm{Four}_{\psi} & \\
\mathrm{Funct}({}_X\mathcal{V}_{m,G}(k)) & \stackrel{(\pi_{\mathcal{U}})_!}{\to} & \mathrm{Funct}(\mathcal{Y}_{P(H)} \times \mathrm{Bun}_G), & &
\end{array}$$

*where the right horizontal arrow is the restriction with respect to* $\mathrm{Bun}_{P(H)\times G} \to \mathrm{Bun}_{H\times G}$. *Besides, $\xi_{\mathcal{U}}\phi_{0,\mathcal{U}}$ is the characteristic function of $\mathcal{V}_{m,G}(k)$.* $\square$

The normalization of the Fourier transform operator $\mathrm{Four}_\psi$ is always that of Section 2.1. It is understood that $(\pi_{cU})_!$ is the summation along the fibres of $\pi_{\mathcal{U}}$.

### 3.2.5 Action on $\mathcal{S}_{\mathcal{L},\psi}$

The group $H(\mathbb{A}) \times \mathrm{GL}(L_0)(\mathbb{A})$ acts on the left on $V_0 \otimes L_0^* \otimes \Omega(\mathbb{A})$ by $(h, a)v = h \circ v \circ a^{-1}$. Here $v : L_0(\mathbb{A}) \to V_0 \otimes \Omega(\mathbb{A})$. So, $(h, a)$ acts on $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$ by

$$((h, a)\phi)(v) = \mid a \otimes h \mid^{\frac{1}{2}} \phi(h^{-1} \circ v \circ a)$$

for $a \in \mathrm{GL}(L_0)(\mathbb{A}), h \in H(\mathbb{A})$.

For $b \in U(G)(\mathbb{A}) \widetilde{\to} (\Omega^{-1} \otimes \mathrm{Sym}^2 L_0)(\mathbb{A})$ the element $(1, b)$ acts on $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$ by

$$((1, b)\phi)(v) = \chi(\langle b, s_{\mathcal{L}}(v)\rangle)\phi(v),$$

where $s_{\mathcal{L}}(v)$ is the composition

$$(\mathrm{Sym}^2 L_0)(\mathbb{A}) \stackrel{\mathrm{Sym}^2 v}{\to} (\Omega^2 \otimes \mathrm{Sym}^2 V_0)(\mathbb{A}) \to \Omega^2(\mathbb{A})$$

So, for $p = \begin{pmatrix} a & b \\ 0 & a^{*-1} \end{pmatrix} \in P(G)(\mathbb{A})$, $h \in H(\mathbb{A})$ the element $(h,p)$ acts on $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))$ by

$$((h,p)\phi)(v) = \mid a \otimes h \mid^{\frac{1}{2}} \chi(\langle a^{-1}b, s_{\mathcal{L}}(h^{-1} \circ v \circ a)\rangle)\phi(h^{-1} \circ v \circ a) \tag{20}$$

and we have

$$\theta_{\mathcal{L},\phi}(h,p) = \sum_{v \in V_0 \otimes L_0^* \otimes \Omega(F)} ((h,p)\phi)(v)$$

*Remark* 1. If the ground field $k$ is algebraically closed then for any connected affine algebraic group $\mathcal{G}$ over $k$, a $\mathcal{G}$-torsor on $X$ is trivial at the generic point (by [5], Theorem 2).

We have a natural inclusion of $H(F)\backslash H(\mathbb{A})/H(\mathcal{O})$ into $\mathrm{Bun}_H(k)$. If the ground field $k$ was algebraically closed (as it will indeed be in Section 8) then, by Remark 1, this inclusion would be a bijection.

Let ${}_X\mathcal{V}^{ex}_{n,H}$ be the stack classifying $L \in \mathrm{Bun}_n$, $V \in \mathrm{Bun}_H$ and a map between the generic fibres $v : L(F) \to V \otimes \Omega(F)$ (here 'ex' stands for 'extended'). Consider the set

$$X_{n,H} = (L_0^* \otimes V_0 \otimes \Omega)(F) \times (\mathrm{GL}(L_0)(\mathbb{A})/\mathrm{GL}(L_0)(\mathcal{O})) \times (H(\mathbb{A})/H(\mathcal{O}))$$

with the diagonal action of $\mathrm{GL}(L_0)(F) \times H(F)$. Then we have a canonical inclusion

$$j_k : (\mathrm{GL}(L_0)(F) \times H(F))\backslash X_{n,H} \hookrightarrow {}_X\mathcal{V}^{ex}_{n,H}(k)$$

Given $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})}$, for $(v, a\,\mathrm{GL}(L_0)(\mathcal{O}), hH(\mathcal{O})) \in X_{n,H}$ the value of

$$\phi(h^{-1} \circ v \circ a)$$

depends only on the image of $(v, a\,\mathrm{GL}(L_0)(\mathcal{O}), hH(\mathcal{O}))$ in ${}_X\mathcal{V}^{ex}_{n,H}(k)$. This gives rise to a map

$$\xi_{\mathcal{L}} : \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})} \to \mathrm{Funct}({}_X\mathcal{V}^{ex}_{n,H}(k)) \tag{21}$$

sending $\phi$ to the function $(v, a\,\mathrm{GL}(L_0)(\mathcal{O}), hH(\mathcal{O})) \mapsto \phi(h^{-1} \circ v \circ a)$, which we extend by zero under $j_k$.

Let ${}_X\mathcal{V}_{n,H} \subset {}_X\mathcal{V}^{ex}_{n,H}$ be the substack given by requiring that the map $s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2$ is regular over $X$.

**Lemma 7.** *For any $\phi \in \mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})}$ its image under (21) is the extension by zero from ${}_X\mathcal{V}_{n,H}(k)$.*

*Proof* The fact that $\phi$ is invariant under the action of $U(G)(\mathcal{O})$ means the following. If $v \in V_0 \otimes L_0^* \otimes \Omega(\mathbb{A})$ and $\phi(v) \neq 0$ then for each $b \in (\Omega^{-1} \otimes \mathrm{Sym}^2 L_0)(\mathcal{O})$ we have $\chi(\langle b, s_{\mathcal{L}}(v)\rangle) = 1$, so $s_{\mathcal{L}}(v) \in \Omega^2 \otimes \mathrm{Sym}^2 L_0^*(\mathcal{O})$.

Let $\eta = (V, L, L(F) \stackrel{v}{\to} V \otimes \Omega(F))$ be a $k$-point of ${}_X\mathcal{V}^{ex}_{n,H}$ that admits a representative

$$(v_0, a\,\mathrm{GL}(L_0)(\mathcal{O}), hH(\mathcal{O})) \in X_{n,H}$$

The condition $s_{\mathcal{L}}(h^{-1} \circ v_0 \circ a) \in \Omega^2 \otimes \mathrm{Sym}^2 L_0^*(\mathcal{O})$ is easily seen to be equivalent to the fact that $\eta \in {}_X\mathcal{V}_{n,H}(k)$. $\square$

Let $\mathcal{Y}_{P(G)}$ be the stack classifying $L \in \mathrm{Bun}_n$ and $s : \mathrm{Sym}^2 L \to \Omega^2$. The stack $\mathrm{Bun}_{P(G)}$ classifies $L \in \mathrm{Bun}_n$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. Thus, $\mathcal{Y}_{P(G)}$ and $\mathrm{Bun}_{P(G)}$ are dual (generalized) vector bundles over $\mathrm{Bun}_n$. Let

$$\pi_{\mathcal{L}} : {}_X\mathcal{V}_{n,H} \to \mathcal{Y}_{P(G)} \times \mathrm{Bun}_H$$

be the map sending $(L, V, L(F) \stackrel{v}{\to} V \otimes \Omega(F))$ to $(V, L, s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2)$.

Let $\mathcal{V}_{n,H} \subset {}_X\mathcal{V}_{n,H}$ be the substack given by the condition that $v : L \to V \otimes \Omega$ is regular over $X$. From (20) one derives the following.

**Lemma 8.** *The diagram of $\mathcal{H}(H)$-modules commutes*

$$\begin{array}{ccccc}
\mathcal{S}(V_0 \otimes L_0^* \otimes \Omega(\mathbb{A}))^{(H\times G)(\mathcal{O})} & \stackrel{\theta_{\mathcal{L}}}{\to} & \mathrm{Funct}(\mathrm{Bun}_{H\times G}(k)) & \to & \mathrm{Funct}(\mathrm{Bun}_{H\times P(G)}(k)) \\
\downarrow \xi_{\mathcal{L}} & & & \nearrow \mathrm{Four}_{\psi} & \\
\mathrm{Funct}({}_X\mathcal{V}_{n,H}(k)) & \stackrel{(\pi_{\mathcal{L}})_!}{\to} & \mathrm{Funct}(\mathrm{Bun}_H \times \mathcal{Y}_{P(G)}(k)), & &
\end{array}$$

*where the right horizontal arrow is the restriction under* $\mathrm{Bun}_{H\times P(G)} \to \mathrm{Bun}_{H\times G}$. *Besides, $\xi_{\mathcal{L}}\phi_{0,\mathcal{L}}$ is the characteristic function of $\mathcal{V}_{n,H}(k)$.* $\square$

It is understood that $(\pi_{\mathcal{L}})_!$ is the summation along the fibres of $\pi_{\mathcal{L}}$.

## 4. Geometric model of the Schwarz space and Hecke functors

4.1 Set $\mathcal{O} = k[[t]] \subset F = k((t))$, write $D^* = \mathrm{Spec}\, F \subset D = \mathrm{Spec}\, \mathcal{O}$. Let $\Omega$ be the completed module of relative differentials of $\mathcal{O}$ over $k$.

For a free $\mathcal{O}$-module $M$ of finite rank we introduce the categories $P(M(F)) \subset \mathrm{D}(M(F))$ as follows. For $N, r \geq 0$ set ${}_{N,r}M = t^{-N}M(\mathcal{O})/t^r M(\mathcal{O})$. Given positive integers $N_1 \geq N_2$, $r_1 \geq r_2$ we have a cartesian diagram

$$\begin{array}{ccc}
{}_{N_2,r_1}M & \stackrel{i}{\hookrightarrow} & {}_{N_1,r_1}M \\
\downarrow p & & \downarrow p \\
{}_{N_2,r_2}M & \stackrel{i}{\hookrightarrow} & {}_{N_1,r_2}M,
\end{array} \qquad (22)$$

where $i$ is the natural closed immersion, and $p$ is the projection.

By ([8], Lemma 4.8), the functor $\mathrm{D}({}_{N,r_2}M) \to \mathrm{D}({}_{N,r_1}M)$ given by $K \mapsto p^*K[\dim.\mathrm{rel}(p)]$ is fully faithful and exact for the perverse t-structures, and similarly for the functor $i_*$. These functors yield a diagram of full triangulated subcategories

$$\begin{array}{ccc} \mathrm{D}({}_{N_2,r_1}M) & \hookrightarrow & \mathrm{D}({}_{N_1,r_1}M) \\ \uparrow & & \uparrow \\ \mathrm{D}({}_{N_2,r_2}M) & \hookrightarrow & \mathrm{D}({}_{N_1,r_2}M) \end{array} \tag{23}$$

We let $\mathrm{P}(M(F))$ (resp., $\mathrm{D}(M(F))$) be the inductive 2-limit of $\mathrm{P}({}_{N,r}M)$ (resp., of $\mathrm{D}({}_{N,r}M)$) as $r, N$ go to infinity.

Set ${}_NM = t^{-N}M(\mathcal{O})$ viewed as a $k$-scheme (not of finite type).

4.2.1 Let $G$ be a connected reductive group over $k$, assume that $M = M_0 \otimes_k \mathcal{O}$, where $M_0$ is a given finite-dimensional representation of $G$.

For $N+r > 0$ the group $G(\mathcal{O})$ acts on ${}_{N,r}M$ via its finite-dimensional quotient $G(\mathcal{O}/t^{N+r}\mathcal{O})$. For $r_1 \ge N+r > 0$ the kernel of $G(\mathcal{O}/t^{r_1}\mathcal{O}) \to G(\mathcal{O}/t^{N+r}\mathcal{O})$ is a contractible unipotent group. So, the projection between the stack quotients

$$q : G(\mathcal{O}/t^{r_1}\mathcal{O})\backslash {}_{N,r}M \to G(\mathcal{O}/t^{N+r}\mathcal{O})\backslash {}_{N,r}M$$

yields an (exact for the perverse t-structures) equivalence of the equivariant derived categories

$$\mathrm{D}_{G(\mathcal{O}/t^{N+r}\mathcal{O})}({}_{N,r}M) \to \mathrm{D}_{G(\mathcal{O}/t^{r_1}\mathcal{O})}({}_{N,r}M)$$

Denote by $\mathrm{D}_{G(\mathcal{O})}({}_{N,r}M)$ the equivariant derived category $\mathrm{D}_{G(\mathcal{O}/t^{r_1}\mathcal{O})}({}_{N,r}M)$ for any $r_1 \ge N+r$.

The stack quotient of (22) by $G(\mathcal{O}/t^{N_1+r_1}\mathcal{O})$ yields a diagram

$$\begin{array}{ccc} \mathrm{D}_{G(\mathcal{O})}({}_{N_2,r_1}M) & \hookrightarrow & \mathrm{D}_{G(\mathcal{O})}({}_{N_1,r_1}M) \\ \uparrow & & \uparrow \\ \mathrm{D}_{G(\mathcal{O})}({}_{N_2,r_2}M) & \hookrightarrow & \mathrm{D}_{G(\mathcal{O})}({}_{N_1,r_2}M), \end{array} \tag{24}$$

where each arrow is a fully faithfull (and exact for the perverse t-structures) functor. Define $\mathrm{D}_{G(\mathcal{O})}(M(F))$ as the inductive 2-limit of $\mathrm{D}_{G(\mathcal{O})}({}_{N,r}M)$ as $N, r$ go to infinity.

Since $G(\mathcal{O}/t^{N+r}\mathcal{O})$ is connected, the category $\mathrm{P}_{G(\mathcal{O})}({}_{N,r}M)$ of $G(\mathcal{O}/t^{N+r}\mathcal{O})$-equivariant perverse sheaves on ${}_{N,r}M$ is a full subcategory of $\mathrm{P}({}_{N,r}M)$. The category $\mathrm{P}_{G(\mathcal{O})}(M(F))$ is defined along the same lines. A similar construction has been used in ([13], Sect. 4).

Since the Verdier duality is compatible with the transition functors in (24) and (23), we have the Verdier duality self-functors $\mathbb{D}$ on $\mathrm{D}_{G(\mathcal{O})}(M(F))$ and on $\mathrm{D}(M(F))$, they preserve perversity.

4.2.2 Write $\mathrm{Gr}_G$ for the affine grassmanian $G(F)/G(\mathcal{O})$ of $G$. Let us define the equivariant derived category $\mathrm{D}_{G(\mathcal{O})}(M(F) \times \mathrm{Gr}_G)$.

For $s_1, s_2 \ge 0$ let

$${}_{s_1,s_2}G(F) = \{g \in G(F) \mid t^{s_1}M(\mathcal{O}) \subset gM(\mathcal{O}) \subset t^{-s_2}M(\mathcal{O})\},$$

it is stable by left and right multiplication by $G(\mathcal{O})$, and ${}_{s_1,s_2}\mathrm{Gr}_G := ({}_{s_1,s_2}G(F))/G(\mathcal{O})$ is closed in $\mathrm{Gr}_G$. For $s'_1 \ge s_1$, $s'_2 \ge s_2$ we have a closed embedding ${}_{s_1,s_2}\mathrm{Gr}_G \hookrightarrow {}_{s'_1,s'_2}\mathrm{Gr}_G$, and the union of all ${}_{s_1,s_2}\mathrm{Gr}_G$ is $\mathrm{Gr}_G$. The map $g \mapsto g^{-1}$ yields an isomorphism ${}_{s_1,s_2}G(F) \widetilde{\to} {}_{s_2,s_1}G(F)$.

Assume for simplicity that $M_0$ is a faithful $G$-module, then the action of $G(\mathcal{O})$ on ${}_{s_1,s_2}\mathrm{Gr}_G$ factors through an action of $G(\mathcal{O}/t^{s_1+s_2})$.

For $N, r, s_1, s_2 \ge 0$ and $s \ge \max\{N+r, s_1+s_2\}$ we have the equivariant derived category

$$\mathrm{D}_{G(\mathcal{O}/t^s)}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G),$$

where the action of $G(\mathcal{O}/t^s)$ on ${}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G$ is diagonal. For $s' \ge s \ge \max\{N+r, s_1+s_2\}$ we have a canonical equivalence (exact for the perverse t-structures)

$$\mathrm{D}_{G(\mathcal{O}/t^s)}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G) \widetilde{\to} \mathrm{D}_{G(\mathcal{O}/t^{s'})}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G)$$

Define $\mathrm{D}_{G(\mathcal{O})}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G)$ as the category $\mathrm{D}_{G(\mathcal{O}/t^s)}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G)$ for any $s$ as above. As in Section 4.2.1, define $\mathrm{D}_{G(\mathcal{O})}(M(F) \times \mathrm{Gr}_G)$ as the inductive 2-limit of $\mathrm{D}_{G(\mathcal{O})}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G)$. The subcategory of perverse sheaves

$$\mathrm{P}_{G(\mathcal{O})}(M(F) \times \mathrm{Gr}_G) \subset \mathrm{D}_{G(\mathcal{O})}(M(F) \times \mathrm{Gr}_G)$$

is defined similarly.

4.3 Let $\mathrm{Sph}_G$ denote the category of spherical perverse sheaves on $\mathrm{Gr}_G$. Remind the canonical equivalence of tensor categories $\mathrm{Loc} : \mathrm{Rep}(\check{G}) \widetilde{\to} \mathrm{Sph}_G$ (cf. [19]).

Let us define an action of the tensor category $\mathrm{Sph}_G$ on $\mathrm{D}_{G(\mathcal{O})}(M(F))$ by Hecke operators. Let $\mathcal{G}\mathrm{r}_G$ denote the stack classifying $G$-torsors $\mathcal{F}_G$, $\mathcal{F}'_G$ on $D$ together with a trivialization $\beta : \mathcal{F}_G \mid_{D^*} \widetilde{\to} \mathcal{F}'_G \mid_{D^*}$. Write $\mathrm{Bun}_{G,D}$ for the stack of $G$-torsors on $D$, this is the classifying stack of $G(\mathcal{O})$. We have a diagram

$$\mathrm{Bun}_{G,D} \stackrel{h^{\leftarrow}}{\leftarrow} \mathcal{G}\mathrm{r}_G \stackrel{h^{\rightarrow}}{\to} \mathrm{Bun}_{G,D},$$

where $h^{\leftarrow}$ (resp., $h^{\rightarrow}$) sends $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$ to $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$).

Let ${}_D\mathcal{W}_G$ be the stack classifying a $G$-torsor $\mathcal{F}_G$ on the disk $D$ together with a section $s : \mathcal{O} \to M_{\mathcal{F}_G} \mid_{D^*}$. Consider the diagram

$${}_D\mathcal{W}_G \stackrel{h^{\leftarrow}_{\mathcal{W}}}{\leftarrow} {}_D\mathcal{W}_G \times_{\mathrm{Bun}_{D,G}} \mathcal{G}\mathrm{r}_G \stackrel{h^{\rightarrow}_{\mathcal{W}}}{\to} {}_D\mathcal{W}_G,$$

where we used $h^{\leftarrow}$ to define the fibre product, the map $h^{\leftarrow}_{\mathcal{W}}$ (resp., $h^{\rightarrow}_{\mathcal{W}}$) sends $(\mathcal{F}_G, \mathcal{F}'_G, \beta, \mathcal{O} \stackrel{s}{\to} M_{\mathcal{F}_G} \mid_{D^*})$ to $(\mathcal{F}_G, s)$ (resp., to $(\mathcal{F}'_G, s' : \mathcal{O} \to M_{\mathcal{F}'_G} \mid_{D^*})$). Here $s' = \beta \circ s$.

Informally, one may think of $\mathrm{D}_{G(\mathcal{O})}(M(F))$ as the category of certain complexes on ${}_D\mathcal{W}_G$. Let $\mathrm{id}^l$ (resp., $\mathrm{id}^r$) be the isomorphism of ${}_D\mathcal{W}_G \times_{\mathrm{Bun}_{D,G}} \mathcal{G}\mathrm{r}_G$ with the twisted product

$$M(F) \times^{G(\mathcal{O})} \mathrm{Gr}_G$$

such that the projection to the first term corresponds to $h^{\leftarrow}_{\mathcal{W}}$ (resp., to $h^{\rightarrow}_{\mathcal{W}}$). Informally speaking, for a complex $K$ on ${}_D\mathcal{W}_G$ and $\mathcal{T} \in \mathrm{Sph}_G$ one may form a twisted external product $(K \tilde{\boxtimes} \mathcal{T})^l$ and

$(K\,\tilde{\boxtimes}\,\mathcal{T})^r$. The Hecke functors $H^{\leftarrow}_G(\cdot,\cdot)$ and $\mathrm{H}^{\rightarrow}_G(\cdot,\cdot)$ from $\mathrm{Sph}_G \times \mathrm{D}_{G(\mathcal{O})}(M(F))$ to $\mathrm{D}_{G(\mathcal{O})}(M(F))$ are informally defined by

$$H^{\leftarrow}_G(\mathcal{T},K) = (h^{\leftarrow}_G)_!(*\mathcal{T}\,\tilde{\boxtimes}\,K)^r \quad\text{and}\quad H^{\rightarrow}_G(\mathcal{T},K) = (h^{\rightarrow}_G)_!(\mathcal{T}\,\tilde{\boxtimes}\,K)^l. \tag{25}$$

Here $* : \mathrm{Sph}_G \to \mathrm{Sph}_G$ is the covariant functor introduced in Section 2.2.1.

Let us give a formal sense to (25). Consider the map $M(F)\times G(F) \widetilde{\to} M(F)\times G(F)$ sending $(m,g)$ to $(g^{-1}m, g)$. Let $(a,b)\in G(\mathcal{O})\times G(\mathcal{O})$ act on the source sending $(m,g)$ to $(am, agb)$. Let it also act on the target sending $(m',g')$ to $(b^{-1}m', ag'b)$. The above map is equivariant for these actions, so yields a morphism of stacks

$${}_q\,\mathrm{act} : G(\mathcal{O})\backslash(M(F)\times \mathrm{Gr}_G) \to (M(F)/G(\mathcal{O}))\times (G(\mathcal{O})\backslash \mathrm{Gr}_G),$$

where the action of $G(\mathcal{O})$ on $M(F)\times \mathrm{Gr}_G$ is the diagonal one.

The connected components of $\mathrm{Gr}_G$ are indexed by $\pi_1(G)$. For $\theta\in\pi_1(G)$ the component $\mathrm{Gr}^\theta_G$ is the one containing $\mathrm{Gr}^\lambda_G$ for any $\lambda\in\Lambda^+_G$ whose image in $\pi_1(G)$ equals $\theta$. For $\theta\in\pi_1$ set ${}_{s_1,s_2}\mathrm{Gr}^\theta_G = \mathrm{Gr}^\theta_G \cap {}_{s_1,s_2}\mathrm{Gr}_G$.

**Lemma 9.** *There exists the 'inverse image' functor*

$${}_q\,\mathrm{act}^*(\cdot,\cdot) : \mathrm{D}_{G(\mathcal{O})}(M(F)) \times \mathrm{D}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}(M(F)\times \mathrm{Gr}_G) \tag{26}$$

*satisfying the following. For $K\in \mathrm{D}_{G(\mathcal{O})}(M(F))$ and $\mathcal{T}\in \mathrm{D}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ we have*

$$\mathbb{D}({}_q\mathrm{act}^*(K,\mathcal{T})) \widetilde{\to} {}_q\,\mathrm{act}^*(\mathbb{D}(K),\mathbb{D}(\mathcal{T}))$$

*naturally. If both $K$ and $\mathcal{T}$ are perverse then ${}_q\,\mathrm{act}^*(K,\mathcal{T})$ is perverse.*

*Proof* For non negative integers $N, r, s_1, s_2$, with $r\ge s_1$ and $s\ge \max\{s_1+s_2, N+r\}$ we have a diagram

$$\begin{array}{ccccc}
 & & {}_{N,r}M \times {}_{s_1,s_2}G(F) & \stackrel{\mathrm{act}}{\to} & {}_{N+s_1,r-s_1}M \\
 & & \downarrow q_G & & \downarrow q_M \\
{}_{N,r}M & \stackrel{\mathrm{pr}}{\leftarrow} & {}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G & \stackrel{\mathrm{act}_q}{\to} & G(\mathcal{O}/t^s)\backslash {}_{N+s_1,r-s_1}M \\
\downarrow & & \downarrow & \nearrow \mathrm{act}_{q,s} & \\
G(\mathcal{O}/t^s)\backslash {}_{N,r}M & \stackrel{\mathrm{pr}}{\leftarrow} & G(\mathcal{O}/t^s)\backslash({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{pr}_2}{\to} & G(\mathcal{O}/t^s)\backslash({}_{s_1,s_2}\mathrm{Gr}_G)
\end{array}$$

Here act sends $(m,g)$ to $g^{-1}m$, the map $q_G$ sends $(m,g)$ to $(m, gG(\mathcal{O}))$, and $\mathrm{pr}, \mathrm{pr}_2$ denote the projections. All the vertical arrows are the stack quotient maps for the action of a corresponding group. One checks that act descends to a map $\mathrm{act}_q$ between the corresponding quotients.

For $s\ge \max\{s_1+s_2, N+r\}$ the group $G(\mathcal{O}/t^s\mathcal{O})$ acts diagonally on ${}_{N,r}M\times {}_{s_1,s_2}\mathrm{Gr}_G$, and $\mathrm{act}_q$ is equivariant with respect to this action. Consider the functors

$$\begin{array}{ccc}
\mathrm{D}_{G(\mathcal{O}/t^s)}({}_{N+s_1,r-s_1}M)\times \mathrm{D}_{G(\mathcal{O}/t^s)}({}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{temp}}{\to} & \mathrm{D}_{G(\mathcal{O}/t^s)}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G) \\
\| & & \| \\
\mathrm{D}_{G(\mathcal{O})}({}_{N+s_1,r-s_1}M)\times \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) & & \mathrm{D}_{G(\mathcal{O})}({}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G)
\end{array}$$

sending $(K, \mathcal{T})$ to

$$\mathrm{act}_{q,s}^*(K) \otimes \mathrm{pr}_2^* \mathcal{T}[s \dim G + s_1 \dim M_0 - c], \tag{27}$$

where $c$ equals to $\langle \theta, \check{\mu} \rangle$ over ${}_{s_1,s_2}\mathrm{Gr}_G^\theta$. Here $\check{\mu} \in \check{\Lambda}_G^+$ denotes the character $\det M_0$.

For $r_1 \geq r_2$ and $s \geq \max\{s_1+s_2, N+r_1\}$ the functors temp are compatible with the transition functors for the diagram

$$\begin{array}{ccc} G(\mathcal{O}/t^s)\backslash({}_{N,r_1}M \times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash_{N+s_1,r_1-s_1}M \\ \downarrow & & \downarrow \\ G(\mathcal{O}/t^s)\backslash({}_{N,r_2}M \times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash_{N+s_1,r_2-s_1}M \end{array}$$

So, they yield a functor

$${}_{N,s_1,s_2}\mathrm{temp} : \mathrm{D}_{G(\mathcal{O})}({}_{N+s_1}M) \times \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}({}_N M \times_{s_1,s_2} \mathrm{Gr}_G)$$

For $N_1 \geq N_2$ and $s \geq \max\{s_1 + s_2, N_1 + r\}$ we have a diagram, where the vertical maps are closed immersions

$$\begin{array}{ccc} G(\mathcal{O}/t^s)\backslash({}_{N_1,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash_{N_1+s_1,r-s_1}M \\ \uparrow & & \uparrow \\ G(\mathcal{O}/t^s)\backslash({}_{N_2,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash_{N_2+s_1,r-s_1}M \end{array}$$

This diagram is not cartesian in general, we come around this as follows. If $K \in \mathrm{D}_{G(\mathcal{O})}({}_N M)$, $\mathcal{T} \in \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G)$ then for any $N_1 \geq N + s_2$ the image of $(K, \mathcal{T})$ under the composition

$$\mathrm{D}_{G(\mathcal{O})}({}_N M) \times \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) \subset \mathrm{D}_{G(\mathcal{O})}({}_{N_1+s_1}M) \times \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) \stackrel{{}_{N_1,s_1,s_2}\mathrm{temp}}{\to}$$
$$\mathrm{D}_{G(\mathcal{O})}({}_{N_1}M \times {}_{s_1,s_2}\mathrm{Gr}_G) \subset \mathrm{D}_{G(\mathcal{O})}(M(F) \times {}_{s_1,s_2}\mathrm{Gr}_G)$$

does not depend on $N_1$, so we get a functor

$${}_{s_1,s_2}\mathrm{temp} : \mathrm{D}_{G(\mathcal{O})}(M(F)) \times \mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}(M(F) \times_{s_1,s_2} \mathrm{Gr}_G)$$

For $s_1' \geq s_1, s_2' \geq s_2$ we have the functors of extension by zero

$$\mathrm{D}_{G(\mathcal{O})}({}_{s_1,s_2}\mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}({}_{s_1',s_2'}\mathrm{Gr}_G)$$

They are compatible with ${}_{s_1,s_2}\mathrm{temp}$. This yields the desired functor (26). A proof of its properties is left to the reader. □

For nonnegative integers $s_1, s_2, N, r$ and $s \geq \max\{N + r, s_1 + s_2\}$ for the projection

$$\mathrm{pr} : G(\mathcal{O}/t^s)\backslash({}_{N,r}M \times_{s_1,s_2} \mathrm{Gr}_G) \to G(\mathcal{O}/t^s)\backslash_{N,r}M$$

the corresponding functors $\mathrm{pr}_! : \mathrm{D}_{G(\mathcal{O})}({}_{N,r}M \times_{s_1,s_2} \mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}({}_{N,r}M)$ are compatible with the transition functors, so yield a functor $\mathrm{pr}_! : \mathrm{D}_{G(\mathcal{O})}(M(F) \times \mathrm{Gr}_G) \to \mathrm{D}_{G(\mathcal{O})}(M(F))$.

Finally, we define the Hecke functor

$$\mathrm{H}^{\leftarrow}_G(\mathcal{T},\cdot) : \mathrm{D}_{G(\mathcal{O})}(M(F)) \to \mathrm{D}_{G(\mathcal{O})}(M(F)) \tag{28}$$

by $\mathrm{H}^{\leftarrow}_G(\mathcal{T},K) = \mathrm{pr}_!({}_q\mathrm{act}^*(K,\mathcal{T}))$ for $\mathcal{T} \in \mathrm{Sph}_G$ and $K \in \mathrm{D}_{G(\mathcal{O})}(M(F))$. By Lemma 9, the functors (28) commute with Verdier duality, namely

$$\mathbb{D}(\mathrm{H}^{\leftarrow}_G(\mathcal{T},K)) \widetilde{\to} \mathrm{H}^{\leftarrow}_G(\mathbb{D}\mathcal{T},\mathbb{D}K)$$

They are also compatible with the tensor structure on $\mathrm{Sph}_G$ (as in [4], Section 3.2.4). For $\mathcal{T} \in \mathrm{Sph}_G$ and $K \in \mathrm{D}_{G(\mathcal{O})}(M(F))$ set $\mathrm{H}^{\rightarrow}_G(\mathcal{T},K) = \mathrm{H}^{\leftarrow}_G(*\mathcal{T},K)$. Then the functors

$$K \mapsto \mathrm{H}^{\leftarrow}_G(\mathcal{T},K) \quad \text{and} \quad K \mapsto \mathrm{H}^{\rightarrow}_G(\mathbb{D}(\mathcal{T}),K)$$

are mutually (both left and right) adjoint.

For a $G$-dominant coweight $\lambda$ we set $\mathrm{H}^{\lambda}_G(\cdot) = \mathrm{H}^{\leftarrow}_G(\mathcal{A}^{\lambda}_G,\cdot)$.

*Remark* 2. Call $K \in \mathrm{P}_{G(\mathcal{O})}({}_RM)$ *smooth* if it comes from a $G(\mathcal{O})$-equivariant local system on ${}_{R,r}M$ for some $r$. Let us make the above definition explicit in this case.

Let us above $\check{\mu} \in \check{\Lambda}^+_G$ denote the character $\det M_0$, so the virtual dimension $\dim(M/gM) = \langle\theta,\check{\mu}\rangle$ for $gG(\mathcal{O}) \in \mathrm{Gr}^{\theta}_G$. Let $\mathcal{T} \in \mathrm{Sph}_G$ be the extension by zero from ${}_{s_1,s_2}\mathrm{Gr}_G$. For $r$ large enough, let ${}_{R,r}M\tilde{\times}_{s_1,s_2}\mathrm{Gr}_G$ be the scheme of pairs $(m, gG(\mathcal{O}))$ with $gG(\mathcal{O}) \in {}_{s_1,s_2}\mathrm{Gr}_G$ and $m \in t^{-R}gM/t^rM$. Set

$${}_{s_1,s_2}\mathrm{Gr}^{\theta}_G = {}_{s_1,s_2}\mathrm{Gr}_G \cap \mathrm{Gr}^{\theta}_G$$

Then ${}_{R,r}M\tilde{\times}_{s_1,s_2}\mathrm{Gr}^{\theta}_G$ is a locally trivial fibration over ${}_{s_1,s_2}\mathrm{Gr}^{\theta}_G$ with fibre an affine space of dimension $(R+r)\dim M_0 - \langle\theta,\check{\mu}\rangle$. We get a diagram

$${}_{R+s_2,r}M \stackrel{\mathrm{pr}}{\leftarrow} {}_{R,r}M\tilde{\times}_{s_1,s_2}\mathrm{Gr}_G \stackrel{\mathrm{act}_q}{\to} G(\mathcal{O}/t^{R+r-s_1})\backslash({}_{R,r-s_1}M),$$

where pr sends $(m, gG(\mathcal{O}))$ to $m$. Let $K\tilde{\boxtimes}\mathcal{T}$ denote the perverse sheaf $\mathrm{act}_q^*K \otimes \mathrm{pr}_2^*\mathcal{T}[\dim]$ on ${}_{R,r}M\tilde{\times}_{s_1,s_2}\mathrm{Gr}_G$. Then $\mathrm{H}^{\leftarrow}_G(\mathcal{T},K) = \mathrm{pr}_!(K\tilde{\boxtimes}\mathcal{T})$. We see once again that indeed the shift in (27) must depend on $\check{\mu}$.

4.4 One can always address the following

**Question.** Let $I_0$ denote the constant sheaf $\bar{\mathbb{Q}}_\ell$ on ${}_{0,0}M$, it is an object of $P_{G(\mathcal{O})}(M(F))$. Describe the submodule over $\mathrm{Sph}_G$ in $\mathrm{D}_{G(\mathcal{O})}(M(F))$ generated by $I_0$.

*Remark* 3. Assume that all the weights of $M_0$ are less or equal to a $G$-dominant weight $\check{\lambda}$. Then for a dominant coweight $\lambda$ of $G$ we have $\mathrm{H}^{\lambda}_G(I_0) \in \mathrm{D}_{G(\mathcal{O})}({}_{N,r}M)$ with $N = \langle -w_0(\lambda),\check{\lambda}\rangle$ and $r = \langle\lambda,\check{\lambda}\rangle$.

Let $M(\mathcal{O})\tilde{\times}\overline{\mathrm{Gr}}^{\lambda}_G$ be the scheme classifying pairs $gG(\mathcal{O}) \in \overline{\mathrm{Gr}}^{\lambda}_G$, $m \in gM(\mathcal{O})$. Let $\pi_{\mathrm{pro}} : M(\mathcal{O})\tilde{\times}\overline{\mathrm{Gr}}^{\lambda}_G \to {}_NM$ be the map sending $(m, gG(\mathcal{O}))$ to $m \in {}_NM$. Write ${}_{0,r}M\tilde{\times}\overline{\mathrm{Gr}}^{\lambda}_G$ for the scheme classifying $gG(\mathcal{O}) \in \overline{\mathrm{Gr}}^{\lambda}_G$, $m \in gM(\mathcal{O})/t^rM(\mathcal{O})$. The map $\pi_{pro}$ gives rise to a proper map

$$\pi : {}_{0,r}M\tilde{\times}\overline{\mathrm{Gr}}^{\lambda}_G \to {}_{N,r}M \tag{29}$$

sending $(m, gG(\mathcal{O}))$ to $m$. By definition, $\mathrm{H}^{\lambda}_G(I_0)$ identifies with $\pi_!(\bar{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^{\lambda}_G)[\dim {}_{0,r}M]$.

4.5 The group of automorphisms of the $k$-algebra $\mathcal{O}$ is naturally the group of $k$-points of a (reduced) affine group scheme $\mathrm{Aut}^0\mathcal{O}$ over $k$. The group scheme $\mathrm{Aut}^0\mathcal{O}$ acts naturally on $M(F)$, $G(F)$, $G(\mathcal{O})$ and $\mathrm{Gr}_G$. We write $\delta:\mathrm{Aut}^0\mathcal{O}\times M(F)\to M(F)$ and $\delta:\mathrm{Aut}^0\mathcal{O}\times G(F)\to G(F)$ for the corresponding action maps, and $G(F)\rtimes\mathrm{Aut}^0\mathcal{O}$ for the corresponding semi-direct product with operation

$$(g_1,c_1)(g_2,c_2)=(g_1\delta(c_1,g_2),c_1c_2),\quad c_i\in\mathbb{G}_m, g_i\in G(F).$$

Then $G(F)\rtimes\mathrm{Aut}^0\mathcal{O}$ acts on $M(F)$ via the map $(G(F)\rtimes\mathrm{Aut}^0\mathcal{O})\times M(F)\to M(F)$ sending $((g,c),m)$ to $g\delta(c,m)$. For $r>0$ we similarly have a semi-direct product $G(\mathcal{O}/t^r)\rtimes\mathrm{Aut}(\mathcal{O}/t^r)$ and a surjective homomoprhism $G(\mathcal{O})\rtimes\mathrm{Aut}(\mathcal{O})\to G(\mathcal{O}/t^r)\rtimes\mathrm{Aut}(\mathcal{O}/t^r)$, whose kernel is pro-unipotent.

Let us define the equivariant derived category $\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))$. As in 4.2, for $r_1\geq N+r>0$ the projection between the stack quotients

$$q:(G(\mathcal{O}/t^{r_1})\rtimes\mathrm{Aut}(\mathcal{O}/t^{r_1}))\backslash_{N,r}M\to(G(\mathcal{O}/t^{N+r})\rtimes\mathrm{Aut}(\mathcal{O}/t^{N+r}))\backslash_{N,r}M$$

yields an (exact for the perverse t-structures) equivalence of the equivariant derived categories

$$\mathrm{D}_{G(\mathcal{O}/t^{N+r})\rtimes\mathrm{Aut}(\mathcal{O}/t^{N+r})}({}_{N,r}M)\to\mathrm{D}_{G(\mathcal{O}/t^{r_1})\rtimes\mathrm{Aut}(\mathcal{O}/t^{r_1})}({}_{N,r}M)$$

Denote by $\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N,r}M)$ the equivariant derived category $\mathrm{D}_{G(\mathcal{O}/t^{r_1})\rtimes\mathrm{Aut}(\mathcal{O}/t^{r_1})}({}_{N,r}M)$ for any $r_1\geq N+r$.

The stack quotient of (22) by $G(\mathcal{O}/t^{N_1+r_1})\rtimes\mathrm{Aut}(\mathcal{O}/t^{N_1+r_1})$ yields a diagram

$$\begin{array}{ccc}\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N_2,r_1}M) & \hookrightarrow & \mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N_1,r_1}M)\\ \uparrow & & \uparrow\\ \mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N_2,r_2}M) & \hookrightarrow & \mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N_1,r_2}M),\end{array}\qquad(30)$$

where each arrow is a fully faithfull (and exact for the perverse t-structures) functor. Define $\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))$ as the inductive 2-limit of $\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}({}_{N,r}M)$ as $N,r$ go to infinity. Similarly, one defines the category of perverse sheaves

$$\mathrm{P}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))\subset\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))$$

As in 4.3, one defines a natural action of $\mathrm{Sph}_G$ on $\mathrm{D}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))$. For our purposes note that the map (29) is $\mathrm{Aut}^0\mathcal{O}$-equivariant, so that all the perverse cohomologies of $\mathrm{H}^\lambda_G(I_0)$ are objects of $\mathrm{P}_{G(\mathcal{O})\rtimes\mathrm{Aut}^0\mathcal{O}}(M(F))$.

4.6 If $X$ is a smooth projective connected curve and $x\in X$ then one can consider the following global version of the category $\mathrm{D}_{G(\mathcal{O})}(M(F))$.

Let ${}_{x,\infty}\mathcal{W}_G$ be stack classifying a $G$-torsor $\mathcal{F}_G$ on $X$ together with a section $\mathcal{O}_X\stackrel{s}{\to}M_{\mathcal{F}_G}(\infty x)$. The stack ${}_{x,\infty}\mathcal{W}_G$ is an ind-algebraic. We have a diagram

$${}_{x,\infty}\mathcal{W}_G\stackrel{h^{\leftarrow}_{\mathcal{W}}}{\leftarrow}{}_{x,\infty}\mathcal{W}_G\times_{\mathrm{Bun}_G}{}_x\mathcal{H}_G\stackrel{h^{\rightarrow}_{\mathcal{W}}}{\to}{}_{x,\infty}\mathcal{W}_G,$$

where we used $h^{\leftarrow}_G$ to define the fibre product, the map $h^{\leftarrow}_{\mathcal{W}}$ (resp., $h^{\rightarrow}_{\mathcal{W}}$) sends $(\mathcal{F}_G, \mathcal{F}'_G, \beta, \mathcal{O}_X \stackrel{s}{\to} M_{\mathcal{F}_G}(\infty x))$ to $(\mathcal{F}_G, s)$ (resp., to $(\mathcal{F}'_G, s')$ with $s' = s\circ\beta$). As in Sections 2.2.1 and 4.3, one defines the Hecke functors

$$\mathrm{H}^{\leftarrow}_G(\cdot,\cdot), \mathrm{H}^{\rightarrow}_G(\cdot,\cdot) : \mathrm{Sph}_G \times \mathrm{D}({}_{x,\infty}\mathcal{W}_G) \to \mathrm{D}({}_{x,\infty}\mathcal{W}_G)$$

Let ${}_{x,\le N}\mathcal{W}_G \subset {}_{x,\infty}\mathcal{W}_G$ be the closed substack given by requiring that $\mathcal{O}_X \stackrel{s}{\to} M_{\mathcal{F}_G}(Nx)$ is regular.

For $r\ge 1$ let $D_{r,x} = \operatorname{Spec} \mathcal{O}_x/t_x^r$, where $\mathcal{O}_x$ is the completed local ring at $x\in X$, and $t_x \in \mathcal{O}_x$ is a local parameter. Pick a trivialization $\mathcal{O}_x \widetilde{\to} \mathcal{O}$. For $N, r\ge 0$ it yields a map

$$_{N,r}p_{\mathcal{W}} : {}_{x,\le N}\mathcal{W}_G \to G(\mathcal{O}/t^{N+r})\backslash_{N,r}M$$

sending $(\mathcal{F}_G, \mathcal{O}_X \stackrel{t}{\to} M_{\mathcal{F}_G}(Nx))$ to $\mathcal{F}_G\mid_{D_{N+r,x}}$ equipped with the induced $G(\mathcal{O}/t^{N+r})$-equivariant map $\mathcal{F}_G\mid_{D_{N+r,x}} \to {}_{N,r}M$. We get a functor $\mathrm{D}_{G(\mathcal{O}/t^{N+r})}({}_{N,r}M) \to \mathrm{D}({}_{x,\le N}\mathcal{W}_G)$ given by

$$K \mapsto {}_{N,r}p^*_{\mathcal{W}}K[a + \dim \mathrm{Bun}_G + N\dim M_0 - \dim G(\mathcal{O}/t^{N+r})\backslash_{N,r}M],$$

here $a$ is a function of a connected component of $\mathrm{Bun}_G$ sending $\mathcal{F}_G$ to $\chi(M_{\mathcal{F}_G})$. The shift in the above formula should be thought of as 'the corrected relative dimension' of $_{N,r}p_{\mathcal{W}}$, over a suitable open substack of ${}_{x,\le N}\mathcal{W}_G$ it is indeed the relative dimension. These functors are compatible with the transition functors in (24), thus we get a well-defined functor

$$\mathrm{glob}_x : \mathrm{D}_{G(\mathcal{O})}(M(F)) \to \mathrm{D}({}_{x,\infty}\mathcal{W}_G),$$

here glob stands for 'globalization'. One checks that it commutes with the functors $\mathrm{H}^{\leftarrow}_G, \mathrm{H}^{\rightarrow}_G$. Along the same lines, one defines a functor $\mathrm{D}_{G(\mathcal{O})\rtimes \mathrm{Aut}^0\mathcal{O}}(M(F)) \to \mathrm{D}({}_{x,\infty}\mathcal{W}_G)$ that does not depend on a choice of a trivialization $\mathcal{O}_x \widetilde{\to} \mathcal{O}$.

### 4.7 Weak analogues of Jacquet functors

4.7.1 Let $P\subset G$ be a parabolic subgroup, $U\subset P$ its unipotent radical and $L = P/U$ the Levi quotient. In classical setting, an important tool is the Jacquet module $\mathcal{S}(M(F))_{U(F)}$ of coinvariants with respect to $U(F)$. We don't know how to geometrize the whole Jacquet module. However, let $V_0\subset M_0$ be a $P$-invariant subspace, on which $U$ acts trivially. Set $V = V_0(\mathcal{O})$. We have a surjective map of $L(F)$-representations $\mathcal{S}(M(F))_{U(F)} \to \mathcal{S}(V(F))$ given by restriction under $V(F)\hookrightarrow M(F)$. We rather geometrize the composition map $\mathcal{S}(M(F)) \to \mathcal{S}(M(F))_{U(F)} \to \mathcal{S}(V(F))$.

As in 4.2, we have the derived categories $\mathrm{D}(V(F))$, $\mathrm{D}_{L(\mathcal{O})}(V(F))$. We are going to define natural functors

$$J^*_P, J^!_P : \mathrm{D}_{G(\mathcal{O})}(M(F)) \to \mathrm{D}_{L(\mathcal{O})}(V(F)) \tag{31}$$

To do so, for $N+r\ge 0$ consider the natural closed embedding $i_{N,r} : {}_{N,r}V \hookrightarrow {}_{N,r}M$. Remind that $_{N,r}V = t^{-N}V/t^rV$. Consider the diagram of stack quotients

$$\begin{array}{ccc} P(\mathcal{O}/t^{N+r})\backslash({}_{N,r}V) & \stackrel{i_{N,r}}{\to} & P(\mathcal{O}/t^{N+r})\backslash({}_{N,r}M) \stackrel{p}{\to} G(\mathcal{O}/t^{N+r})\backslash({}_{N,r}M) \\ \downarrow q & & \\ L(\mathcal{O}/t^{N+r})\backslash({}_{N,r}V), & & \end{array} \tag{32}$$

where $p$ comes from the inclusion $P\subset G$ and $q$ is the natural quotient map. Using (32), define functors

$$J^*_P, J^!_P : \mathrm{D}_{G(\mathcal{O}/t^{N+r})}({}_{N,r}M)\to \mathrm{D}_{L(\mathcal{O}/t^{N+r})}({}_{N,r}V)$$

by

$$q^*\circ J^*_P[\dim.\mathrm{rel}(q)] = (i_{N,r})^* p^*[\dim.\mathrm{rel}(p)-ra]$$

$$q^*\circ J^!_P[\dim.\mathrm{rel}(q)] = (i_{N,r})^! p^*[\dim.\mathrm{rel}(p)+ra]$$

Since $q^*[\dim.\mathrm{rel}(q)] : \mathrm{D}_{L(\mathcal{O}/t^{N+r})}({}_{N,r}V)\to \mathrm{D}_{P(\mathcal{O}/t^{N+r})}({}_{N,r}V)$ is an equivalence (exact for the perverse t-structures), the functors $J^*_P, J^!_P$ are well-defined. Here we have set $a=\dim M_0-\dim V_0$.

Further, $J^*_P, J^!_P$ are compatible with the transition functors in (24), so give rise to the desired functors (31). We underline that $J^*_P, J^!_P$ do not depend on a choice of a section of $P\to P/U$. Note also that $\mathbb{D}\circ J^*_P\widetilde{\to} J^!_P\circ\mathbb{D}$ naturally.

4.7.2 Due to its importance, remind the definition of the geometric restriction functor $\mathrm{gRes}^G_L : \mathrm{Sph}_G\to\mathrm{Sph}_L$ from ([4], Proposition 4.3.3). The diagram $L\leftarrow P\to G$ yields by functoriality the diagram

$$\mathrm{Gr}_L\stackrel{\mathfrak{t}_P}{\leftarrow}\mathrm{Gr}_P\stackrel{\mathfrak{t}_G}{\to}\mathrm{Gr}_G$$

The connected components of $\mathrm{Gr}_G$ are indexed by $\pi_1(G)$. For $\theta\in\pi_1(G)$ the component $\mathrm{Gr}^\theta_G$ is the one containing $\mathrm{Gr}^\lambda_G$ for any $\lambda\in\Lambda^+_G$ whose image in $\pi_1(G)$ equals $\theta$.

For $\theta\in\pi_1(M)$ let $\mathrm{Gr}^\theta_P$ be the preimage of $\mathrm{Gr}^\theta_L$ under $\mathfrak{t}_P$. The following strengthened version of ([4], Proposition 4.3.3) is derived from the results of [19].

**Proposition 3.** *For any $\mathcal{S}\in\mathrm{Sph}_G$ and $\theta\in\pi_1(L)$ the complex*

$$(\mathfrak{t}_P)_!(\mathcal{S}\mid_{\mathrm{Gr}^\theta_P})[\langle\theta, 2(\check\rho-\check\rho_L\rangle]$$

*lies in $\mathrm{Sph}_L$. The functor $\mathrm{gRes}^G_L:\mathrm{Sph}_G\to\mathrm{Sph}_L$ given by*

$$\mathcal{S}\mapsto\bigoplus_{\theta\in\pi_1(L)}(\mathfrak{t}_P)_!(\mathcal{S}\mid_{\mathrm{Gr}^\theta_P})[\langle\theta, 2(\check\rho-\check\rho_L\rangle]$$

*has a natural structure of a tensor functor. The following diagram is 2-commutative*

$$\begin{array}{ccc}\mathrm{Sph}_G & \stackrel{\mathrm{gRes}^G_L}{\to} & \mathrm{Sph}_L\\ \uparrow \mathrm{Loc} & & \uparrow\mathrm{Loc}\\ \mathrm{Rep}(\check G) & \stackrel{\mathrm{Res}^G_L}{\to} & \mathrm{Rep}(\check L)\end{array}$$

□

For the purposes of Lemma 10 below we renormalize $\mathrm{gRes}^G_L$ as follows. We let $\mathrm{g}\underline{\mathrm{Res}}^G_L : \mathrm{Sph}_G\to\mathrm{D}\,\mathrm{Sph}_L$ be given by $\mathrm{g}\underline{\mathrm{Res}}^G_L(\mathcal{T})=(\mathfrak{t}_P)_!\mathfrak{t}^*_G\mathcal{T}$.

**Corollary 2.** *The diagram is 2-commutative*

$$\begin{array}{ccc} \mathrm{Sph}_G & \overset{\mathrm{gRes}^G_L}{\to} & \mathrm{D}\,\mathrm{Sph}_L \\ \uparrow \mathrm{Loc} & & \uparrow \mathrm{Loc}^{\mathfrak{r}} \\ \mathrm{Rep}(\check{G}) & \overset{\mathrm{Res}^{\kappa_0}}{\to} & \mathrm{Rep}(\check{L}\times\mathbb{G}_m), \end{array}$$

*where $\kappa_0 : \check{L}\times\mathbb{G}_m \to \check{G}$ is the map whose first component is a Levi factor $\check{L} \overset{i_L}{\to} \check{G}$, and the second is*

$$\mathbb{G}_m \overset{2(\check{\rho}-\check{\rho}_L)}{\to} Z(\check{L}) \hookrightarrow \check{L} \overset{i_L}{\hookrightarrow} \check{G}$$

*Here $Z(\check{L})$ is the center of $\check{L}$.* $\square$

Write $\check{\mu} = \det M_0$ and $\check{\nu} = \det V_0$, view them as cocharacters of the center $Z(\check{L})$ of $\check{L}$. Let $\kappa : \check{L}\times\mathbb{G}_m \to \check{G}$ be the homomorphism, whose first component is $i_L : \check{L}\to\check{G}$, and the second component is $2(\check{\rho}-\check{\rho}_L)+\check{\mu}-\check{\nu}$. Let $\mathrm{gRes}^{\kappa} : \mathrm{Sph}_G \to \mathrm{D}\,\mathrm{Sph}_L$ denote the corresponding geometric restriction functor.

**Lemma 10.** *For $\mathcal{T}\in\mathrm{Sph}_G, K\in \mathrm{D}_{G(\mathcal{O})}(M(F))$ there is a filtration in the derived category on $J_P^*\mathrm{H}^{\leftarrow}_G(\mathcal{T},K)$ such that the corresponding graded complex identifies with*

$$\mathrm{H}^{\leftarrow}_L(\mathrm{gRes}^{\kappa}(\mathcal{T}), J_P^*(K))$$

*In particular, for $P=G$ and a $G$-subrepresentation $V_0\subset M_0$ we have canonically*

$$J_P^*\mathrm{H}^{\leftarrow}_G(\mathcal{T},K)\widetilde{\to}\mathrm{H}^{\leftarrow}_L(\mathrm{gRes}^{\kappa}(\mathcal{T}), J_P^*(K))$$

*Proof* For $s_1,s_2\ge 0$ let

$$_{s_1,s_2}P(F) = \{p\in P(F) \mid t^{s_1}M(\mathcal{O})\subset pM(\mathcal{O})\subset t^{-s_2}M(\mathcal{O})\},$$

it is stable by left and right multiplication by $P(\mathcal{O})$, and $_{s_1,s_2}\mathrm{Gr}_P := (_{s_1,s_2}P(F))/P(\mathcal{O})$ is closed in $\mathrm{Gr}_P$. We have a natural map $_{s_1,s_2}\mathrm{Gr}_P\to {}_{s_1,s_2}\mathrm{Gr}_G$, and at the level of reduced schemes the connected components of $_{s_1,s_2}\mathrm{Gr}_P$ form a stratification of $_{s_1,s_2}\mathrm{Gr}_G$. Set

$$_{s_1,s_2}\mathrm{Gr}_L = \{x\in L(\mathcal{O})\backslash L(F) \mid t^{s_1}V(\mathcal{O})\subset xV(\mathcal{O})\subset t^{-s_2}V(\mathcal{O})\}$$

The map $\mathfrak{t}_P : \mathrm{Gr}_P\to\mathrm{Gr}_L$ yields a map still denoted $\mathfrak{t}_P : {}_{s_1,s_2}\mathrm{Gr}_P\to {}_{s_1,s_2}\mathrm{Gr}_L$.

Let $N, r\ge 0$, assume that $\mathcal{T}$ is the extension by zero from $_{s_1,s_2}\mathrm{Gr}_G$, and $K\in\mathrm{D}_{G(\mathcal{O})}(_{N+s_1,r-s_1}M)$. For the diagram

$$_{N,r}M \overset{\mathrm{pr}}{\leftarrow} {}_{N,r}M\times {}_{s_1,s_2}\mathrm{Gr}_G \overset{\mathrm{act}_q}{\to} G(\mathcal{O}/t^{N+r})\backslash_{N+s_1,r-s_1}M$$

we calculate the direct image

$$\mathrm{pr}_!(\mathrm{act}_q^*K\otimes\mathrm{pr}_2^*\mathcal{T})[\dim]$$

with respect to the stratification of ${}_{s_1,s_2}\mathrm{Gr}_G$ by the connected components of ${}_{s_1,s_2}\mathrm{Gr}_P$. We have the diagram

$$\begin{array}{ccccc}
 & & {}_{N,r}V \times {}_{s_1,s_2}P(F) & \overset{\mathrm{act}}{\to} & {}_{N+s_1,r-s_1}V \\
 & & \downarrow q_P & & \downarrow q_U \\
{}_{N,r}V & \overset{\mathrm{pr}}{\leftarrow} & {}_{N,r}V \times {}_{s_1,s_2}\mathrm{Gr}_P & \overset{\mathrm{act}_{q,P}}{\to} & P(\mathcal{O}/t^{N+r})\backslash {}_{N+s_1,r-s_1}V \\
\downarrow i_{N,r} & & \downarrow i_{N,r}\times \mathrm{id} & & \downarrow i_{N+s_1,r-s_1} \\
{}_{N,r}M & \overset{\mathrm{pr}}{\leftarrow} & {}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_P & \overset{\mathrm{act}_{q,P}}{\to} & P(\mathcal{O}/t^{N+r})\backslash {}_{N+s_1,r-s_1}M \\
 & & \downarrow & & \downarrow p \\
 & & {}_{N,r}M \times {}_{s_1,s_2}\mathrm{Gr}_G & \overset{\mathrm{act}_q}{\to} & G(\mathcal{O}/t^{N+r})\backslash {}_{N+s_1,r-s_1}M,
\end{array}$$

where act sends $(m,p)$ to $p^{-1}m$, the map $q_P$ sends $(m,p)$ to $(m, pP(\mathcal{O}))$, and $q_U$ is the stack quotient under the action of $P(\mathcal{O}/t^{N+r})$. Moreover, $\mathrm{act}_{q,P}$ fits into the diagram

$$\begin{array}{ccc}
{}_{N,r}V \times {}_{s_1,s_2}\mathrm{Gr}_P & \overset{\mathrm{act}_{q,P}}{\to} & P(\mathcal{O}/t^{N+r})\backslash {}_{N+s_1,r-s_1}V \\
\downarrow \mathrm{id}\times \mathfrak{t}_P & & \downarrow q \\
{}_{N,r}V \times {}_{s_1,s_2}\mathrm{Gr}_L & \overset{\mathrm{act}_{q,L}}{\to} & L(\mathcal{O}/t^{N+r})\backslash {}_{N+s_1,r-s_1}V,
\end{array}$$

Our assertion follows (the shifts can be checked using Remark 2). □

Let $\delta_U : \mathbb{G}_m \times M_0 \to M_0$ be an action, whose fixed points set is $V_0$. Assume that $\delta_U$ contracts $M_0$ onto $V_0$. We will apply Lemma 10 under the following form.

**Corollary 3.** *Let $K \in \mathrm{P}_{G(\mathcal{O})}(M(F))$ be $\mathbb{G}_m$-equivariant for $\delta_U$-action on $M(F)$. Assume that $K$ admits a $k'$-structure for some finite subfield $k' \subset k$ and, as such, is pure of weight zero. Then $J_P^*(K)$ is also pure of weight zero, and there is an isomorphism*

$$J_P^*\mathrm{H}^{\leftarrow}_G(\mathcal{T}, K) \widetilde{\to} \mathrm{H}^{\leftarrow}_L(\mathrm{gRes}^{\kappa}(\mathcal{T}), J_P^*(K))$$

*in $\mathrm{D}_{L(\mathcal{O})}(V(F))$.*

*Proof* Under our assumptions, $J_P^*$ is the hyperbolic localization functor with respect to the $\delta_U$-action on $M(F)$, the assertion follows from ([3], Theorem 2) and Lemma 10. □

### 4.8 Fourier transform

Remind the notation $\Omega$ from Section 4.1. Let us define the Fourier transform functors $\mathrm{Four}_\psi : \mathrm{D}(M(F)) \to \mathrm{D}(M^* \otimes \Omega(F))$ and

$$\mathrm{Four}_\psi : \mathrm{D}_{G(\mathcal{O})}(M(F)) \to \mathrm{D}_{G(\mathcal{O})}(M^* \otimes \Omega(F)) \tag{33}$$

We actually will use the following a bit more general functor. Given a decomposition $M_0 \widetilde{\to} M_1 \oplus M_2$ into direct sum of vector spaces, one defines the Fourier transform

$$\mathrm{Four}_\psi : \mathrm{D}(M(F)) \to \mathrm{D}(M_1^* \otimes \Omega(F) \oplus M_2(F)) \tag{34}$$

as follows. For $N\geq 0$ we have a natural evaluation map $ev: {}_{N,N}M_1\times {}_{N,N}(M_1^*\otimes\Omega)\to \mathbb{A}^1$ sending $(m,m^*)$ to $\operatorname{Res}\langle m, m_1\rangle$. It gives rise to the usual Fourier transform functor

$$\operatorname{Four}_\psi: \mathrm{D}({}_{N,N}M)\widetilde{\to}\mathrm{D}({}_{N,N}M_1^*\otimes\Omega\oplus {}_{N,N}M_2)$$

For $N'\geq N$ these functors are compatible with the transition functors $\mathrm{D}({}_{N,N}M)\to \mathrm{D}({}_{N',N'}M)$ in (23), so give rise to the desired functor (34). From the usual properties of the Fourier transform we learn that (34) is an equivalence of triangulated categories, which preserves the perversity.

Assume in addition that $M_0\widetilde{\to} M_1\oplus M_2$ is a decomposition of $M_0$ into a direct sum of $G$-modules. Then similarly the usual Fourier transform functors

$$\operatorname{Four}_\psi: \mathrm{D}_{G(\mathcal{O})}({}_{N,N}M)\widetilde{\to}\mathrm{D}_{G(\mathcal{O})}({}_{N,N}M_1^*\otimes\Omega\oplus {}_{N,N}M_2),$$

being compatible with the transition functors in (23), give rise to the functor

$$\operatorname{Four}_\psi: \mathrm{D}_{G(\mathcal{O})}(M(F))\widetilde{\to}\mathrm{D}_{G(\mathcal{O})}(M_1^*\otimes\Omega(F)\oplus M_2(F)), \tag{35}$$

which satisfies the same formal properties.

*Remark* 4. The following diagram commutes

$$\begin{array}{ccc}
\mathrm{D}_{G(\mathcal{O})}(M(F)) & \stackrel{\operatorname{Four}_\psi}{\to} & \mathrm{D}_{G(\mathcal{O})}(M_1^*\otimes\Omega(F)\oplus M_2(F)) \\
& \searrow \operatorname{Four}_\psi & \downarrow \operatorname{Four}_\psi \\
& & \mathrm{D}_{G(\mathcal{O})}(M^*\otimes\Omega(F)),
\end{array}$$

that is, the composition of two partial Fourier trasforms identifies with the complete Fourier tranform.

**Lemma 11.** *The functor (33) commutes with Hecke operators. Namely, there is an isomorphism functorial in $\mathcal{T}\in\operatorname{Sph}_G$ and $K\in\mathrm{D}_{G(\mathcal{O})}(M(F))$*

$$\operatorname{Four}_\psi \mathrm{H}^{\leftarrow}_G(\mathcal{T},K)\widetilde{\to}\mathrm{H}^{\leftarrow}_G(\mathcal{T},\operatorname{Four}_\psi(K))$$

*Proof*

**Step 1**. Pick $s_1,s_2\geq 0$ so that $\mathcal{T}$ is the extension by zero from ${}_{s_1,s_2}\mathrm{Gr}_G$. Pick $r,r_1,N,N_1$ large enough compared to $s_i$ and $K$. In particular, we assume

$$r-N_1\geq s_1+s_2 \quad\text{and}\quad r_1-N\geq s_1+s_2 \tag{36}$$

Let $s\geq\max\{s_1+s_2, N+r, N_1+r_1\}$. Consider the diagram

$$\begin{array}{ccc}
& & G(\mathcal{O}/t^s)\backslash_{N+s_1,r-s_1}M \\
& & \uparrow \operatorname{act}_{q,s} \\
G(\mathcal{O}/t^s)\backslash_{N,r}M & \stackrel{\mathrm{pr}}{\leftarrow} & G(\mathcal{O}/t^s)\backslash({}_{N,r}M\times {}_{s_1,s_2}\mathrm{Gr}_G) \\
\uparrow \alpha & & \uparrow \\
G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times_{N,r}M) & \stackrel{\mathrm{pr}}{\leftarrow} & G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times_{N,r}M\times {}_{s_1,s_2}\mathrm{Gr}_G) \\
\downarrow \beta & & \\
G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)), & &
\end{array}$$

where the square is cartesian, all the quotients are taken in the stack sense, the action of $G(\mathcal{O}/t^s)$ on all the involved schemes is diagonal. We have denoted by $\alpha$ and $\beta$ are the projections.

By our assumptions,

$$\mathrm{H}^{\leftarrow}_G(\mathcal{T}, K)\widetilde{\to}\, \mathrm{pr}_!(\mathcal{T}\otimes \mathrm{act}^*_{q,s} K)[\dim]$$

for a suitable shift. Assuming $K\in \mathrm{D}_{G(\mathcal{O})}({}_{\tilde N,\tilde r}M)$ with $N_i, r_i$ sufficiently large with respect to $\tilde N, \tilde r$, we get

$$\mathrm{Four}_\psi(\mathrm{H}^{\leftarrow}_G(\mathcal{T}, K))\widetilde{\to}\,\beta_!(ev^*\mathcal{L}_\psi\otimes \alpha^*\mathrm{H}^{\leftarrow}_G(\mathcal{T}, K))[\dim.\mathrm{rel}(\alpha)]$$

Here $ev: G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times_{N,r} M)\to\mathbb{A}^1$ is the evaluation map, it is correctly defined because $r-N_1$ and $r_1-N$ are nonnegative.

Consider the diagram

$$\begin{array}{ccc}
 & & G(\mathcal{O}/t^s)\backslash_{N+s_1,r-s_1}M \\
 & & \uparrow \alpha' \\
G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times_{N,r} M\times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash({}_{N_1+s_2,r_1-s_2}(M^*\otimes\Omega)\times_{N+s_1,r-s_1} M) \\
\downarrow & & \downarrow \beta' \\
G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times {}_{s_1,s_2}\mathrm{Gr}_G) & \stackrel{\mathrm{act}'_{q,s}}{\to} & G(\mathcal{O}/t^s)\backslash_{N_1+s_2,r_1-s_2}(M^*\otimes\Omega) \\
\downarrow \mathrm{pr}' & & \\
G(\mathcal{O}/t^s)\backslash_{N_1,r_1}(M^*\otimes\Omega) & &
\end{array}$$

where $\alpha', \beta', \mathrm{pr}'$ are the projections. The square in the above diagram is not cartesian, write

$$b:\mathcal{Y}\to G(\mathcal{O}/t^s)\backslash({}_{N_1+s_2,r_1-s_2}(M^*\otimes\Omega)\times_{N+s_1,r-s_1} M)$$

for the map obtained from $\mathrm{act}'_{q,s}$ by the base change $\beta'$. Then

$$G(\mathcal{O}/t^s)\backslash({}_{N_1,r_1}(M^*\otimes\Omega)\times_{N,r} M\times {}_{s_1,s_2}\mathrm{Gr}_G)\hookrightarrow \mathcal{Y} \tag{37}$$

is naturally a closed substack. Let

$$ev: G(\mathcal{O}/t^s)\backslash({}_{N_1+s_2,r_1-s_2}(M^*\otimes\Omega)\times_{N+s_1,r-s_1} M)\to\mathbb{A}^1$$

be the evaluation map, it is correctly defined due to (36). By our assumptions,

$$\mathrm{Four}_\psi(K)\widetilde{\to}\,\beta'_!(ev^*\mathcal{L}_\psi\otimes \alpha'^*K)[\dim.\mathrm{rel}(\alpha')]$$

and

$$\mathrm{H}^{\leftarrow}_G(\mathcal{T}, \mathrm{Four}_\psi(K))\widetilde{\to}\,\mathrm{pr}'_!(\mathcal{T}\otimes(\mathrm{act}'_{q,s})^*\,\mathrm{Four}_\psi(K))[\dim]$$

Since $N, r$ are large enough compared to $\tilde N, \tilde r$, it follows that $b^*(\alpha')^*K$ is the extension by zero under (37). The desired result follows now from the base change theorem. □

Note that for the functor (33) we have $\mathrm{Four}_\psi(I_0)\widetilde{\to} I_0$ canonically.

### 4.9 Extensions of actions

Let $G$ be a connected reductive group, $P$ and $P^-$ two opposite parabolic subgroups in $G$ with commun Levi subgroup $L=P\cap P^-$.

**Lemma 12.** *Let $Y$ be a scheme (of finite type over $k$) with a $G$-action. Then we have a diagram of equivariant categories of perverse sheaves*

$$\begin{array}{ccc} P_P(Y) & \subset & P_L(Y) \\ \cup & & \cup \\ P_G(Y) & \subset & P_{P^-}(Y), \end{array}$$

*where all the functors are fully faithful embeddings. Moreover, $P_P(Y) \cap P_{P^-}(Y) = P_G(Y)$, that is, if an object $K \in P_L(Y)$ lies in both $P_P(Y)$ and $P_{P^-}(Y)$ then $K \in P_G(Y)$.*

*Proof* The natural maps between the stack quotients $Y/L \to Y/P \to Y/G$ are smooth of fixed relative dimension, surjective, and have connected fibres. By ([8], Lemma 4.8), they induce the corresponding fully faithful embeddings of categories.

Now assume $K$ is an object of $P_P(Y) \cap P_{P^-}(Y)$. Let $W$ be the image of the product map $m : P \times P^- \to G$. We have a diagram

$$\begin{array}{ccc} P \times P^- \times Y & \stackrel{\mathrm{act}'}{\to} & Y \\ \downarrow m\times \mathrm{id} & & \downarrow \mathrm{id} \\ W \times Y & \stackrel{\mathrm{act}_W}{\to} & Y \end{array}$$

The map $m : P \times P^- \to W$ is smooth and surjective with connected fibres. So, by *loc.cit.*, the equivariance isomorphism $(\mathrm{act}')^* K \widetilde{\to} \bar{\mathbb{Q}}_\ell \boxtimes K$ descends to an isomoprhism $\mathrm{act}_W^* K \widetilde{\to} \bar{\mathbb{Q}}_\ell \boxtimes K$ over $W \times Y$. Further, the product map $m : W \times W \to G$ is smooth and surjective with connected fibres. So, for the action map $\mathrm{act}_{W\times W} : W \times W \times Y \to Y$ the equivariance isomorphism $\mathrm{act}_{W\times W}^* K \widetilde{\to} \bar{\mathbb{Q}}_\ell \boxtimes K$ descends to the desired isomorphism $\mathrm{act}^* K \widetilde{\to} \bar{\mathbb{Q}}_\ell \boxtimes K$ over $G \times Y$. □

Now assume $M_0$ is a finite-dimensional representation of $G$, set $M = M_0 \otimes_k \mathcal{O}$. Let $U$ be the unipotent radical of $P$. The following result will be used in Section 6.2.

**Lemma 13.** *i) We have a diagram of fully faithful embeddings of categories*

$$\begin{array}{ccc} P_{P(\mathcal{O})}(M(F)) & \subset & P_{L(\mathcal{O})}(M(F)) \\ \cup & & \cup \\ P_{G(\mathcal{O})}(M(F)) & \subset & P_{P^-(\mathcal{O})}(M(F)), \end{array}$$

*The intersection $P_{P(\mathcal{O})}(M(F)) \cap P_{P^-(\mathcal{O})}(M(F))$ inside $P_{L(\mathcal{O})}(M(F)$ equals $P_{G(\mathcal{O})}(M(F))$.*
*i) We have fully faithful embeddings $P_{L(\mathcal{O})}(M(F)) \subset P(M(F)) \supset P_{U(\mathcal{O})}(M(F))$. The intersection $P_{L(\mathcal{O})}(M(F)) \cap P_{U(\mathcal{O})}(M(F))$ equals $P_{P(\mathcal{O})}(M(F))$.*

*Proof* i) Given $N, r \geq 0$, by ([8], Lemma 4.8), we get a diagram of fully faithful embeddings

$$\begin{array}{ccc} P_{P(\mathcal{O}/t^s)}({}_{N,r}M) & \subset & P_{L(\mathcal{O}/t^s)}({}_{N,r}M) \\ \cup & & \cup \\ P_{G(\mathcal{O}/t^s)}({}_{N,r}M) & \subset & P_{P^-(\mathcal{O}/t^s)}({}_{N,r}M) \end{array}$$

with $s=N+r$. Let $K$ be an object of $P_{P(\mathcal{O}/t^s)}({}_{N,r}M)\cap P_{P^-(\mathcal{O}/t^s)}({}_{N,r}M)$.

Let $m: P\times P^-\to W$ and $m: W\times W\to G$ be as in Lemma 12. The induced maps $m: P(\mathcal{O}/t^s)\times P^-(\mathcal{O}/t^s)\to W(\mathcal{O}/t^s)$ and $m: W(\mathcal{O}/t^s)\times W(\mathcal{O}/t^s)\to G(\mathcal{O}/t^s)$ are again smooth and surjective. Indeed, if $Y_1\to Y_2$ is a smooth surjective morphism of affine algebraic varieties, $A$ is an Artin $k$-algebra then $Y_1(A)$ is a scheme, and the induced map $Y_1(A)\to Y_2(A)$ is smooth and surjective. As in Lemma 12, one shows now that $K\in P_{G(\mathcal{O}/t^s)}({}_{N,r}M)$. The first assertion follows.

ii) is left to the reader. $\square$

## 5. Geometric model of the Weil representation of $\mathrm{GL}_m\times\mathrm{GL}_n$

5.1 Let $U_0=k^m, L_0=k^n$ be the standard $k$-vector spaces of dimensions $m$ and $n$. For Section 5 we let $G=\mathrm{GL}(L_0)$ and $H=\mathrm{GL}(U_0)$. Let $\Pi_0=U_0\otimes L_0$, this is a spherical $G\times H$-variety, as the open $G\times H$-orbit in $\Pi_0$ satisfies the following.

*Remark* 5. Assume $G_1,G_2$ are connected reductive groups over $k$, $P_i\subset G_i$ a parabolic subgroup and $M_i$ is the Levi factor of $P_i$. Assume given an isomorphism $M_1\times R\widetilde{\to}M_2$, where $R$ is another group over $k$. Let $Q\subset P_1\times P_2$ be the subgroup obtained by the base change diag : $M_1\hookrightarrow M_1\times M_1$ from the projection $P_1\times P_2\to M_1\times M_1$. Then $(G_1\times G_2)/Q$ is a spherical $G_1\times G_2$-variety.

Set $U=U_0(\mathcal{O})$, $L=L_0(\mathcal{O})$ and $\Pi=\Pi_0(\mathcal{O})$. Let $T_G\subset B_G\subset G$ be the torus of diagonal matrices and the Borel subgroup of upper-triangular matrices. We identify $\Lambda_G\widetilde{\to}\mathbb{Z}^n$ in the usual way. Write $\check\omega_i\in\check\Lambda^+_G$ be the h.w. of the representation $\wedge^i L_0$ of $G$. The objects $T_H\subset B_H\subset H$ are defined similarly for $H$. By some abuse of notation, $\check\omega_i\in\check\Lambda^+_H$ will also denote the h.w. of the $H$-representation $\wedge^i U_0$. Keep the notations of Section 4, in particular we write ${}_{N,r}\Pi=t^{-N}\Pi/t^r\Pi$, and $I_0$ is the constant sheaf on ${}_{0,0}\Pi$.

We are going to describe the submodule over $\mathrm{Sph}_G$ (resp., over $\mathrm{Sph}_H$) in $\mathrm{D}_{(G\times H)(\mathcal{O})}(\Pi(F))$ generated by $I_0$. Assume $m\ge n$.

Let $U_1\oplus U_2\widetilde{\to}U_0$ be the direct sum decomposition, where $U_1$ (resp., $U_2$) is generated by the first $n$ (resp., last $m-n$) base vectors. Let $P\subset H$ be the parabolic subgroup preserving $U_1$, $U_H\subset P$ be its unipotent radical. Let $M=\mathrm{GL}(U_1)\times\mathrm{GL}(U_2)\subset P$ be the standard Levi factor. Let $\kappa:\check G\times\mathbb{G}_m\to\check H$ be the composition

$$\check G\times\mathbb{G}_m\stackrel{\mathrm{id}\times 2\check\rho_{\mathrm{GL}(U_2)}}{\to}\check G\times\check{\mathrm{GL}}(U_2)=\check M\hookrightarrow\check H$$

Write $\mathrm{gRes}^\kappa:\mathrm{Sph}_H\to\mathrm{D}\,\mathrm{Sph}_G$ for the functor corresponding (in view of Loc and $\mathrm{Loc}^{\mathfrak{r}}$) to the restriction $\mathrm{Rep}(\check H)\to\mathrm{Rep}(\check G\times\mathbb{G}_m)$ with respect to $\kappa$. Here is the main result of Section 5.

**Proposition 4.** *The two functors* $\mathrm{Sph}_H\to\mathrm{D}_{(G\times H)(\mathcal{O})}(\Pi(F))$ *given by*

$$\mathcal{T}\mapsto\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0)\quad\textit{and}\quad\mathcal{T}\mapsto\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^\kappa(\mathcal{T}),I_0)\tag{38}$$

*are isomorphic.*

Let $N, r \in \mathbb{Z}$ with $N+r \geq 0$. Think of $v \in \Pi(F)$ as a map $v: U^*(F) \to L(F)$. For $v \in {}_{N,r}\Pi$ let $U_{v,r} = v(U^*) + t^r L$, this is a $\mathcal{O}$-lattice in $L(F)$. For $\lambda \in \Lambda^+_G$ satisfying

$$\langle -w_0^G(\lambda), \check{\omega}_1\rangle \leq N \quad \text{and} \quad \langle \lambda, \check{\omega}_1\rangle \leq r \tag{39}$$

let ${}_{\lambda,r}\Pi^0 \subset {}_{N,r}\Pi$ be the locally closed subscheme of those $v \in {}_{N,r}\Pi$ for which $t^{a_n}L/(v(U^*)+t^{a_1}L)$ is isomorphic to $\mathcal{O}/t^{a_1-a_n} \oplus \ldots \oplus \mathcal{O}/t^{a_n-a_n}$ as $\mathcal{O}$-module. Here $\lambda = (a_1 \geq \ldots \geq a_n)$. In other words, for $v \in {}_{N,r}\Pi$ we have $v \in {}_{\lambda,r}\Pi^0$ iff $U_{v,r} \in \mathrm{Gr}^\lambda_G$.

One checks that the $G(\mathcal{O}) \times H(\mathcal{O})$-orbits on ${}_{N,r}\Pi$ are exactly ${}_{\lambda,r}\Pi^0$ for $\lambda \in \Lambda^+_G$ satisfying (39).

Given $\lambda \in \Lambda^+_G$ let now $N = \langle -w_0^G(\lambda), \check{\omega}_1\rangle$ and $r = \langle \lambda, \check{\omega}_1\rangle$. By Remark 3, $\mathrm{H}^\lambda_G(I_0) \in \mathrm{D}_{(G\times H)(\mathcal{O})}({}_{N,r}\Pi)$. Define the closed subscheme ${}_\lambda\Pi \subset {}_N\Pi$ as follows. A point $v \in {}_N\Pi$ lies in ${}_\lambda\Pi$ iff for $i = 1, \ldots, n$ the map

$$\wedge^i U^* \stackrel{\wedge^i v}{\to} (\wedge^i L)(-\langle w_0(\lambda), \check{\omega}_i\rangle)$$

is regular. The scheme ${}_\lambda\Pi$ is stable under translations by $t^r\Pi(\mathcal{O})$, so there is a unique closed subscheme ${}_{\lambda,r}\Pi \subset {}_{N,r}\Pi$ such that ${}_\lambda\Pi$ is the preimage of ${}_{\lambda,r}\Pi$ under the projection ${}_N\Pi \to {}_{N,r}\Pi$. Under our assumptions the map (29) factors as

$${}_{0,r}\Pi\tilde{\times}\overline{\mathrm{Gr}}^\lambda_G \stackrel{\pi}{\to} {}_{\lambda,r}\Pi \hookrightarrow {}_{N,r}\Pi$$

**Proposition 5.** *For $\lambda \in \Lambda^+_G$ we have a canonical isomorphism $\mathrm{H}^\lambda_G(I_0) \widetilde{\to} \mathrm{IC}({}_{\lambda,r}\Pi^0)$ with the intersection cohomology sheaf of ${}_{\lambda,r}\Pi^0$.*

*Proof* Note that ${}_{\lambda,r}\Pi^0 \subset {}_{\lambda,r}\Pi$ is an open subscheme. The map ${}_{0,r}\Pi\tilde{\times}\overline{\mathrm{Gr}}^\lambda_G \stackrel{\pi}{\to} {}_{\lambda,r}\Pi$ is an isomorphism over ${}_{\lambda,r}\Pi^0$, in particular $\dim {}_{\lambda,r}\Pi^0 = rnm + \langle \lambda, 2\check{\rho}_G - m\check{\omega}_n\rangle$.

The scheme ${}_{\lambda,r}\Pi$ is stratified by locally closed subschemes ${}_{\mu,r}\Pi^0$, where $\mu \in \Lambda^+_G$ satisfies (39) and

$$\langle w_0^G(\lambda-\mu), \check{\omega}_i\rangle \leq 0 \tag{40}$$

for $i = 1, \ldots, n$. Further, ${}_{0,r}\Pi\tilde{\times}\overline{\mathrm{Gr}}^\lambda_G$ is stratified by locally closed subschemes ${}_{0,r}\Pi\tilde{\times}\mathrm{Gr}^\mu_G$ with $\mu \in \Lambda^+_G$, $\mu \leq \lambda$. Let us show that $\pi$ is stratified small (in the sense of [19]) with respect to these stratifications.

Let $\mu \in \Lambda^+_G$ satisfy (39) and (40), take $v \in {}_{\mu,r}\Pi^0$. Let $Y$ be the fibre of $\pi : {}_{0,r}\Pi\tilde{\times}\mathrm{Gr}^\lambda_G \to {}_{\lambda,r}\Pi$ over $v$. We must show that $2\dim Y \leq \langle \lambda-\mu, 2\check{\rho}_G - m\check{\omega}_n\rangle$.

From (40) it follows that $\langle \lambda-\mu, \check{\omega}_n\rangle \leq 0$. So, to finish the proof it suffices to show that $2\dim Y \leq \langle \lambda-\mu, 2\check{\rho}_G - n\check{\omega}_n\rangle$.

The scheme $Y$ classifies $\mathcal{O}$-lattices $L' \subset L(F)$ such that $L' \in \mathrm{Gr}^\lambda_G$ and $U_{v,r} \subset L'$. Stratify $Y$ by locally closed subschemes $Y_\tau$ indexed by $\tau \in \Lambda^+_G$, which are *very positive*. We call

$$\tau = (b_1 \geq \ldots \geq b_n)$$

very positive iff $b_n \geq 0$. By definition, the subscheme $Y_\tau$ classifies $L' \in Y$ such that $U_{v,r}$ is in the position $\tau$ with respect to $L'$. Now by ([19], Lemma 4.4), if $Y_\tau$ is nonempty then

$$\dim Y_\tau \leq \langle \lambda+\tau-\mu, \check{\rho}_G\rangle$$

So, we have to show that $\langle \tau, 2\check{\rho}_G\rangle \le \langle \lambda-\mu, -n\check{\omega}_n\rangle$. The formula for virtual dimensions $\dim(L/L') + \dim(L'/U_{v,r}) = \dim(L/U_{v,r})$ reads $\langle \tau+\lambda-\mu, \check{\omega}_n\rangle = 0$. Thus, we are reduced to show that

$$\langle \tau, n\check{\omega}_n - 2\check{\rho}_G\rangle \ge 0 \tag{41}$$

This inequality follows from the fact that $\tau$ is very positive, because

$$n\check{\omega}_n - 2\check{\rho}_G = (1,3,5,\ldots,2n-1)$$

is very positive. Moreover, the inequality (41) is strict unless $\tau = 0$. Since $\tau = 0$ iff $\lambda = \mu$, we are done. $\square$

**Corollary 4.** *i) The functor* $\mathrm{Sph}_G \to \mathrm{D}_{(G\times H)(\mathcal{O})}(\Pi(F))$ *given by* $\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_G(\mathcal{T}, I_0)$ *takes values in* $\mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$*. The corresponding functor*

$$\mathrm{Sph}_G \to \mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$$

*is fully faithful, its image is the full subcategory* $\mathrm{P}^{ss}_{(G\times H)(\mathcal{O})}(\Pi(F))$ *of semi-simple objects in* $\mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$*.*
*ii) For any* $\lambda \in \Lambda^+_G$ *we have* $\mathrm{Ext}^1_{(G\times H)(\mathcal{O})}(\mathrm{IC}({}_{\lambda,r}\Pi^0), \mathrm{IC}({}_{\lambda,r}\Pi^0)) = 0$.

*Proof* For $\lambda \in \Lambda^+_G$ let $r = \langle \lambda, \check{\omega}_1\rangle$ and $N = \langle -w_0^G(\lambda), \check{\omega}_1\rangle$. Pick $s \ge N + r$. The stabilizor, say $K$, in $(G\times H)(\mathcal{O}/t^s)$ of a point of ${}_{\lambda,r}\Pi^0$ is connected. So, the irreducible objects of $\mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$ are exactly $\mathrm{IC}({}_{\lambda,r}\Pi^0)$, $\lambda \in \Lambda^+_G$. Part i) follows.

We have a canonical equivalence $\mathrm{P}_{(G\times H)(\mathcal{O})}({}_{\lambda,r}\Pi^0) \widetilde{\to} \mathrm{P}_K(\mathrm{Spec}\, k)$. By ([8], Lemma 4.8), the connectedness of $K$ implies that $\mathrm{P}_K(\mathrm{Spec}\, k)$ is equivalent to the category of vector spaces. If $0 \to \mathrm{IC}({}_{\lambda,r}\Pi^0) \to \mathcal{K} \to \mathrm{IC}({}_{\lambda,r}\Pi^0) \to 0$ is an exact sequence in $\mathrm{P}_{(G\times H)(\mathcal{O}/t^s)}({}_{N,r}\Pi)$ then $\mathcal{K}$ is the intermediate extension from ${}_{\lambda,r}\Pi^0$. Part ii) follows. $\square$

*Remark* 6. The category $\mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$ is not semi-simple in general. To have an example, take $n = m = 2$ and $\lambda = (1,0)$. Let $Y \subset {}_{0,1}\Pi$ be the support of $\mathrm{IC}({}_{\lambda,1}\Pi^0)$ then $\dim Y = 3$ and $\dim {}_{0,1}\Pi = 4$. The restriction to $Y$ yields a nontrivial map $I_0 \to \mathrm{IC}({}_{\lambda,1}\Pi^0)[1]$ in $\mathrm{D}_{(G\times H)(\mathcal{O})}({}_{0,1}\Pi)$.

*Proof of Proposition 4*
**Step 1**. Assume first $n = m$. Interchenging $U_0$ and $L_0$, one derives from Proposition 5 that the functors $\mathrm{Rep}(\mathrm{GL}_n) \to \mathrm{P}_{(G\times H)(\mathcal{O})}(\Pi(F))$ given by

$$V \mapsto \mathrm{H}^{\leftarrow}_H(V, I_0) \quad \text{and} \quad V \mapsto \mathrm{H}^{\leftarrow}_G(V, I_0)$$

are isomorphic. For $n = m$ we are done.
**Step 2**. For $m \ge n$ consider the Jacquet functors

$$J_P^* : \mathrm{D}_{(G\times H)(\mathcal{O})}(\Pi(F)) \to \mathrm{D}_{(G\times M)(\mathcal{O})}(U_1 \otimes L_0(F))$$

We have $J_P^*(I_0) \widetilde{\to} I_0$ canonically. The action of $\mathrm{GL}(U_2)$ on $U_1 \otimes L_0$ is trivial, so $\mathcal{S} \in \mathrm{Sph}_{\mathrm{GL}(U_2)}$ acts on $I_0 \in \mathrm{D}_{(G\times M)(\mathcal{O})}(U_1^* \otimes L_0(F))$ as

$$\mathrm{H}^{\leftarrow}_{\mathrm{GL}(U_2)}(\mathcal{S}, I_0) \widetilde{\to} I_0 \otimes \mathrm{R}\Gamma(\mathrm{Gr}_{\mathrm{GL}(U_2)}, \mathcal{S})$$

As a representation of $H$, $\det(U_0\otimes L_0)$ is the character $n\check{\omega}_m\in\check{\Lambda}^+_H$. As a representation of $M$, $\det(U_1\otimes L_0)$ is the character $n\check{\omega}_n\in\check{\Lambda}^+_H$. Thus, let $\kappa_1:\check{\mathrm{GL}}(U_1)\times\mathbb{G}_m\to\check{H}$ be the composition

$$\check{\mathrm{GL}}(U_1)\times\mathbb{G}_m\stackrel{i\times(2\check{\rho}_H-2\check{\rho}_{\mathrm{GL}(U_1)}-n\check{\omega}_n+n\check{\omega}_m)}{\to}\check{M}\hookrightarrow\check{H},$$

where $i:\check{GL}(U_1)\hookrightarrow\check{M}$ is the natural inclusion. Let $\mathrm{gRes}^{\kappa_1}:\mathrm{Sph}_H\to\mathrm{D}\,\mathrm{Sph}_{\mathrm{GL}(U_1)}$ be the corresponding restriction functor. From Corollary 3 we get for $\mathcal{T}\in\mathrm{Sph}_H$ an isomorphism

$$J_P^*\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{\mathrm{GL}(U_1)}(\mathrm{gRes}^{\kappa_1}(\mathcal{T}),I_0)$$

Let $\kappa_2:\check{G}\times\mathbb{G}_m\to\check{H}$ be the map obtained from $\kappa_1$ via the canonical identification $\check{\mathrm{GL}}(U_1)\widetilde{\to}\check{G}$. By Step 1, we have an isomorphism

$$\mathrm{H}^{\leftarrow}_{\mathrm{GL}(U_1)}(\mathrm{gRes}^{\kappa_1}(\mathcal{T}),I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa_2}(\mathcal{T}),I_0)$$

in $\mathrm{D}_{(G\times M)(\mathcal{O})}(U_1\otimes L_0(F))$.

Further, we may think of $J_P^*$ as the Jacquet functor corresponding to the parabolic subgroup $P\times G$ of $H\times G$. As a representation of $G$, $\det(U_0\otimes L_0)\otimes\det(U_1\otimes L_0)^{-1}$ is the character $(m-n)\check{\omega}_n\in\check{\Lambda}^+_G$. So, let $\kappa_3:\check{G}\times\mathbb{G}_m\to\check{G}\times\mathbb{G}_m$ be the map, whose second component $\check{G}\times\mathbb{G}_m\to\mathbb{G}_m$ is the projection, and the first component is

$$\check{G}\times\mathbb{G}_m\stackrel{(\mathrm{id},(m-n)\check{\omega}_n)}{\to}\check{G}$$

Write $\mathrm{gRes}^{\kappa_3}:\mathrm{D}\,\mathrm{Sph}_G\to\mathrm{D}\,\mathrm{Sph}_G$ for the corresponding geometric restriction functor. By Corollary 3, for $\mathcal{S}\in\mathrm{D}\,\mathrm{Sph}_G$ we get an isomorphism

$$J_P^*\mathrm{H}^{\leftarrow}_G(\mathcal{S},I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa_3}(\mathcal{S}),I_0)$$

in $\mathrm{D}_{(G\times M)(\mathcal{O})}(U_1\otimes L_0(F))$. From Corollary 4 we conclude that

$$J_P^*:\mathrm{DP}_{(G\times H)(\mathcal{O})}(\Pi(F))\to\mathrm{DP}_{(\mathrm{GL}(U_1)\times G)(\mathcal{O})}(U_1\otimes L_0(F))\tag{42}$$

yields an equivalence of the corresponding semi-simple categories. The equality

$$2\check{\rho}_H-2\check{\rho}_{\mathrm{GL}(U_1)}-2\check{\rho}_{\mathrm{GL}(U_2)}+n\check{\omega}_m-m\check{\omega}_n=0$$

shows that the composition $\check{G}\times\mathbb{G}_m\stackrel{\kappa_3}{\to}\check{G}\times\mathbb{G}_m\stackrel{\kappa}{\to}\check{H}$ equals $\kappa_2$.

Summarizing, for $\mathcal{T}\in\mathrm{Sph}_H$ we get an isomorphism

$$J_P^*\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0)\widetilde{\to}J_P^*\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(\mathcal{T}),I_0)$$

in $\mathrm{D}_{(G\times M)(\mathcal{O})}(U_1\otimes L_0(F))$, and (42) garantees that this isomorphism can be lifted to the desired isomorphism of functors (38). □

We will need the following version of Proposition 4. Set $\Pi_1=U_0^*\otimes L_0$. Remind the functor $*:\mathrm{Sph}(\check{H})\widetilde{\to}\mathrm{Sph}(\check{H})$ from Section 2.2.1.

**Corollary 5.** *The two functors* $\mathrm{Sph}_H \to \mathrm{D}_{(G\times H)(\mathcal{O})}(\Pi_1(F))$ *given by*

$$\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_H(*\mathcal{T}, I_0) = \mathrm{H}^{\rightarrow}_H(\mathcal{T}, I_0) \quad \textit{and} \quad \mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(\mathcal{T}), I_0)$$

*are isomorphic.* □

## 6. Geometric model of the Weil representation of $\mathbb{SO}_{2m} \times \mathbb{Sp}_{2n}$

6.1 Let $U_0 = k^m, L_0 = k^n$. Set $V_0 = U_0 \oplus U_0^*$, we equip it with the symmetric form $\mathrm{Sym}^2 V_0 \to k$ as in Section 3.2. Set $H = \mathbb{SO}(V_0)$.

Let $P_H \subset H$ be the parabolic subgroup preserving $U_0$, $U_H \subset P_H$ be its unipotent radical. Write $Q_H = \mathrm{GL}(U_0) \widetilde{\to} \mathrm{GL}_m$ for the standard Levi factor of $P_H$. We equip it with the maximal torus $T_H$ of diagonal matrices and the Borel subgroup of upper-triangular matrices (its preimage in $P(H)$ is a Borel subgroup, which yields our choice of positive roots).

Set $M_0 = L_0 \oplus L_0^*$, we equip it with the symplectic form $\wedge^2 M_0 \to k$ arising from pairing between $L_0$ and $L_0^*$, so $L_0$ and $L_0^*$ are lagrangian subspaces in $M_0$. Set $G = \mathbb{Sp}(M_0)$. Let $P_G \subset G$ be the parabolic subgroup preserving $L_0$, $U_G \subset P_G$ its unipotent radical. Write $Q_G = \mathrm{GL}(L_0) \widetilde{\to} \mathrm{GL}_n$ for the natural Levi factor of $P_G$. We equip $Q_G$ with the maximal torus $T_G$ of diagonal matrices and the Borel subgroup of upper-triangular matrices (the preimage of the latter in $P_G$ is a Borel subgroup, which yields our choice of positive roots).

Keep the notation of Section 4, in particular, $\mathcal{O} = k[[t]]$ and $\Omega$ is the completed module of relative differential of $\mathcal{O}$ over $k$. Set $L = L_0(\mathcal{O}), U = U_0(\mathcal{O}), V = V_0(\mathcal{O})$ and $M = L \oplus L^* \otimes \Omega$. The isomorphism $\mathcal{O} \widetilde{\to} \Omega$ sending 1 to $dt$ yields an isomorphism of group schemes $G \widetilde{\to} \mathbb{Sp}(M)$ over $\mathrm{Spec}\, \mathcal{O}$. So, we offen think of $G$ as the group acting on $M$.

Set $\Upsilon = L^* \otimes V \otimes \Omega$ and $\Pi = U^* \otimes M$.

*Remark* 7. In general, $L_0^* \otimes V_0$ is not a spherical $Q_G \times H$-variety. By ([11], Theorem 1.1.1), in this case the set of $Q_G(\mathcal{O}) \times H(\mathcal{O})$-orbits on $\Upsilon(F)$ is not countable. Indeed, already for the open $Q_G \times H$-orbit $(Q_G \times H)/R$ in $L_0^* \otimes V_0$, the set of $R(F)$-orbits on $\mathrm{Gr}_{Q_G \times H}$ is not countable.

Similarly, in general $U_0^* \otimes M_0$ is not a spherical $Q_H \times G$-variety, and the set of $Q_H(\mathcal{O}) \times G(\mathcal{O})$-orbits on $\Pi(F)$ is not countable.

6.2 As in Section 3.2, we define the functor

$$\zeta : \mathrm{D}_{(Q_G \times Q_H)(\mathcal{O})}(\Upsilon(F)) \widetilde{\to} \mathrm{D}_{(Q_H \times Q_G)(\mathcal{O})}(\Pi(F)) \tag{43}$$

as the partial Fourier transform (35) with respect to the decomposition $\Upsilon(F) \widetilde{\to} L^* \otimes U \otimes \Omega(F) \oplus L^* \otimes U^* \otimes \Omega(F)$.

**Definition 4.** The *Weil category for* $G \times H$ is the category $\mathcal{W}\mathrm{eil}_{G,H}$ of triples $(\mathcal{F}_1, \mathcal{F}_2, \beta)$, where $\mathcal{F}_1 \in \mathrm{P}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F))$, $\mathcal{F}_2 \in \mathrm{P}_{(Q_H \times G)(\mathcal{O})}(\Pi(F))$, and $\beta : \zeta(f(\mathcal{F}_1)) \widetilde{\to} f(\mathcal{F}_2)$ is an isomorphism for the diagram

$$\begin{array}{ccc} \mathrm{P}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F)) & & \mathrm{P}_{(Q_H \times G)(\mathcal{O})}(\Pi(F)) \\ \downarrow f & & \downarrow f \\ \mathrm{P}_{(Q_G \times Q_H)(\mathcal{O})}(\Upsilon(F)) & \stackrel{\zeta}{\to} & \mathrm{P}_{(Q_H \times Q_G)(\mathcal{O})}(\Pi(F)), \end{array}$$

where $f$ are forgetful functors. Write

$$f_G : \mathcal{W}\mathrm{eil}_{G,H} \to \mathrm{P}_{(Q_H\times G)(\mathcal{O})}(\Pi(F)) \quad \text{and} \quad f_H : \mathcal{W}\mathrm{eil}_{G,H} \to \mathrm{P}_{(Q_G\times H)(\mathcal{O})}(\Upsilon(F))$$

for the functors sending $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ to $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively. Write $\mathrm{D}\mathcal{W}\mathrm{eil}_{G,H}$ for the category obtained by replacing in the above definition P by DP everywhere.

By ([8], Lemma 4.8), both functors $f$ in the above diagram are full embeddings, and their image is stable under subquotients. It follows that $\mathcal{W}\mathrm{eil}_{G,H}$ is abelian, and both $f_G$ and $f_H$ are full embeddings. Write $\mathcal{W}\mathrm{eil}^{ss}_{G,H} \subset \mathcal{W}\mathrm{eil}_{G,H}$ for the full subcategory of semi-simple objects. We write

$$\mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H} \subset \mathrm{D}\mathcal{W}\mathrm{eil}_{G,H}$$

for the full subcategory of objects of the form $\oplus_{i\in\mathbb{Z}} K_i[i]$ with $K_i \in \mathcal{W}\mathrm{eil}^{ss}_{G,H}$ for all $i$.

Since $\zeta(I_0) \widetilde{\to} I_0$ canonically, $I_0$ is naturally an object of $\mathcal{W}\mathrm{eil}^{ss}_{G,H}$. Combining the decomposition theorem ([2]) with the fact that $G$ and $H$-actions in the Weil representation commute with each other, one gets the following.

**Proposition 6.** *There exist natural functors* $\mathrm{D}\,\mathrm{Sph}_G \to \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ *and* $\mathrm{D}\,\mathrm{Sph}_H \to \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ *such that the diagrams commute*

$$\begin{array}{ccc} \mathrm{D}\,\mathrm{Sph}_G & \to & \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H} \\ & \searrow a_G & \downarrow f_G \\ & & \mathrm{DP}^{ss}_{(Q_H\times G)(\mathcal{O})}(\Pi(F)) \end{array} \qquad \begin{array}{ccc} \mathrm{D}\,\mathrm{Sph}_H & \to & \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H} \\ & \searrow a_H & \downarrow f_H \\ & & \mathrm{DP}^{ss}_{(Q_G\times H)(\mathcal{O})}(\Upsilon(F)) \end{array}$$

*Here the functor* $a_G$ *(resp.,* $a_H$*) sends* $\mathcal{T}$ *to* $\mathrm{H}^{\leftarrow}_G(\mathcal{T}, I_0)$ *(resp., to* $\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0)$*).*

*Proof* The arguments for both functors being similar, we give a proof only for the second one. Given $\mathcal{T} \in \mathrm{Sph}_H$, by decomposition theorem $\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0) \in \mathrm{D}_{(Q_G\times H)(\mathcal{O})}(\Upsilon(F))$ identifies with the direct sum of its (shifted) perverse cohomology sheaves. It suffices to show that each perverse cohomology sheaf $K$ of $\zeta(f\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0))$ actually lies in the full subcategory $\mathrm{P}_{(Q_H\times G)(\mathcal{O})}(\Pi(F))$ of $\mathrm{P}_{(Q_H\times Q_G)(\mathcal{O})}(\Pi(F))$.

Denote by $P^-_G \subset G$ the parabolic subgroup preserving $L^*_0$, write $U^-_G$ for its unipotent radical. By Lemma 13, it suffices to show that $K$ admits a $U_G(\mathcal{O})$ and $U^-_G(\mathcal{O})$-equivariant structures. For $v \in \Upsilon(F)$ write $s_{\mathcal{L}}(v)$ for the composition

$$\mathrm{Sym}^2 L(F) \stackrel{\mathrm{Sym}^2 v}{\to} \mathrm{Sym}^2(V\otimes\Omega)(F) \to \Omega^2(F)$$

Let $\mathrm{Char}(\Upsilon) \subset \Upsilon(F)$ be the ind-subscheme of $v \in \Upsilon(F)$ such that $s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2$ is regular. The $U_G(\mathcal{O})$-equivariance of $K$ is equivalent to the fact that $\zeta^{-1}(K)$ is the extension by zero from $\mathrm{Char}(\Upsilon)$. But the complex $\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0)$ itself satisfies this property, so its direct summand also does.

To get a $U^-_G(\mathcal{O})$-action on $K$, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{P}_{(Q_G\times Q_H)(\mathcal{O})}(\Upsilon(F)) & \stackrel{\zeta}{\to} & \mathrm{P}_{(Q_H\times Q_G)(\mathcal{O})}(\Pi(F)) \\ & \searrow \mathrm{Four}_\psi & \downarrow \zeta_1 \\ & & P_{(Q_G\times Q_H)(\mathcal{O})}(L\otimes V(F)), \end{array}$$

where $\zeta_1$ is a partial Fourier transform (with respect to $\psi$), and $\mathrm{Four}_\psi$ is the complete Fourier transform (cf. Remark 4). By Lemma 11,

$$\zeta_1\zeta(f\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0)) \widetilde{\to} \mathrm{Four}_\psi(f\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0)) \widetilde{\to} f_1\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0),$$

where we have denote by $f_1 : \mathrm{P}_{(Q_G\times H)(\mathcal{O})}(L\otimes V(F)) \to \mathrm{P}_{(Q_G\times Q_H)(\mathcal{O})}(L\otimes V(F))$ the forgetful functor.

Let $\mathrm{Char}(L\otimes V(F)) \subset L\otimes V(F)$ be the ind-subscheme of $v \in L\otimes V(F)$ such that the composition $\mathrm{Sym}^2 L^* \stackrel{\mathrm{Sym}^2 v}{\to} \mathrm{Sym}^2 V(F) \to F$ factors through $\mathcal{O} \subset F$. Note that

$$\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0) \in \mathrm{P}_{(Q_G\times H)(\mathcal{O})}(L\otimes V(F))$$

is the extension by zero from $\mathrm{Char}(L\otimes V(F))$. The $U^-_G(\mathcal{O})$-equivariance of $K$ is equivalent to the fact that $\zeta_1(K)$ is the extension by zero from $\mathrm{Char}(L\otimes V(F))$. We are done. □

By abuse of notation, we simply write $\mathrm{H}^{\leftarrow}_G(\mathcal{T}, I_0) \in \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ (resp., $\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0) \in \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$) for $\mathcal{T} \in \mathrm{D}\,\mathrm{Sph}_G$ (resp., $\mathcal{T} \in \mathrm{D}\,\mathrm{Sph}_H$).

Remind the definition of the homomorphisms $\kappa$ from Section 2.4.2. For $m \le n$ we have $\kappa : \check{H}\times\mathbb{G}_m \to \check{G}$. We write $\mathrm{gRes}^\kappa : \mathrm{Sph}_G \to \mathrm{D}\,\mathrm{Sph}_H$ for the corresponding geometric restriction functor.

For $m > n$ we have $\kappa : \check{G}\times\mathbb{G}_m \to \check{H}$. Write $\mathrm{gRes}^\kappa : \mathrm{Sph}_H \to \mathrm{D}\,\mathrm{Sph}_G$ for the corresponding geometric restriction functor.

Here is our main local result.

**Theorem 7.** *1) Assume $m \le n$. The functors $\mathrm{Sph}_G \to \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ given by*

$$\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_G(\mathcal{S}, I_0) \quad \textit{and} \quad \mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_H(*\,\mathrm{gRes}^\kappa(\mathcal{S}), I_0) \tag{44}$$

*are isomorphic.*
*2) Assume $m > n$. The two functors $\mathrm{Sph}_H \to \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ given by*

$$\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0) \quad \textit{and} \quad \mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^\kappa(*\mathcal{T}), I_0) \tag{45}$$

*are isomorphic.*

The proof will be given in Section 6.4.

6.3.1 In this subsection we assume $m \le n$ and analyse the action of $\mathrm{Sph}_H$ on $\mathrm{D}_{(Q_G\times H)(\mathcal{O})}(\Upsilon(F))$ in more details.

Write $V^{\check{\lambda}}$ for the irreducible $H$-module with h.w. $\check{\lambda} \in \check{\Lambda}^+_H$. For $1 \le i < m$ let $\check{\alpha}_i \in \check{\Lambda}^+_H$ denote the h.w. of the $H$-module $\wedge^i V_0$. Remind that

$$\wedge^m V_0 \widetilde{\to} V^{\check{\alpha}_m} \oplus V^{\check{\alpha}'_m}$$

is a direct sum of two irreducible representations, this is our definition of $\check{\alpha}_m, \check{\alpha}'_m$. Say that a maximal isotropic subspace $\mathcal{L} \subset V_0$ is *$\check{\alpha}_m$-oriented* (resp., *$\check{\alpha}'_m$-oriented*) if $\wedge^m\mathcal{L} \subset V^{\check{\alpha}_m}$ (resp.,

$\wedge^m \mathcal{L} \subset V^{\check{\alpha}'_m}$). The group $H$ has two orbits on the scheme of maximal isotropic subspaces in $V_0$ given by their orientation.

For $v \in \Upsilon(F)$ let $s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2(F)$ be the composition

$$\mathrm{Sym}^2 L \stackrel{\mathrm{Sym}^2 v}{\to} \mathrm{Sym}^2(V \otimes \Omega)(F) \to \Omega^2(F)$$

For $\lambda \in \Lambda^+_H$ let $N = \langle \lambda, \check{\alpha}_1 \rangle$, define a closed subscheme ${}_\lambda\Upsilon \subset {}_N\Upsilon = t^{-N}\Upsilon$ as follows. A point $v \in {}_N\Upsilon$ lies in ${}_\lambda\Upsilon$ iff the following conditions hold:

C1) $s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2$ is regular;

C2) for $1 \le i < m$ the map $\wedge^i L \stackrel{\wedge^i v}{\to} (\Omega^i \otimes \wedge^i V)(\langle -w_0(\lambda), \check{\alpha}_i \rangle)$ is regular;

C3) the map $\wedge^m L \stackrel{v_m \oplus v'_m}{\to} (\Omega^m \otimes V^{\check{\alpha}_m})(\langle -w_0(\lambda), \check{\alpha}_m \rangle) \oplus (\Omega^m \otimes V^{\check{\alpha}'_m})(\langle -w_0(\lambda), \check{\alpha}'_m \rangle)$ induced by $\wedge^m v$ is regular.

The scheme ${}_\lambda\Upsilon$ is stable under translations by ${}_{-N}\Upsilon$, so there is a closed subscheme ${}_{\lambda,N}\Upsilon \subset {}_{N,N}\Upsilon$ whose preimage under the projection ${}_N\Upsilon \to {}_{N,N}\Upsilon$ is ${}_\lambda\Upsilon$.

As in Remark 3, we have a map $\pi : {}_{0,N}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^\lambda_H \to {}_{N,N}\Upsilon$, it factors through the closed immersion ${}_{\lambda,N}\Upsilon \hookrightarrow {}_{N,N}\Upsilon$, and

$$\mathrm{H}^\lambda_H(I_0) \widetilde{\to} \pi_!(\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda_H)[\dim {}_{0,N}\Upsilon] \in \mathrm{D}_{(Q_G \times H)(\mathcal{O})}({}_{N,N}\Upsilon)$$

Let $\mathrm{Char}(\Upsilon) \subset \Upsilon(F)$ be the ind-subscheme of $v \in \Upsilon(F)$ satisfying C1). Note that $\mathrm{Char}(\Upsilon)$ is preserved by the $H(F)$-action. For $v \in \mathrm{Char}(\Upsilon)$ let $L_v = v(L) + V \otimes \Omega$ and

$$L_v^\perp = \{v \in V \otimes \Omega \mid \langle v, u \rangle \in \Omega^2 \quad \text{for any} \quad u \in L_v\}$$

Let $V_v \subset V(F)$ be defined by $V_v \otimes \Omega = v(L) + L_v^\perp$, then $V_v$ is an orthogonal lattice in $V(F)$, that is, a point of $\mathrm{Gr}_H$. We stratify $\mathrm{Char}(\Upsilon)$ by locally closed subschemes ${}_\lambda \mathrm{Char}(\Upsilon)$ indexed by $\lambda \in \Lambda^+_H$. Namely, for $v \in \mathrm{Char}(\Upsilon)$ we let $v \in {}_\lambda \mathrm{Char}(\Upsilon)$ iff $V_v \in \mathrm{Gr}^\lambda_H$.

We have ${}_\lambda \mathrm{Char}(\Upsilon) \subset {}_\lambda\Upsilon$. Moreover, for $N = \langle \lambda, \check{\alpha}_1 \rangle$ there is a unique open subscheme ${}_{\lambda,N}\Upsilon^0 \subset {}_{\lambda,N}\Upsilon$ whose preimage under the projection ${}_\lambda\Upsilon \to {}_{\lambda,N}\Upsilon$ identifies with ${}_\lambda \mathrm{Char}(\Upsilon)$.

Write $\mathrm{IC}({}_{\lambda,N}\Upsilon^0) \in \mathrm{P}_{(Q_G \times H)(\mathcal{O})}({}_{N,N}\Upsilon)$ for the intersection cohomology sheaf of ${}_{\lambda,N}\Upsilon^0$.

**Lemma 14.** *Assume $m \le n$. 1) The map*

$$\pi : {}_{0,N}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^\lambda_H \to {}_{\lambda,N}\Upsilon$$

*is an isomorphism over the open subscheme ${}_{\lambda,N}\Upsilon^0$. So, $\dim {}_{\lambda,N}\Upsilon^0 = 2Nnm + \langle \lambda, 2\check{\rho}_H \rangle$.*
*2) For $\lambda \in \Lambda^+_H$ we have $\mathrm{H}^\lambda_H(I_0) \widetilde{\to} \mathrm{IC}({}_{\lambda,N}\Upsilon^0)$ canonically.*

*Proof* 1) The fibre of $\pi$ over $v \in {}_{\lambda,N}\Upsilon^0$ is the scheme classifying orthogonal lattices $V' \subset V(F)$ such that $V' \in \overline{\mathrm{Gr}}^\lambda_H$ and $v(L) \subset V' \otimes \Omega$. Given such lattice $V'$, the inclusion $v(L) + V \otimes \Omega \subset V' \otimes \Omega + V \otimes \Omega$ must be an equality, because for $V' \in \mathrm{Gr}^\mu_H$ with $\mu \le \lambda$ we get

$$\dim(V' + V)/V = \epsilon(\mu) \le \epsilon(\lambda) = \dim(v(L) + V \otimes \Omega / V \otimes \Omega)$$

We have set here $\epsilon(\mu) = \max\{\langle \mu, \check{\alpha}_m \rangle, \langle \mu, \check{\alpha}'_m \rangle\}$. Thus, $V_v \in \mathrm{Gr}^\lambda_H$ is the unique preimage of $v$ under $\pi$. The first assertion follows.

2) From 1) we learn that $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$ appears in $\mathrm{H}^\lambda_H(I_0)$ with multiplicity one. So, it suffices to show that

$$\mathrm{Hom}(\mathrm{H}^\lambda_H(I_0), \mathrm{H}^\lambda_H(I_0)) = \bar{\mathbb{Q}}_\ell,$$

where Hom is taken in the derived category $\mathrm{D}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F))$. By adjointness,

$$\mathrm{Hom}(\mathrm{H}^\lambda_H(I_0), \mathrm{H}^\lambda_H(I_0)) \widetilde{\to} \mathrm{Hom}(\mathrm{H}^{-w_0(\lambda)}_H \mathrm{H}^\lambda_H(I_0), I_0).$$

So, it suffices to show that for any $\lambda \in \Lambda^+_H$ with $\lambda \ne 0$ one has $\mathrm{Hom}(\mathrm{H}^\lambda_H(I_0), I_0) = 0$ in $\mathrm{D}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F))$. As above, set $N = \langle \lambda, \check{\alpha}_1 \rangle$. Further, it suffices to show that

$$\mathrm{Hom}(\mathrm{H}^\lambda_H(I_0), I_0) = 0$$

in $\mathrm{D}({}_{N,N}\Upsilon(F))$, this is what we are going to do.

Let $i : {}_{0,N}\Upsilon \to {}_{N,N}\Upsilon$ denote the natural closed immersion. Remind that $I_0 = i_! p^! \bar{\mathbb{Q}}_\ell[-2Nnm]$ on ${}_{N,N}\Upsilon$, where $p : {}_{0,N}\Upsilon \to \mathrm{Spec}\, k$ is the projection. By adjointness, we are reduced to show that

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Spec}\, k)}(p_! i^* \mathrm{H}^\lambda_H(I_0)[2Nnm], \bar{\mathbb{Q}}_\ell) = 0.$$

Let us show that $p_! i^* \mathrm{H}^\lambda_H(I_0)[2Nnm]$ is placed in degrees $< 0$.

Denote by $\mathcal{Y}^\lambda$ the preimage of ${}_{0,N}\Upsilon$ under $\pi : {}_{0,N}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^\lambda_H \to {}_{N,N}\Upsilon$. Then $\mathcal{Y}^\lambda$ is the scheme classifying $V' \in \overline{\mathrm{Gr}}^\lambda_H$ and $v \in {}_{0,N}\Upsilon$ such that $v(L) \subset (V'/t^N V) \otimes \Omega$.

Stratify $\mathcal{Y}^\lambda$ by locally closed subschemes $\mathcal{Y}^{\lambda,\mu}$ indexed by $\mu \in \Lambda^+_H$ with $\mu \le \lambda$. The subscheme $\mathcal{Y}^{\lambda,\mu} \subset \mathcal{Y}^\lambda$ is given by the condition $V' \in \mathrm{Gr}^\mu_H$.

Remind that $\mathrm{H}^\lambda_H(I_0) = \pi_!(I_0 \tilde{\boxtimes} \mathcal{A}^\lambda_H)$, where $I_0 \tilde{\boxtimes} \mathcal{A}^\lambda_H$ is perverse. It remains to show that for each stratum $\mathcal{Y}^{\lambda,\mu}$ the complex

$$\mathrm{R}\Gamma_c(\mathcal{Y}^{\lambda,\mu}, (I_0 \tilde{\boxtimes} \mathcal{A}^\lambda_H) \mid_{\mathcal{Y}^{\lambda,\mu}})[2Nnm] \tag{46}$$

is placed in degrees $< 0$. The key observation is that the map $\mathcal{Y}^{\lambda,\mu} \to \mathrm{Gr}^\mu_H$ sending $(v, V')$ to $V'$ is a vector bundle, its rank equals $n(2mN - \epsilon(\mu))$. Here $\epsilon(\mu)$ is the expression defined in 1). So, (46) identifies with

$$\mathrm{R}\Gamma_c(\mathrm{Gr}^\mu_G, \mathcal{A}^\lambda_H \mid_{\mathrm{Gr}^\mu_H})[2n\epsilon(\mu)]. \tag{47}$$

By definition of the intersection cohomology sheaf, $\mathcal{A}^\lambda_H \mid_{\mathrm{Gr}^\mu_H}$ has usual cohomology sheaves in degrees $\le -\langle \mu, 2\check{\rho}_H \rangle$, and the inequality is strict unless $\mu = \lambda$. So, (47) is placed in degrees $\le \langle \mu, 2\check{\rho}_H \rangle - 2n\epsilon(\mu)$, and the inequality is strict unless $\mu = \lambda$.

One checks that for any $\tau \in \Lambda_H^+$ we have $\langle \tau, 2\check{\rho}_H \rangle - 2m\epsilon(\tau) \le 0$, and the inequality is strict unless $\tau = 0$. Our assertion follows, because $n \ge m$.

For the convenience of the reader remind that $T_H \widetilde{\to} \mathbb{G}_m^m$ is the torus of diagonal matrices in $\mathrm{GL}(U_0)$, and $\Lambda_H^+ = \{\tau = (a_1 \ge \ldots \ge a_m) \in \mathbb{Z}^m \mid a_{m-1} \ge |a_m|\}$. In this notation

$$\langle \tau, 2\check{\rho}_H \rangle - 2m\epsilon(\tau) = -2a_1 - 4a_2 - \ldots - (2m-2)a_{m-1} - 2m|a_m|$$

□

**Proposition 7.** *Assume $m \le n$. The irreducible objects of $\mathcal{W}\mathrm{eil}_{G,H}$ are exactly $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$, $\lambda \in \Lambda_H^+$. The functor $\mathcal{T} \mapsto \mathrm{H}_H^{\leftarrow}(\mathcal{T}, I_0)$ yields an equivalence of categories*

$$\mathrm{Sph}_H \widetilde{\to} \mathcal{W}\mathrm{eil}_{G,H}^{ss}$$

*Proof* By Lemma 14, $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$ is an irreducible object of $\mathcal{W}\mathrm{eil}_{G,H}$. For any finite subfield $k_0 \subset k$ we have the $\bar{\mathbb{Q}}_\ell$-vector space $\mathcal{W}\mathrm{eil}_{G,H}(k_0)$ as in Section 3.2.1. Write $\phi_\lambda \in \mathcal{W}\mathrm{eil}_{G,H}(k_0)$ for the function 'trace of Frobenius' of $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$. By Proposition 2, the set $\{\phi_\lambda\}_{\lambda \in \Lambda_H^+}$ is a base of $\mathcal{W}\mathrm{eil}_{G,H}(k_0)$.

Let $K \in \mathcal{W}\mathrm{eil}_{G,H}$ be an irreducible object. Then $K$ is an irreducible perverse sheaf on ${}_{N,N}\Upsilon$ for some $N$. Assume that $K$ is not isomorphic to any $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$, $\lambda \in \Lambda_H^+$. Let $\phi$ be the function trace of Frobenius of $K$. Then $\{\phi, \phi_\lambda\}_{\lambda \in \Lambda_H^+}$ are linearly independent in $K_0(\Upsilon(F))$, so $\phi$ does not lie in $\mathcal{W}\mathrm{eil}_{G,H}(k_0)$. This is a contracition, so $K$ is isomorphic to $\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$ for some $\lambda \in \Lambda_H^+$.

The second assertion also follows from Lemma 14. □

6.3.2 In this subsection we assume $m \ge n$ and analyse the action of $\mathrm{Sph}_G$ on $\mathrm{D}_{(Q_H \times G)(\mathcal{O})}(\Pi(F))$ in more details.

Let $\check{\omega}_i \in \check{\Lambda}_G^+$ be the h.w. of the representation $\wedge^i M_0$ of $G$. For $v \in \Pi(F)$ write $s_{\mathcal{U}}(v) : \wedge^2 U(F) \to \Omega(F)$ for the composition

$$\wedge^2 U(F) \stackrel{\wedge^2 v}{\to} \wedge^2 M(F) \to \Omega(F)$$

Write $\mathrm{Char}(\Pi) \subset \Pi(F)$ for the ind-subscheme of $v \in \Pi(F)$ such that $s_{\mathcal{U}}(v) : \wedge^2 U \to \Omega$ is regular.

For $\lambda \in \Lambda_G^+$ let $N = \langle \lambda, \check{\omega}_1 \rangle$, define the closed subscheme ${}_\lambda\Pi \subset {}_N\Pi = t^{-N}\Pi$ as follows. A point $v \in {}_N\Pi$ lies in ${}_\lambda\Pi$ if the following conditions hold:

C1) $v \in \mathrm{Char}(\Pi)$;

C2) for $i = 1, \ldots, n$ the map $\wedge^i U \stackrel{\wedge^i v}{\to} \wedge^i M(-\langle w_0(\lambda), \check{\omega}_i \rangle)$ is regular.

The scheme ${}_\lambda\Pi$ is stable under the translations by $t^N\Pi$, so there is a closed subscheme ${}_{\lambda,N}\Pi \subset {}_{N,N}\Pi$ such that ${}_\lambda\Pi$ is the preimage of ${}_{\lambda,N}\Pi$ under the projection ${}_N\Pi \to {}_{N,N}\Pi$. As in Remark 3, we have a map

$$\pi : {}_{0,N}\Pi \tilde{\times} \overline{\mathrm{Gr}}_G^\lambda \to {}_{N,N}\Pi \tag{48}$$

and, by definition,

$$\mathrm{H}^{\lambda}_G(I_0)\widetilde{\to}\pi_!(\bar{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^{\lambda}_G)[\dim {}_{0,N}\Pi]\in \mathrm{D}_{(Q_H\times G)(\mathcal{O})}({}_{N,N}\Pi)$$

Since all the weights of the $G$-module $U_0^*\otimes M_0$ are less or equal to $\check{\omega}_i$, the map (48) factors through the closed subscheme ${}_{\lambda,N}\Pi\hookrightarrow {}_{N,N}\Pi$.

For $v\in \mathrm{Char}(\Pi)$ let $U_v=v(U)+M$ and

$$U_v^{\perp}=\{m\in M(F)\mid \langle m,m_1\rangle\in\Omega \;\text{ for any }\; m_1\in U_v\}$$

Let $M_v=v(U)+U_v^{\perp}$. Note that $U_v/U_v^{\perp}$ is naturally a symplectic vector space, and $M_v/U_v^{\perp}\subset U_v/U_v^{\perp}$ is a lagrangian subspace. So, $M_v\subset M(F)$ is a symplectic lattice, that is, $M_v\in \mathrm{Gr}_G$.

Stratify $\mathrm{Char}(\Pi)$ by locally closed subschemes ${}_\lambda\mathrm{Char}(\Pi)$ indexed by $\lambda\in\Lambda^+_G$. Namely, for $v\in\mathrm{Char}(\Pi)$ we let $v\in{}_\lambda\mathrm{Char}(\Pi)$ iff $M_v\in\mathrm{Gr}^{\lambda}_G$. The condition $M_v\in\mathrm{Gr}^{\lambda}_G$ is also equivalent to requiring that there is an isomorphism of $\mathcal{O}$-modules

$$U_v/M\widetilde{\to}\mathcal{O}/t^{a_1}\oplus\ldots\oplus\mathcal{O}/t^{a_n}$$

for $\lambda=(a_1\ge\ldots\ge a_n\ge 0)\in\Lambda^+_G$. So, the stratification in question is by the isomorphism classes of the $\mathcal{O}$-module $U_v/M$.

Clearly, ${}_\lambda\mathrm{Char}(\Pi)\subset{}_\lambda\Pi$, and there is a unique open subscheme ${}_{\lambda,N}\Pi^0\subset{}_{\lambda,N}\Pi$ whose preimage under the projection ${}_\lambda\Pi\to{}_{\lambda,N}\Pi$ identifies with ${}_\lambda\mathrm{Char}(\Pi)$.

Write $\mathrm{IC}({}_{\lambda,N}\Pi^0)$ for the intersection cohomology sheaf of ${}_{\lambda,N}\Pi^0$.

**Lemma 15.** *Assume $m>n$. 1) For any $\lambda\in\Lambda^+_G$ the map*

$$\pi: {}_{0,N}\Pi\tilde{\times}\overline{\mathrm{Gr}}^{\lambda}_G\to{}_{\lambda,N}\Pi$$

*is an isomorphism over the open subscheme ${}_{\lambda,N}\Pi^0$. So, $\dim{}_{\lambda,N}\Pi^0=2Nmn+\langle\lambda,2\check{\rho}_G\rangle$.*
*2) For $\lambda\in\Lambda^+_G$ we have $\mathrm{H}^{\lambda}_G(I_0)\widetilde{\to}\mathrm{IC}({}_{\lambda,N}\Pi^0)$ canonically.*

*Proof* 1) The fibre of $\pi$ over $v\in{}_{\lambda,N}\Pi^0$ is the scheme classifying symplectic lattices $M'\subset M(F)$ such that $M'\in\overline{\mathrm{Gr}}^{\lambda}_G$ and $v(U)\subset M'$. Given such lattice $M'$, the inclusion $U_v\subset M'+M$ must be an equality, because for $M'\in\mathrm{Gr}^{\mu}_G$ with $\mu\le\lambda$ we get

$$\dim(M'+M/M)=\epsilon(\mu)\le\epsilon(\lambda)=\dim(U_v/M).$$

We have denoted here $\epsilon(\mu):=\langle\mu,\check{\omega}_n\rangle$ for $\mu\in\Lambda^+_G$. Thus, $M'=M_v$ is the unique preimage of $v$ under $\pi$. The first assertion follows.
2) is completely analogous to the proof of the second part of Lemma 14. $\square$

The following result is completely analogous to Proposition 7, its proof is omitted.

**Proposition 8.** *Assume $m>n$. The irreducible objects of $\mathcal{W}\mathrm{eil}_{G,H}$ are exactly $\mathrm{IC}({}_{\lambda,N}\Pi^0)$, $\lambda\in\Lambda^+_G$. The functor $\mathcal{T}\mapsto\mathrm{H}^{\leftarrow}_G(\mathcal{T},I_0)$ yields an equivalence of categories*

$$\mathrm{Sph}_G\widetilde{\to}\mathcal{W}\mathrm{eil}^{ss}_{G,H}$$

*Remark* 8. i) For $n=1$ and $m>n$ the isomorphism $\mathrm{H}_G^\lambda(I_0)\widetilde{\to}\mathrm{IC}({}_{\lambda,N}\Pi^0)$ for $\lambda\in\Lambda_G^+$ can also be obtained from Proposition 5. Indeed, in this case $\mathrm{Gr}_G$ identifies with a connected component of $(\mathrm{Gr}_{\mathrm{GL}_2})_{red}$. The desired irreducibility of $\mathrm{H}_G^\lambda(I_0)$ becomes a particular case of Proposition 5.
ii) For $m=1$ and $m\le n$ it is evident that $\mathrm{H}_H^\lambda(I_0)\widetilde{\to}\mathrm{IC}({}_{\lambda,N}\Upsilon^0)$ for $\lambda\in\Lambda_H^+$.

6.4 *Proof of Theorem 7*

**Step 1**. The following property of the Fourier transform functors is standard. If $\mathcal{V}\to S\leftarrow\mathcal{V}^*$ is a diagram of dual vector bundles over a scheme $S$, let $\mathcal{V}'\to S'\leftarrow\mathcal{V}'^*$ be the diagram obtained from it by the base change with respect to a closed immersion $S'\hookrightarrow S$. Then for the inclusions $i_1:\mathcal{V}'\hookrightarrow\mathcal{V}$ and $i_2:\mathcal{V}'^*\hookrightarrow\mathcal{V}^*$ we have $i_2^*\circ\mathrm{Four}_\psi\widetilde{\to}\mathrm{Four}_\psi\circ i_1^*$. Thus, the following diagram of functors commutes

$$\begin{array}{ccc} & \mathrm{D}\mathcal{W}\mathrm{eil}_{G,H} & \\ & \swarrow f_G \qquad \searrow f_H & \\ \mathrm{DP}_{(Q_G\times H)(\mathcal{O})}(\Upsilon(F)) & & \mathrm{DP}_{(Q_H\times G)(\mathcal{O})}(\Pi(F)) \\ \downarrow J^*_{P_H} & & \downarrow J^*_{P_G} \\ \mathrm{D}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F)) & \stackrel{\mathrm{Four}_\psi}{\to} & \mathrm{D}_{(Q_G\times Q_H)(\mathcal{O})}(U^*\otimes L(F))\end{array}$$

Let $\kappa_H:\check{Q}_H\times\mathbb{G}_m\to\check{H}$ be the map, whose first component $\check{Q}_H\to\check{H}$ is the natural inclusion, and second component $\mathbb{G}_m\to\check{H}$ is $2(\check{\rho}_H-\check{\rho}_{Q_H})-n\check{\omega}_m$. Here $\check{\omega}_m$ is the h.w. of the $Q_H$-module $\det U_0$. The corresponding geometric restriction functor is denoted by

$$\mathrm{gRes}^{\kappa_H}:\mathrm{Sph}_H\to\mathrm{D}\,\mathrm{Sph}_{Q_H}$$

Let $\kappa_G:\check{Q}_G\times\mathbb{G}_m\to\check{G}$ be the map, whose first component is the natural inclusion $\check{Q}_G\hookrightarrow\check{G}$, and the second component is $2(\check{\rho}_G-\check{\rho}_{Q_G})-m\check{\omega}_n$. Here $\check{\omega}_n$ is the h.w. of the $Q_G$-module $\det L_0$. The corresponding geometric restriction functor is denoted by

$$\mathrm{gRes}^{\kappa_G}:\mathrm{Sph}_G\to\mathrm{D}\,\mathrm{Sph}_{Q_G}$$

Note that $J^*_{P_H}(I_0)\widetilde{\to}I_0$, $J^*_{P_G}(I_0)\widetilde{\to}I_0$ and $\mathrm{Four}_\psi(I_0)\widetilde{\to}I_0$ canonically. Combining Corollary 3 with Lemma 11, for $\mathcal{T}\in\mathrm{Sph}_H$ and $\mathcal{S}\in\mathrm{Sph}_G$ we get isomorphisms

$$\mathrm{Four}_\psi\, J^*_{P_H}\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_H}(\mathrm{gRes}^{\kappa_H}(\mathcal{T}),I_0)$$

and

$$J^*_{P_G}\mathrm{H}^{\leftarrow}_G(\mathcal{S},I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_G}(\mathrm{gRes}^{\kappa_G}(\mathcal{S}),I_0)$$

in $\mathrm{D}_{(Q_G\times Q_H)(\mathcal{O})}(U^*\otimes L(F))$.

Set $\mathcal{P}=J^*_{P_H}\circ f_G$. To summarize, for $\mathcal{T}\in\mathrm{Sph}_H$ and $\mathcal{S}\in\mathrm{Sph}_G$ we get isomorphisms

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0))\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_H}(\mathrm{gRes}^{\kappa_H}(\mathcal{T}),I_0) \qquad (49)$$

and

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_G(\mathcal{S},I_0))\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_G}(\mathrm{gRes}^{\kappa_G}(\mathcal{S}),I_0) \qquad (50)$$

in $\mathrm{DP}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))$.

According to our conventions in Section 2.1, write

$$\mathrm{DP}^{ss}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))=\underset{i\in\mathbb{Z}}{\oplus}P^{ss}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))[i],$$

it as a full subcategory in $\mathrm{DP}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))$.

**Step 2**. CASE $m\le n$. Remind the Satake equivalence $\mathrm{Loc}^{\mathfrak{r}}:\mathrm{Rep}(\check{Q}_H\times\mathbb{G}_m)\widetilde{\to}\mathrm{D\,Sph}_{Q_H}$. By Corollary 4, we have an equivalence of categories

$$\mathrm{D\,Sph}_{Q_H}\widetilde{\to}\mathrm{DP}^{ss}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))$$

sending $\mathcal{T}$ to $\mathrm{H}^{\leftarrow}_{Q_H}(\mathcal{T},I_0)$. Let

$$\kappa_Q:\check{Q}_H\times\mathbb{G}_m\to\check{Q}_G\times\mathbb{G}_m$$

be the map whose second component $\check{Q}_H\times\mathbb{G}_m\to\mathbb{G}_m$ is the projection, and the first component $\check{Q}_H\times\mathbb{G}_m\to\check{Q}_G$ is the composition

$$\check{Q}_H\times\mathbb{G}_m\stackrel{\mathrm{id}\times 2\check{\rho}_{\mathrm{GL}_{n-m}}}{\to}\check{Q}_H\times\mathrm{GL}_{n-m}\stackrel{\mathrm{Levi}}{\hookrightarrow}\check{Q}_G$$

Write $\mathrm{gRes}^{\kappa_Q}:\mathrm{D\,Sph}_{Q_G}\to\mathrm{D\,Sph}_{Q_H}$ for the corresponding geometric restriction functor.

Now (50) and Corollary 5 yield for $\mathcal{S}\in\mathrm{Sph}_G$ isomorphisms

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_G(\mathcal{S},I_0))\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_G}(\mathrm{gRes}^{\kappa_G}(\mathcal{S}),I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_H}(\mathrm{gRes}^{\kappa_Q}(*\,\mathrm{gRes}^{\kappa_G}(\mathcal{S})),I_0)\widetilde{\to}\mathrm{gRes}^{\kappa_Q}(*\,\mathrm{gRes}^{\kappa_G}(\mathcal{S}))$$

in $\mathrm{D\,Sph}_{Q_H}$. On the other hand, (49) yields an isomorphism

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_H(*\,\mathrm{gRes}^{\kappa}(\mathcal{S}),I_0))\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_H}(\mathrm{gRes}^{\kappa_H}(*\,\mathrm{gRes}^{\kappa}(\mathcal{S})),I_0)$$

Let $\sigma:\check{Q}_G\widetilde{\to}\check{Q}_G$ be the automorphism sending $g$ to ${}^tg^{-1}$ for $g\in\check{Q}_G=\mathrm{GL}_n$. The restriction functor with respect to $\sigma\times\mathrm{id}:\check{Q}_G\times\mathbb{G}_m\widetilde{\to}\check{Q}_G\times\mathbb{G}_m$ identifies with $*:\mathrm{D\,Sph}_{Q_G}\widetilde{\to}\mathrm{D\,Sph}_{Q_G}$.

We will define an automorphism $\sigma_H$ of $\check{H}$ inducing $*:\mathrm{Rep}(\check{H})\widetilde{\to}\mathrm{Rep}(\check{H})$ and $\kappa$ making the following diagram commmutative

$$\begin{array}{ccccc}\check{H}\times\mathbb{G}_m & \stackrel{\sigma_H\times\mathrm{id}}{\to} & \check{H}\times\mathbb{G}_m & \stackrel{\kappa}{\to} & \check{G}\\ \uparrow\kappa_H & & & & \uparrow\kappa_G\\ \check{Q}_H\times\mathbb{G}_m & \stackrel{\kappa_Q}{\to} & \check{Q}_G\times\mathbb{G}_m & \stackrel{\sigma\times\mathrm{id}}{\to} & \check{Q}_G\times\mathbb{G}_m,\end{array}\qquad(51)$$

This will yield for $\mathcal{S}\in\mathrm{Sph}_G$ an isomorphism

$$\mathcal{P}\mathrm{H}^{\leftarrow}_G(\mathcal{S},I_0)\widetilde{\to}\mathcal{P}\mathrm{H}^{\leftarrow}_H(*\,\mathrm{gRes}^{\kappa}(\mathcal{S}),I_0)\qquad(52)$$

Let $W_0=\bar{\mathbb{Q}}_\ell^n$, let $W_0=W_1\oplus W_2$ be the decomposition, where $W_1$ (resp., $W_2$) is generated by the first $m$ (resp., last $n-m$) base vectors. Equip $W_0\oplus W_0^*\oplus\bar{\mathbb{Q}}_\ell$ with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0\\ E_n & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

where $E_n\in \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ is the unity. Realize $\check{G}$ as $\mathbb{SO}(W_0\oplus W_0^*\oplus\bar{\mathbb{Q}}_\ell)$.

Equip the subspace $W_1\oplus W_1^*\subset W_0\oplus W_0^*\oplus\bar{\mathbb{Q}}_\ell$ with the induced symmetric form, realize $\check{H}$ as $\mathbb{SO}(W_1\oplus W_1^*)$, this yields the inclusion $\check{H}\hookrightarrow\check{G}$. Let $\sigma_H:\check{H}\to\check{H}$ be the automorphism sending $g$ to ${}^tg^{-1}$. It is understood that $\check{Q}_G=\mathrm{Aut}(W_0)$ and $\check{Q}_H=\mathrm{Aut}(W_1)$ canonically. Let $\kappa$ be the composition

$$\check{H}\times\mathbb{G}_m\stackrel{\mathrm{id}\times\alpha_\kappa}{\to}\check{H}\times\mathrm{GL}(W_2)\hookrightarrow\check{G},$$

where $\alpha_\kappa:\mathbb{G}_m\to\mathrm{GL}(W_2)$ is $(n-m+1)(\check{\omega}_n-\check{\omega}_m)-2\check{\rho}_{\mathrm{GL}_{n-m}}$. The equality

$$\alpha_\kappa-2(\check{\rho}_H-\check{\rho}_{Q_H})+n\check{\omega}_m=2(\check{\rho}_G-\check{\rho}_{Q_G})-2\rho_{\mathrm{GL}(W_2)}-m\check{\omega}_n$$

shows that (51) commutes. If $m=n$ then $\kappa$ is trivial on $\mathbb{G}_m$.

The functor $\mathcal{P}$ fits into diagram

$$\begin{array}{ccc}\mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H} & \widetilde{\to} & \mathrm{D}\,\mathrm{Sph}_H\\ \downarrow\mathcal{P} & & \downarrow \mathrm{gRes}^{\kappa_H}\\ \mathrm{DP}^{ss}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F)) & \widetilde{\to} & \mathrm{D}\,\mathrm{Sph}_{Q_H},\end{array}$$

and $\mathrm{gRes}^{\kappa_H}$ is faithfull. So, (52) can be lifted to the desired isomorphism (44).

CASE $m>n$. Remind the Satake equivalence $\mathrm{Loc}^{\mathfrak{r}}:\mathrm{Rep}(\check{Q}_G\times\mathbb{G}_m)\widetilde{\to}\mathrm{D}\,\mathrm{Sph}_{Q_G}$. By Corollary 4, we have an equivalence of categories

$$\mathrm{D}\,\mathrm{Sph}_{Q_G}\widetilde{\to}\mathrm{DP}^{ss}_{(Q_G\times Q_H)(\mathcal{O})}(U\otimes L^*\otimes\Omega(F))$$

sending $\mathcal{S}$ to $\mathrm{H}^{\leftarrow}_{Q_G}(\mathcal{S},I_0)$. Let

$$\kappa_Q:\check{Q}_G\times\mathbb{G}_m\to\check{Q}_H\times\mathbb{G}_m$$

be the map whose second component $\check{Q}_G\times\mathbb{G}_m\to\mathbb{G}_m$ is the projection, and the first component $\check{Q}_G\times\mathbb{G}_m\to\check{Q}_H$ is the composition

$$\check{Q}_G\times\mathbb{G}_m\stackrel{\mathrm{id}\times 2\check{\rho}_{\mathrm{GL}_{m-n}}}{\to}\check{Q}_G\times\mathrm{GL}_{m-n}\stackrel{\mathrm{Levi}}{\to}\check{Q}_H$$

Write $\mathrm{gRes}^{\kappa_Q}:\mathrm{D}\,\mathrm{Sph}_{Q_H}\to\mathrm{D}\,\mathrm{Sph}_{Q_G}$ for the corresponding geometric restriction functor.

Now (49) and Corollary 5 yield for $\mathcal{T}\in\mathrm{Sph}_H$ isomorphisms

$$\mathcal{P}\mathrm{H}^{\leftarrow}_H(\mathcal{T},I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_H}(\mathrm{gRes}^{\kappa_H}(\mathcal{T}),I_0)\widetilde{\to}\mathrm{H}^{\leftarrow}_{Q_G}(\mathrm{gRes}^{\kappa_Q}(*\,\mathrm{gRes}^{\kappa_H}(\mathcal{T})),I_0)\widetilde{\to}\mathrm{gRes}^{\kappa_Q}(*\,\mathrm{gRes}^{\kappa_H}(\mathcal{T}))$$

in $\mathrm{D\,Sph}_{Q_G}$. On the other hand, (50) yields an isomorphism

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{Q_G}(\mathrm{gRes}^{\kappa_G}\, \mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)$$

Let $\sigma$ be the automorphism of $\check{Q}_H = \mathrm{GL}_m$ sending $g$ to ${}^t g^{-1}$. The automorphism $\sigma \times \mathrm{id}$ of $\check{Q}_H \times \mathbb{G}_m$ induces the equivalence $* : \mathrm{D\,Sph}_{Q_H} \widetilde{\to} \mathrm{D\,Sph}_{Q_H}$. We will define an automorphism $\sigma_H$ of $\check{H}$ inducing $* : \mathrm{Rep}(\check{H}) \widetilde{\to} \mathrm{Rep}(\check{H})$ and $\kappa$ making the following diagram commutative

$$\begin{array}{ccccc} \check{G} \times \mathbb{G}_m & \stackrel{\kappa}{\to} & \check{H} & \stackrel{\sigma_H}{\to} & \check{H} \\ \uparrow \kappa_G & & & & \uparrow \kappa_H \\ \check{Q}_G \times \mathbb{G}_m & \stackrel{\kappa_Q}{\to} & \check{Q}_H \times \mathbb{G}_m & \stackrel{\sigma\times\mathrm{id}}{\to} & \check{Q}_H \times \mathbb{G}_m \end{array} \qquad (53)$$

This will provide for $\mathcal{T} \in \mathrm{Sph}_H$ an isomorphism

$$\mathcal{P}\mathrm{H}^{\leftarrow}_H(\mathcal{T}, I_0) \widetilde{\to} \mathcal{P}\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0) \qquad (54)$$

Let $W_0 = \bar{\mathbb{Q}}_\ell^m$, let $W_1$ (resp., $W_2$) be the subspace of $W_0$ spanned by the first $n$ (resp., last $m-n$) base vectors. Equip $W_0 \oplus W_0^*$ with the symmetric form given by the matrix

$$\left(\begin{array}{cc} 0 & E_m \\ E_m & 0 \end{array}\right),$$

where $E_m \in \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$ is the unity. Realize $\check{H}$ as $\mathbb{SO}(W_0 \oplus W_0^*)$. Let $\sigma_H$ be the automorphism of $\check{H}$ sending $g$ to ${}^t g^{-1}$. Let $\bar{W} \subset W_2 \oplus W_2^*$ be the subspace spanned by $e_{n+1} + e^*_{n+1}$, equip $W_1 \oplus W_1^* \oplus \bar{W}$ with the induced form and realize $\check{G}$ as $\mathbb{SO}(W_1 \oplus W_1^* \oplus \bar{W})$. Thus, the inclusion $i_\kappa : \check{G} \hookrightarrow \check{H}$ is fixed. There is a unique $\alpha_\kappa : \mathbb{G}_m \to \check{H}$ such that for $\kappa = (i_\kappa, \alpha_\kappa) : \check{G} \times \mathbb{G}_m \to \check{H}$ the diagram (53) commutes. Actually, $\alpha_\kappa = 2\rho_{\mathrm{GL}(W_2)} + (m-n-1)(\check{\omega}_n - \check{\omega}_m)$. Note that if $m = n+1$ then $\alpha_\kappa$ is trivial.

The functor $\mathcal{P}$ fits into diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{W}\mathrm{eil}^{ss}_{G,H} & \widetilde{\to} & \mathrm{D\,Sph}_G \\ \downarrow \mathcal{P} & & \downarrow \mathrm{gRes}^{\kappa_G} \\ \mathrm{DP}^{ss}_{(Q_G \times Q_H)(\mathcal{O})}(U \otimes L^* \otimes \Omega(F)) & \widetilde{\to} & \mathrm{D\,Sph}_{Q_G}, \end{array}$$

and $\mathrm{gRes}^{\kappa_G}$ is faithfull. So, (54) can be lifted to the desired isomorphism (45). □

*Remark* 9. In the special case $m = 1$ we have $H = Q_H$. So, in this case $\mathcal{W}\mathrm{eil}_{G,H}$ is equivalent to the category $\mathrm{P}_{(H\times G)(\mathcal{O})}(\Pi(F))$, and one need not glue the categories as in Definition 4. The proof of Theorem 7 can be simplified in this case. For an integer $N$ let ${}_N\mathrm{IC} \in \mathrm{P}_{(H\times G)(\mathcal{O})}(\Pi(F))$ denote the constant perverse sheaf on $t^{-N}\Pi$. The irreducible objects of $\mathcal{W}\mathrm{eil}_{G,H}$ in this case are exactly ${}_N\mathrm{IC}$, $N \in \mathbb{Z}$. For a dominant coweight $\lambda = N$ of $H$ in this case we get $\mathrm{H}^{\lambda}_H(I_0) \widetilde{\to} {}_N\mathrm{IC}$.

## 7. Global theta-lifting for the dual pair $\mathrm{GL}_n, \mathrm{GL}_m$

In this section we prove Theorems 5 and 6.

*Proof of Theorem 6*

Remind the notation $U_0 = k^m$, $L_0 = k^n$, and the groups $G = \mathrm{GL}(L_0), H = \mathrm{GL}(U_0)$. Set $M_0 = L_0 \otimes U_0$ and $M = M_0(\mathcal{O})$ for $\mathcal{O} = k[[t]]$. Viewing $M_0$ as a representation of $G \times H$, one defines for $x \in X$ the functor $\mathrm{glob}_x : \mathrm{D}_{(G\times H)(\mathcal{O})}(M(F)) \to \mathrm{D}({}_{x,\infty}\mathcal{W}_{n,m})$ as in Section 4.6. Then $\mathrm{glob}_x(I_0) \widetilde{\to} \mathcal{I}$ canonically, where $\mathcal{I}$ is given by (13). The Hecke functors (11) and (12) are a particular case of those defined in Section 4.6. Since $\mathrm{glob}_x$ commutes with Hecke functors, our assertion follows from Proposition 4. □

*Proof of Theorem 5*

The argument below mimics that of ([4], Section 4.1.8). To simplify notation, we will establish for $\mathcal{S} \in \mathrm{Sph}_H$ and $K \in \mathrm{D}(\mathrm{Bun}_n)$ an isomorphism

$$ {}_x\mathrm{H}^{\leftarrow}_H(\mathcal{S}, F_{n,m}(K)) \widetilde{\to} F_{n,m}({}_x\mathrm{H}^{\rightarrow}_G(\mathrm{Res}^{\kappa}(\mathcal{S}), K)) \tag{55} $$

for a given $x \in X$. The proof of the original statement is completely similar.

Let ${}_{x,\infty}Z_H$ denote the stack classifying $(U, U', \beta : U' \widetilde{\to} U \mid_{X-x}) \in {}_x\mathcal{H}_H$, $L \in \mathrm{Bun}_n$ and $s : \mathcal{O}_X \to L \otimes U'(\infty x)$. We have a diagram, where both squares are cartesian

$$\begin{array}{ccccc} {}_{x,\infty}\mathcal{W}_{n,m} & \stackrel{h^{\leftarrow}_{Z,H}}{\leftarrow} & {}_{x,\infty}Z_H & \stackrel{h^{\rightarrow}_{Z,H}}{\rightarrow} & {}_{x,\infty}\mathcal{W}_{n,m} \\ \downarrow h_m & & \downarrow & & \downarrow h_m \\ \mathrm{Bun}_m & \stackrel{h^{\leftarrow}_H}{\leftarrow} & {}_x\mathcal{H}_H & \stackrel{h^{\rightarrow}_H}{\rightarrow} & \mathrm{Bun}_m \end{array}$$

Here $h^{\rightarrow}_H$ (resp., $h^{\leftarrow}_H$) sends $(U, U')$ to $U'$ (resp., to $U$). The map $h^{\rightarrow}_{Z,H}$ (resp., $h^{\leftarrow}_{Z,H}$) sends $(U, U', L, s : \mathcal{O} \to L \otimes U'(\infty x))$ to $(L, U', s : \mathcal{O} \to L \otimes U'(\infty x))$ (resp., to $(L, U, \beta \circ s : \mathcal{O} \to L \otimes U(\infty x))$).

Using base change and the projection formula, one gets an isomorphism

$$ {}_x\mathrm{H}^{\leftarrow}_H(\mathcal{S}, F_{n,m}(K)) \widetilde{\to} (h_m)_!(h_n^*K \otimes {}_x\mathrm{H}^{\leftarrow}_H(\mathcal{S}, \mathcal{I}))[-\dim \mathrm{Bun}_n] $$

Here $h_n : {}_{x,\infty}\mathcal{W}_{n,m} \to \mathrm{Bun}_n$ is the corresponding projection. By Theorem 6, this complex identifies with

$$ (h_m)_!(h_n^*K \otimes {}_x\mathrm{H}^{\leftarrow}_G(\mathrm{gRes}^{\kappa}(\mathcal{S}), \mathcal{I}))[-\dim \mathrm{Bun}_n] \tag{56} $$

Let ${}_{x,\infty}Z_G$ be the stack classifying $(L, L', \beta : L' \widetilde{\to} L \mid_{X-x}) \in {}_x\mathcal{H}_G$, $U \in \mathrm{Bun}_m$ and $s : \mathcal{O}_X \to L' \otimes U(\infty x)$. As above, one has the diagram

$$\begin{array}{ccccc} {}_{x,\infty}\mathcal{W}_{n,m} & \stackrel{h^{\leftarrow}_{Z,G}}{\leftarrow} & {}_{x,\infty}Z_G & \stackrel{h^{\rightarrow}_{Z,G}}{\rightarrow} & {}_{x,\infty}\mathcal{W}_{n,m} \\ \downarrow h_n & & \downarrow & & \downarrow h_n \\ \mathrm{Bun}_n & \stackrel{h^{\leftarrow}_G}{\leftarrow} & {}_x\mathcal{H}_G & \stackrel{h^{\rightarrow}_G}{\rightarrow} & \mathrm{Bun}_n \end{array}$$

Here $h^{\rightarrow}_G$ (resp., $h^{\leftarrow}_G$) sends $(L, L')$ to $L'$ (resp., to $L$). The map $h^{\rightarrow}_{Z,G}$ (resp., $h^{\leftarrow}_{Z,G}$) sends $(U, L', L, s : \mathcal{O} \to L' \otimes U(\infty x))$ to $(L', U, s : \mathcal{O} \to L' \otimes U(\infty x))$ (resp., to the collection $(L, U, \beta \circ s : \mathcal{O} \to L \otimes U(\infty x))$).

The maps $h_m \circ h^{\leftarrow}_{Z,G}$ and $h_m \circ h^{\rightarrow}_{Z,G}$ coincide. So, by base change and projection formula, (56) identifies with

$$F_{n,m}({}_x\mathrm{H}^{\rightarrow}_G(\mathrm{gRes}^{\kappa}(\mathcal{S}), K))$$

This yields the desired isomorphism (55). □

## 8. Global theta-lifting for the dual pair $\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n}$

8.1 In this subsection we derive Theorem 3 from Theorem 4. We give the argument for $m \le n$ (the case $m > n$ is completely similar).

By base change theorem, for $\mathcal{S} \in \mathrm{Sph}_G$, $K \in \mathrm{D}(\mathrm{Bun}_H)$ we get

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, F_G(K)) \widetilde{\to} (\mathrm{id} \times \mathfrak{p})_!(\mathfrak{q}^* K \otimes \mathrm{H}^{\leftarrow}_G(\mathcal{S}, \mathrm{Aut}_{G,H}))[-\dim \mathrm{Bun}_H],$$

where $\mathrm{id} \times \mathfrak{p} : X \times \mathrm{Bun}_{G,H} \to X \times \mathrm{Bun}_G$ and $\mathfrak{q} : \mathrm{Bun}_{G,H} \to \mathrm{Bun}_H$ are the projections. By Theorem 4, the latter complex identifies with

$$(\mathrm{id} \times \mathfrak{p})_!(\mathfrak{q}^* K \otimes \mathrm{H}^{\rightarrow}_H(\mathrm{gRes}^{\kappa}(\mathcal{S}), \mathrm{Aut}_{G,H}))[-\dim \mathrm{Bun}_H] \qquad (57)$$

Now the diagram

$$\begin{array}{ccccc}
X \times \mathrm{Bun}_H & \stackrel{\mathrm{supp} \times h^{\leftarrow}_H}{\leftarrow} & \mathcal{H}_H & \stackrel{h^{\rightarrow}_H}{\to} & \mathrm{Bun}_H \\
\uparrow \mathrm{id} \times \mathfrak{q} & & \uparrow & & \uparrow \mathfrak{q} \\
X \times \mathrm{Bun}_{G,H} & \stackrel{\mathrm{supp} \times h^{\leftarrow}_H}{\leftarrow} & \mathcal{H}_H \times \mathrm{Bun}_G & \stackrel{\mathrm{supp} \times h^{\rightarrow}_H}{\to} & X \times \mathrm{Bun}_{G,H} \\
& & & & \downarrow \mathrm{id} \times \mathfrak{p} \\
& & & & X \times \mathrm{Bun}_G
\end{array}$$

and the projection formula show that (57) identifies with

$$(\mathrm{id} \times \mathfrak{p})_!((\mathrm{id} \times \mathfrak{q})^* \mathrm{H}^{\leftarrow}_H(\mathrm{gRes}^{\kappa}(\mathcal{S}), K) \otimes \mathrm{Aut}_{G,H})[-\dim \mathrm{Bun}_H]$$

This is what we had to prove. □

8.2 In this section we we derive Theorem 4 from Theorem 7. To simplify notations, fix $x \in X$, we will establish isomorphisms (8) and (9) over $x \times \mathrm{Bun}_{G,H}$. The fact that these isomorphisms depend on $x$ as expected is left to the reader.

Keep notations of Section 2. As in Section 6.1, let $L$ be a free $\mathcal{O}_x$-module of rank $n$, set $M = L \oplus L \otimes \Omega_x$ with the corresponding symplectic form $\wedge^2 M \to \Omega_x$. Let $U$ be a free $\mathcal{O}_x$-module of rank $m$, set $V = U \oplus U^*$ with the corresponding symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}_x$. Sometimes we view $M$ (resp., $V$) as a $G$-torsor (resp., $H$-torsor) over $\mathrm{Spec}\, \mathcal{O}_x$.

In view of Theorem 7, Theorem 4 is reduced to the following result, which we actually prove.

**Proposition 9.** *There is a natural functor* $\mathrm{LW} : \mathrm{D}\mathcal{W}\mathrm{eil}_{G,H} \to \mathrm{D}(\mathrm{Bun}_{G,H})$ *commuting with the actions of both* $\mathrm{D}\,\mathrm{Sph}_G$ *and* $\mathrm{D}\,\mathrm{Sph}_H$*. There is an isomorphism* $\mathrm{LW}(I_0) \widetilde{\to} \mathrm{Aut}_{G,H}$*.*

8.2.1 The proof is based on the following construction from [18]. Let $\mathcal{L}_d(M \otimes V(F_x))$ denote the scheme of discrete lagrangian lattices in $M \otimes V(F_x)$. Let $\mathcal{A}_d$ be the line bundle on $\mathcal{L}_d(M \otimes V(F_x))$ with fibre $\det(M \otimes V : R)$ at $R \in \mathcal{L}_d(M \otimes V(F_x))$ (cf. *loc.cit.* for the definition of this relative determinant). Note that $\mathcal{A}_d$ is $(G \times H)(\mathcal{O}_x)$-equivariant, so it can be viewed as a line bundle on the stack quotient

$$\mathcal{L}_d(M \otimes V(F_x))/(G \times H)(\mathcal{O}_x)$$

Let $\widetilde{\mathcal{L}}_d(M \otimes V(F_x))$ denote the $\mu_2$-gerb of square roots of $\mathcal{A}_d$. Write $\widetilde{\mathcal{L}}_d(M \otimes V(F_x))/(G \times H)(\mathcal{O}_x)$ for the corresponding $\mu_2$-gerb over $\mathcal{L}_d(M \otimes V(F_x))/(G \times H)(\mathcal{O}_x)$.

We have a morphism of stacks

$$\xi_x : \mathrm{Bun}_{G,H} \to \mathcal{L}_d(M \otimes V(F_x))/(G \times H)(\mathcal{O}_x)$$

sending $\mathcal{M} \in \mathrm{Bun}_G, \mathcal{V} \in \mathrm{Bun}_H$ to the Tate space $\mathcal{M} \otimes \mathcal{V}(F_x)$ equipped with lagrangian c-lattice $\mathcal{M} \otimes \mathcal{V}(\mathcal{O}_x)$ and lagrangian d-lattice $\mathrm{H}^0(X - x, \mathcal{M} \otimes \mathcal{V})$. It is understood that one first picks isomorphisms

$$\mathcal{M} \mid_{\mathrm{Spec}\,\mathcal{O}_x} \widetilde{\to} M \mid_{\mathrm{Spec}\,\mathcal{O}_x} \quad \text{and} \quad \mathcal{V} \mid_{\mathrm{Spec}\,\mathcal{O}_x} \widetilde{\to} V \mid_{\mathrm{Spec}\,\mathcal{O}_x}$$

of the corresponding $G$-torsors and $H$-torsors over $\mathrm{Spec}\,\mathcal{O}_x$ and then takes the stack quotients by $(G \times H)(\mathcal{O}_x)$.

We have canonically $\xi_x^*\mathcal{A}_d \widetilde{\to} \tau^*\mathcal{A}_{G_{2nm}}$, where $\tau$ is defined in Section 2.4.1. We lift $\xi_x$ to a morphism

$$\tilde{\xi}_x : \mathrm{Bun}_{G,H} \to \widetilde{\mathcal{L}}_d(M \otimes V(F_x))/(G \times H)(\mathcal{O}_x)$$

sending $(\mathcal{M} \in \mathrm{Bun}_G, \mathcal{V} \in \mathrm{Bun}_H)$ to $(\xi_x(\mathcal{M}, \mathcal{V}), \mathcal{B})$, where

$$\mathcal{B} = \frac{\det \mathrm{R}\Gamma(X, \mathcal{V})^n \otimes \det \mathrm{R}\Gamma(X, \mathcal{M})^m}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm}}$$

is equipped with an isomorphism $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V})$ given by Lemma 1.

For $r \geq 0$ write ${}_{rx}\mathrm{Bun}_{G,H} \subset \mathrm{Bun}_{G,H}$ for the open substack given by $\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{V}(-rx)) = 0$ for $\mathcal{M} \in \mathrm{Bun}_G, \mathcal{V} \in \mathrm{Bun}_H$. If $r' \geq r$ then ${}_{rx}\mathrm{Bun}_{G,H} \subset {}_{r'x}\mathrm{Bun}_{G,H}$ is an open substack, and we consider the projective 2-limit

$$\underset{r \to \infty}{2\text{–}\lim}\, \mathrm{D}({}_{rx}\mathrm{Bun}_{G,H})$$

8.2.2 As in ([18], Section 7.2) one defines the restriction functor

$$\tilde{\xi}_x^* : \mathrm{D}_{(G \times H)(\mathcal{O}_x)}(\widetilde{\mathcal{L}}_d(M \otimes V(F_x))) \to \underset{r \to \infty}{2\text{–}\lim}\, \mathrm{D}({}_{rx}\mathrm{Bun}_{G,H}) \tag{58}$$

as follows. For $N, r \in \mathbb{Z}$ with $N + r \geq 0$ and a free $\mathcal{O}_x$-module $\mathcal{L}$ of finite rank write ${}_{N,r}\mathcal{L} = t^{-N}\mathcal{L}/t^r\mathcal{L}$. Let $\mathcal{L}({}_{N,N}M \otimes V)$ denote the scheme of lagrangian subspaces in the symplectic $k$-space ${}_{N,N}M \otimes V$. For $N \geq r \geq 0$ let

$${}_r\mathcal{L}({}_{N,N}M \otimes V) \subset \mathcal{L}({}_{N,N}M \otimes V)$$

be the open subscheme of $R \in \mathcal{L}({}_{N,N}M \otimes V)$ such that $R \cap {}_{-r,N}(M \otimes V) = 0$. For $r_1 \geq 2N$ let $\mathcal{A}_N$ be the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on the stack quotient

$$ {}_r\mathcal{L}({}_{N,N}M \otimes V)/(G \times H)(\mathcal{O}/t^{r_1}) $$

whose fibre at a lagragian subspace $R$ is $\det({}_{0,N}M \otimes V) \otimes \det R$. Write

$$ ({}_r\mathcal{L}({}_{N,N}M \otimes V)/(G \times H)(\mathcal{O}_x/t^{r_1}))\tilde{} $$

for the gerb of square roots of this line bundle. The derived categories on these gerbs for all $r_1 \geq 2N$ are canonically equivalent to each other and are denoted

$$ \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}({}_{N,N}M \otimes V)) $$

For $N_1 \geq N \geq r \geq 0$ we have a projection

$$ p : {}_r\mathcal{L}({}_{N_1,N_1}M \otimes V) \to {}_r\mathcal{L}({}_{N,N}M \otimes V) $$

sending $R$ to $R \cap {}_{N,N_1}M$. There is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $p^*\mathcal{A}_N \widetilde{\to} \mathcal{A}_{N_1}$. For $r_1 \geq 2N_1$ it yields a morphism of stacks

$$ \tilde{p} : ({}_r\mathcal{L}({}_{N_1,N_1}M \otimes V)/(G \times H)(\mathcal{O}_x/t^{r_1}))\tilde{} \to ({}_r\mathcal{L}({}_{N,N}M \otimes V)/(G \times H)(\mathcal{O}_x/t^{r_1}))\tilde{} $$

The latter gives rise to the transition functor

$$ \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}({}_{N,N}M \otimes V)) \to \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}({}_{N_1,N_1}M \otimes V)) \tag{59} $$

sending $K$ to $\tilde{p}^*K[\dim.\mathrm{rel}(\tilde{p})]$, it is exact for the perverse t-structures and a fully faithful embedding. The inductive 2-limit of

$$ \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}({}_{N,N}M \otimes V)) $$

as $N$ goes to infinity is denoted $\mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}_d(M \otimes V(F_x)))$. For $N \geq r' \geq r$ and $r_1 \geq 2N$ we have an open immersion

$$ \tilde{j} : ({}_r\mathcal{L}({}_{N,N}M \otimes V)/(G \times H)(\mathcal{O}_x/t^{r_1}))\tilde{} \hookrightarrow ({}_{r'}\mathcal{L}({}_{N,N}M \otimes V)/(G \times H)(\mathcal{O}_x/t^{r_1}))\tilde{} $$

hence the restriction functors

$$ \tilde{j}^* : \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_{r'}\widetilde{\mathcal{L}}({}_{N,N}M \otimes V)) \to \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}({}_{N,N}M \otimes V)) $$

compatible with the transition functors (59). Passing to the limit as $N$ goes to infinity we get the functors

$$ \tilde{j}_{r',r} : \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_{r'}\widetilde{\mathcal{L}}_d(M \otimes V(F_x))) \to \mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}_d(M \otimes V(F_x))) $$

By definition, $\mathrm{D}_{(G\times H)(\mathcal{O}_x)}(\widetilde{\mathcal{L}}_d(M\otimes V(F_x)))$ is the projective 2-limit of

$$\mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}_d(M\otimes V(F_x)))$$

as $r$ goes to infinity (cf. also *loc.cit.*).

The map $p$ fits into the diagram

$$\begin{array}{ccc} {}_{rx}\operatorname{Bun}_{G,H} & \stackrel{\xi_N}{\to} & {}_r\mathcal{L}({}_{N,N}M\otimes V)/(G\times H)(\mathcal{O}_x/t^{r_1}) \\ & \searrow \xi_{N_1} & \uparrow p \\ & & {}_r\mathcal{L}({}_{N_1,N_1}M\otimes V)/(G\times H)(\mathcal{O}_x/t^{r_1}) \end{array}$$

where $\xi_N$ sends $(\mathcal{M},\mathcal{V})$ to the lagrangian subspace $\mathrm{H}^0(X,\mathcal{M}\otimes\mathcal{V}(Nx))\subset {}_{N,N}M\otimes V$. As above, it is understood that one first picks a trivialization

$$\mathcal{M}\otimes\mathcal{V}\mid_{\operatorname{Spec}\mathcal{O}_x/t^{r_1}}\widetilde{\to} M\otimes V\mid_{\operatorname{Spec}\mathcal{O}_x/t^{r_1}}$$

of the corresponding $G\times H$-torsor over $\operatorname{Spec}\mathcal{O}_x/t^{2N}$ and takes stack quotients by $(G\times H)(\mathcal{O}_x/t^{r_1})$.

We have canonically $\xi_N^*\mathcal{A}_N\widetilde{\to}\tau^*\mathcal{A}_{G_{2nm}}$. So, we get a similar diagram between the gerbs

$$\begin{array}{ccc} {}_{rx}\operatorname{Bun}_{G,H} & \stackrel{\tilde\xi_N}{\to} & ({}_r\mathcal{L}({}_{N,N}M\otimes V)/(G\times H)(\mathcal{O}/t^{r_1}))\tilde{} \\ & \searrow \tilde\xi_{N_1} & \uparrow p \\ & & ({}_r\mathcal{L}({}_{N_1,N_1}M\otimes V)/(G\times H)(\mathcal{O}/t^{r_1}))\tilde{} \end{array}$$

The functors $K\mapsto\tilde\xi_N^*K[\dim.\operatorname{rel}(\xi_N)]$ are compatible with the transition functors (59) so yield a functor

$${}_r\xi_x^*:\mathrm{D}_{(G\times H)(\mathcal{O}_x)}({}_r\widetilde{\mathcal{L}}_d(M\otimes V(F_x)))\to\mathrm{D}({}_{rx}\operatorname{Bun}_{G,H})$$

Passing to the limit by $r$, one gets the desired functor (58). It commutes with the actions of $\mathrm{D}\operatorname{Sph}_G$ and $\mathrm{D}\operatorname{Sph}_H$ on both sides.

8.2.3 Remind that $\mathrm{P}(\operatorname{Bun}_{G,H})\widetilde{\to}\text{2–}\lim_{r\to\infty}\mathrm{P}({}_{rx}\operatorname{Bun}_{G,H})\subset\text{2–}\lim_{r\to\infty}\mathrm{D}({}_{rx}\operatorname{Bun}_{G,H})$ is a full subcategory. Let $S_{M\otimes V(F_x)}$ denote the theta-sheaf on $\widetilde{\mathcal{L}}_d(M\otimes V(F_x))$ introduced in ([18], Section 6.5). It is naturally $(G\times H)(\mathcal{O}_x)$-equivariant, and we have $\tilde\xi_x^*S_{M\otimes V(F_x)}\widetilde{\to}\operatorname{Aut}_{G,H}$ by (*loc.cit.*, Theorem 3).

8.2.4 As in ([18], Section 5.4) let $\widetilde{\mathbb{S}\mathrm{p}}(M\otimes V)(F_x)$ denote the metaplectic group corresponding to the c-lattice $M\otimes V$ in $M\otimes V(F_x)$. This is a group stack classifying collections

$$(g\in\mathbb{S}\mathrm{p}(M\otimes V)(F_x),\mathcal{B},\ \mathcal{B}^2\widetilde{\to}\det(M\otimes V:g(M\otimes V))),$$

where $\mathcal{B}$ is a 1-dimensional $k$-vector space. The product map sends

$$(g_1,\mathcal{B}_1,\sigma_1:\mathcal{B}_1^2\widetilde{\to}\det(M\otimes V:gM\otimes V)),(g_2,\mathcal{B}_2,\sigma_2:\mathcal{B}_2^2\widetilde{\to}\det(M\otimes V:gM\otimes V))$$

to $(g_1g_2,\mathcal{B},\sigma:\mathcal{B}^2\widetilde{\to}\det(M\otimes V:g_1g_2M\otimes V))$, where $\mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2$ and $\sigma$ is the composition

$$(\mathcal{B}_1\otimes\mathcal{B}_2)^2\stackrel{\sigma_1\otimes\sigma_2}{\to}\det(M\otimes V:g_1M\otimes V)\otimes\det(M\otimes V:g_2M\otimes V)\stackrel{\mathrm{id}\otimes g_1}{\to}$$
$$\det(M\otimes V:g_1M\otimes V)\otimes\det(g_1M\otimes V:g_1g_2M\otimes V)\widetilde{\to}\det(M\otimes V:g_1g_2M\otimes V)$$

**Lemma 16.** *Let $M_i$ be a free $\mathcal{O}_x$-module of rank $n_i$. Then for $g \in \mathrm{Gr}_{\mathrm{SL}(M_0)}$ there is a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded lines*

$$\det(M_0 \otimes M_1 : (gM_0) \otimes M_1) \widetilde{\to} \det(M_0 : gM_0)^{n_1}$$

*Proof* Let $\mathcal{A}_0$ be the line bundle on $\mathrm{Gr}_{\mathrm{SL}(M_0)}$ with fibre $\det(M_0 : gM_0)$ at $g \in \mathrm{Gr}_{\mathrm{SL}(M_0)}$. It is known that $\mathrm{Pic}\, \mathrm{Gr}_{\mathrm{SL}(M_0)} \widetilde{\to} \mathbb{Z}$ is generated by $\mathcal{A}_0$. Let $\mathcal{L}$ be the line bundle on $\mathrm{Gr}_{\mathrm{SL}(M_0)}$ with fibre

$$\det(M_0 \otimes M_1 : gM_0 \otimes M_1)$$

at $g$. A choice of a base in $M_1$ yields a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\mathcal{L} \widetilde{\to} \mathcal{A}_0^{n_1}$. Thus, the line bundle $\mathcal{L} \otimes \mathcal{A}_0^{-n_1}$ on $\mathrm{Gr}_{\mathrm{SL}(M_0)}$ is constant. Its fibre at $1 \in \mathrm{Gr}_{\mathrm{SL}(M_0)}$ is canonically trivialized, so the line bundle itself is canonically trivialized. □

By Lemma 16, for $g \in G(F_x), h \in H(F_x)$ we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det(M \otimes V : gM \otimes hV) \widetilde{\to} \det(M \otimes V : gM \otimes V) \otimes \det(gM \otimes V : gM \otimes hV) \widetilde{\to} \det(M : gM)^{2m} \otimes \det(V : hV)^{2n}$$

It yields a canonical section $(G \times H)(F_x) \to \widetilde{\mathbb{S}\mathrm{p}}(M \otimes V)(F_x)$ sending $(g \in G(F_x), h \in H(F_x)$ to $(g \otimes h, \mathcal{B}, \mathcal{B}^2 \widetilde{\to} \det(M \otimes V : gM \otimes hV))$, where

$$\mathcal{B} = \det(M : gM)^m \otimes \det(V : hV)^n$$

The canonical sections $(G \times H)(F_x) \to \widetilde{\mathbb{S}\mathrm{p}}(M \otimes V)(F_x)$ and $\mathbb{S}\mathrm{p}(M \otimes V)(\mathcal{O}_x) \to \widetilde{\mathbb{S}\mathrm{p}}(M \otimes V)(F_x)$ are compatible over $(G \times H)(\mathcal{O}_x)$.

8.2.5 *Proof of Proposition 9*

Remind the notation $\Upsilon = L^* \otimes V \otimes \Omega_x$ and $\Pi = U^* \otimes M$ from Section 6.1. Consider the decomposition $M \otimes V = (L \otimes V) \oplus (L^* \otimes V \otimes \Omega_x)$. In ([18], Definition 5) we associated to this decomposition a functor

$$\mathcal{F}_{L \otimes V(F_x)} : \mathrm{D}(\Upsilon) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M \otimes V(F_x)))$$

exact for the perverse t-structures. Let also

$$\mathcal{F}_{M \otimes U(F_x)} : \mathrm{D}(\Pi(F_x)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M \otimes V(F_x)))$$

denote the corresponding functor for the decompositioon $M = (M \otimes U) \oplus (M \otimes U^*)$. By construction, the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathrm{D}(\Upsilon) & \stackrel{\mathcal{F}_{L \otimes V(F_x)}}{\to} & \mathrm{D}(\widetilde{\mathcal{L}}_d(M \otimes V(F_x))) \\ \downarrow \zeta & \nearrow \mathcal{F}_{M \otimes U(F_x)} & \\ \mathrm{D}(\Pi(F_x)), & & \end{array}$$

where $\zeta$ is the partial Fourier transform (43).

Let $(G\times H)(F_x)$ act on $\widetilde{\mathcal{L}}_d(M\otimes V(F_x))$ via the canonical section

$$(G\times H)(F_x)\to \widetilde{\mathbb{S}\mathrm{p}}(M\otimes V)(F_x)$$

Then $\mathcal{F}_{L\otimes V(F_x)}$ commutes with the action of $H(F_x)$, and $\mathcal{F}_{M\otimes U(F_x)}$ commutes with the action of $G(F_x)$ (cf. [18], Section 6.6). So, if $(\mathcal{F}_1,\mathcal{F}_2,\beta)$ is an object of $\mathcal{W}\mathrm{eil}_{G,H}$ as in Definition 4 then $\mathcal{F}_{L\otimes V(F_x)}(\mathcal{F}_1)$ is $H(\mathcal{O}_x)$-equivariant, and $\mathcal{F}_{M\otimes U(F_x)}(\mathcal{F}_2)$ is $G(\mathcal{O}_x)$-equivariant. We get a functor

$$\mathrm{LW}_d:\mathcal{W}\mathrm{eil}_{G,H}\to \mathrm{P}_{(G\times H)(\mathcal{O}_x)}(\widetilde{\mathcal{L}}_d(M\otimes V(F_x)))$$

sending $(\mathcal{F}_1,\mathcal{F}_2,\beta)$ to $\mathcal{F}_{L\otimes V(F_x)}(\mathcal{F}_1)$. By definition, we get $\mathrm{LW}_d(I_0)\widetilde{\to} S_{M\otimes V(F_x)}$. Extend $\mathrm{LW}_d$ to a functor

$$\mathrm{LW}_d:\mathrm{D}\mathcal{W}\mathrm{eil}_{G,H}\to \mathrm{D}_{(G\times H)(\mathcal{O}_x)}(\widetilde{\mathcal{L}}_d(M\otimes V(F_x)))$$

by $\mathrm{LW}_d(K[r])=\mathrm{LW}_d(K)[r]$. Then $\mathrm{LW}_d$ commutes with the actions of both $\mathrm{D}\,\mathrm{Sph}_G$ and $\mathrm{D}\,\mathrm{Sph}_H$. Finally, we set $\mathrm{LW}=\tilde{\xi}_x^*\circ \mathrm{LW}_d$. Our assertion follows.

Thus, Proposition 9 and Theorem 4 are proved. □

8.3 In this subsection we give another proof of a somewhat weaker statement than Theorem 4. Namely, fixing $x\in X$, we will establish isomorphisms (8) and (9) after restriction to $\mathrm{Bun}_{P(G)\times P(H)}$.

8.3.1 Model with $\mathrm{Sph}_G$-action

Let ${}_X\mathcal{V}^{ex}_{m,G}$ be the stack classifying $U\in\mathrm{Bun}_m, M\in\mathrm{Bun}_G$ and $v\in U^*\otimes M(F)$. For a point $v\in {}_X\mathcal{V}^{ex}_{m,G}$ let $s_{\mathcal{U}}(v):\wedge^2U(F)\to\Omega(F)$ be the composition

$$\wedge^2U(F)\stackrel{\wedge^2 v}{\to}\wedge^2M(F)\to\Omega(F)$$

Let ${}_{x,\infty}\mathcal{V}^{ex}_{m,G}\subset {}_X\mathcal{V}^{ex}_{m,G}$ be the substack given by the condition that $v:U\to M(\infty x)$ is regular over $X-x$. Denote by ${}_{x,\infty}\mathcal{V}_{m,G}\subset {}_{x,\infty}\mathcal{V}^{ex}_{m,G}$ the substack given by the condition that

$$s_{\mathcal{U}}(v):\wedge^2U\to\Omega$$

is regular over $X$. The stack ${}_{x,\infty}\mathcal{V}^{ex}_{m,G}$ is ind-algebraic. As in Section 4.6, one defines the Hecke functors

$$\mathrm{H}^{\leftarrow}_G,\mathrm{H}^{\rightarrow}_G:\mathrm{Sph}_G\times\mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{m,G})\to\mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{m,G})$$

and

$$\mathrm{H}^{\leftarrow}_G,\mathrm{H}^{\rightarrow}_G:\mathrm{Sph}_G\times\mathrm{D}({}_{x,\infty}\mathcal{V}_{m,G})\to\mathrm{D}({}_{x,\infty}\mathcal{V}_{m,G})$$

Let $\mathcal{V}_{m,G}\subset {}_{x,\infty}\mathcal{V}_{m,G}$ be the closed substack given by the condition that $v:U\to M$ is regular over $X$.

We freely use notations of Section 6. Consider the category $\mathrm{P}_{(Q_H\times G)(\mathcal{O})}(\Pi(F))$ from Definition 4. As in Section 4.6, one defines the globalization functor

$$\mathrm{glob}_x : \mathrm{P}_{(Q_H\times G)(\mathcal{O})}(\Pi(F)) \to \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{m,G})$$

Denote by $\mathrm{glob}_{x,G} : \mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{m,G})$ the functor sending $(\mathcal{F}_1, \mathcal{F}_2, \beta) \in \mathcal{W}\mathrm{eil}^{ss}_{G,H}$ to $\mathrm{glob}_x(\mathcal{F}_2)$. One knows (for example, from Theorem 7) that $\mathrm{glob}_x(\mathcal{F}_2)$ is the extension by zero from ${}_{x,\infty}\mathcal{V}_{m,G}$. In this sense we may think of $\mathrm{glob}_{x,G}$ as a functor

$$\mathrm{glob}_{x,G} : \mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\infty}\mathcal{V}_{m,G})$$

By construction, $\mathrm{glob}_{x,G}$ commutes with Hecke functors $\mathrm{H}^{\leftarrow}_G, \mathrm{H}^{\rightarrow}_G$. Remind the distinguished object $I_0 \in \mathcal{W}\mathrm{eil}^{ss}_{G,H}$. One checks that $\mathrm{glob}_{x,G}(I_0)$ is the extension by zero from $\mathcal{V}_{m,G}$ and identifies caninically with

$$(\bar{\mathbb{Q}}_\ell \mid_{\mathcal{V}_{m,G}})[\dim \mathrm{Bun}_G + \dim \mathrm{Bun}_m + a_G],$$

where $a_G$ is a function of a connected component of $\mathcal{V}_{m,G}$ sending $(U, M, v)$ to $\chi(U^* \otimes M)$.

Let $\mathcal{Y}_{P(H)}$ be the stack classifying $U \in \mathrm{Bun}_m$, $s : \wedge^2 U \to \Omega$. Remind that $\mathrm{Bun}_{P(H)}$ classifies $U \in \mathrm{Bun}_m$ and an exact sequence $0 \to \wedge^2 U \to ? \to \mathcal{O} \to 0$ on $X$. So, $\mathcal{Y}_{P(H)}$ and $\mathrm{Bun}_{P(H)}$ are dual (generalized) vector bundles over $\mathrm{Bun}_m$. Let

$$\pi_{\mathcal{U}} : {}_{x,\infty}\mathcal{V}_{m,G} \to \mathcal{Y}_{P(H)} \times \mathrm{Bun}_G$$

be the map sending $(M, U, U \stackrel{v}{\to} M(\infty x))$ to $(M, U, s_{\mathcal{U}}(v) : \wedge^2 U \to \Omega)$. To summarize, we get the following (remind that $\mathrm{Four}_\psi$ is normalized as in Section 2.1).

**Lemma 17.** *The functors in the diagram*

$$\begin{array}{ccccccc} \mathcal{W}\mathrm{eil}^{ss}_{G,H} & \stackrel{\mathrm{glob}_{x,G}}{\to} & \mathrm{D}({}_{x,\infty}\mathcal{V}_{m,G}) & \stackrel{(\pi_{\mathcal{U}})_!}{\to} & \mathrm{D}(\mathcal{Y}_{P(H)} \times \mathrm{Bun}_G) & \stackrel{\mathrm{Four}_\psi}{\to} & \mathrm{D}(\mathrm{Bun}_{P(H)\times G}) \\ & & & & & & \uparrow \nu^*_{P(H),G} \\ & & & & & & \mathrm{D}(\mathrm{Bun}_{H\times G}) \end{array}$$

*are equipped with an additional structure, namely they commute with Hecke functors $\mathrm{H}^{\leftarrow}_G, \mathrm{H}^{\rightarrow}_G$ at $x$ (this means that the commutation isomorphisms are provided). Here $\nu_{P(H),G} : \mathrm{Bun}_{P(H)\times G} \to \mathrm{Bun}_{H\times G}$ is the projection.*

The following is proved in ([16], Proposition 1).

**Proposition 10.** *There exists an isomorphism*

$$\mathrm{Four}_\psi (\pi_{\mathcal{U}})_! \, \mathrm{glob}_{x,G}(I_0) \widetilde{\to} \nu^*_{P(H),G} \, \mathrm{Aut}_{G,H}[\dim.\mathrm{rel}(\nu_{P(H),G})] \quad (60)$$

*Once $\sqrt{-1} \in k$ is chosen, this isomorphism is well-defined up to a sign.*

*Remark* 10. For $d \geq 0$ write ${}_d\mathcal{W}\mathrm{eil}^{ss}_{G,H} \subset \mathcal{W}\mathrm{eil}^{ss}_{G,H}$ for the full subcategory given by the condition

$$\mathcal{F}_1 \in \mathrm{P}_{(Q_G \times H)(\mathcal{O})}({}_{d,d}\Upsilon) \quad \text{or, equivalently} \quad \mathcal{F}_2 \in \mathrm{P}_{(Q_H \times G)(\mathcal{O})}({}_{d,d}\Pi)$$

Let ${}_{x,\leq d}\mathcal{V}_{m,G} \subset {}_{x,\infty}\mathcal{V}_{m,G}$ be the substack given by the condition that $v : U \to M(dx)$ is regular over $x$. Then for $K \in {}_d\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ the complex $\mathrm{glob}_{x,G}(K)$ is the extension by zero from ${}_{x,\leq d}\mathcal{V}_{m,G}$, in this sense $\mathrm{glob}_{x,G}$ restricts to a functor

$$\mathrm{glob}_{x,G} : {}_d\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\leq d}\mathcal{V}_{m,G}) \tag{61}$$

8.3.2 Model with $\mathrm{Sph}_H$-action

Let ${}_X\mathcal{V}^{ex}_{n,H}$ be the stack classifying $L \in \mathrm{Bun}_n, V \in \mathrm{Bun}_H$ and $v \in L^* \otimes \Omega \otimes V(F)$. For a point $v \in {}_X\mathcal{V}^{ex}_{n,H}$ let $s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L(F) \to \Omega^2(F)$ denote the composition

$$\mathrm{Sym}^2 L(F) \stackrel{\mathrm{Sym}^2 v}{\to} \mathrm{Sym}^2(\Omega \otimes V)(F) \to \Omega^2(F)$$

Let ${}_{x,\infty}\mathcal{V}^{ex}_{n,H} \subset {}_X\mathcal{V}^{ex}_{n,H}$ be the substack given by the condition that $v : L \to \Omega \otimes V(\infty x)$ is regular over $X - x$. Denote by ${}_{x,\infty}\mathcal{V}_{n,H} \subset {}_{x,\infty}\mathcal{V}^{ex}_{n,H}$ the substack given by requiring that

$$s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2$$

is regular over $X$. The stack ${}_{x,\infty}\mathcal{V}^{ex}_{n,H}$ is ind-algebraic. As above, one defines the Hecke functors

$$\mathrm{H}^{\leftarrow}_H, \mathrm{H}^{\rightarrow}_H : \mathrm{Sph}_H \times \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{n,H}) \to \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{n,H})$$

and

$$\mathrm{H}^{\leftarrow}_H, \mathrm{H}^{\rightarrow}_H : \mathrm{Sph}_H \times \mathrm{D}({}_{x,\infty}\mathcal{V}_{n,H}) \to \mathrm{D}({}_{x,\infty}\mathcal{V}_{n,H})$$

Let $\mathcal{V}_{n,H} \subset {}_{x,\infty}\mathcal{V}_{n,H}$ be the substack given by the condition that $v : L \to V \otimes \Omega$ is regular over $X$.

Consider the category $\mathrm{P}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F))$ from Definition 4. As in Section 4.6, one defines the globalization functor

$$\mathrm{glob}_x : \mathrm{P}_{(Q_G \times H)(\mathcal{O})}(\Upsilon(F)) \to \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{n,H})$$

Denote by $\mathrm{glob}_{x,H} : \mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\infty}\mathcal{V}^{ex}_{n,H})$ the functor sending $(\mathcal{F}_1, \mathcal{F}_2, \beta) \in \mathcal{W}\mathrm{eil}^{ss}_{G,H}$ to $\mathrm{glob}_x(\mathcal{F}_1)$. As in the previous section, $\mathrm{glob}_x(\mathcal{F}_1)$ is the extension by zero from ${}_{x,\infty}\mathcal{V}_{n,H}$, and we may think of $\mathrm{glob}_{x,H}$ as a functor

$$\mathrm{glob}_{x,H} : \mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\infty}\mathcal{V}_{n,H})$$

For $d \geq 0$ let ${}_{x,\leq d}\mathcal{V}_{n,H} \subset {}_{x,\infty}\mathcal{V}_{n,H}$ be the substack given by the condition that $v : L \to \Omega \otimes V(dx)$ is regular over $X$. Then for $K \in {}_d\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ the complex $\mathrm{glob}_{x,H}(K)$ is the extension by zero from ${}_{x,\leq d}\mathcal{V}_{n,H}$, in this sense $\mathrm{glob}_{x,H}$ restricts to a functor

$$\mathrm{glob}_{x,H} : {}_d\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_{x,\leq d}\mathcal{V}_{n,H}) \tag{62}$$

One checks that $\mathrm{glob}_{x,H}(I_0)$ is the extension by zero from $\mathcal{V}_{n,H}$ and identifies canonically with

$$(\bar{\mathbb{Q}}_\ell\mid_{\mathcal{V}_{n,H}})[\dim \mathrm{Bun}_H + \dim \mathrm{Bun}_n + a_H],$$

where $a_H$ is a function of a connected component of $\mathcal{V}_{n,H}$ sending $(L, V, v)$ to $\chi(L^* \otimes \Omega \otimes V)$.

Remind that $\mathrm{Bun}_{P(G)}$ is the stack classifying $L \in \mathrm{Bun}_n$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$ on $X$. Let $\mathcal{Y}_{P(G)}$ be the stack classifying $L \in \mathrm{Bun}_n$ and $s : \mathrm{Sym}^2 L \to \Omega^2$. So, $\mathcal{Y}_{P(G)}$ and $\mathrm{Bun}_{P(G)}$ are (generalized) dual vector bundles over $\mathrm{Bun}_n$. Let

$$\pi_{\mathcal{L}} : {}_{x,\infty}\mathcal{V}_{n,H} \to \mathcal{Y}_{P(G)} \times \mathrm{Bun}_H$$

be the map sending $(V, L, L \stackrel{v}{\to} V \otimes \Omega(\infty x))$ to $(V, L, s_{\mathcal{L}}(v) : \mathrm{Sym}^2 L \to \Omega^2)$. As in the previous section, this can be summarized as follows.

**Lemma 18.** *The functors in the diagram*

$$\begin{array}{ccccccc} \mathcal{W}\mathrm{eil}^{ss}_{G,H} & \stackrel{\mathrm{glob}_{x,H}}{\to} & \mathrm{D}({}_{x,\infty}\mathcal{V}_{n,H}) & \stackrel{(\pi_{\mathcal{L}})_!}{\to} & \mathrm{D}(\mathcal{Y}_{P(G)} \times \mathrm{Bun}_H) & \stackrel{\mathrm{Four}_\psi}{\to} & \mathrm{D}(\mathrm{Bun}_{P(G)\times H}) \\ & & & & & & \uparrow \nu^*_{P(G),H} \\ & & & & & & \mathrm{D}(\mathrm{Bun}_{G\times H}) \end{array}$$

*are equipped with an additional structure, namely they commute with Hecke functors* $\mathrm{H}^{\leftarrow}_H, \mathrm{H}^{\rightarrow}_H$ *at* $x$. *Here* $\nu_{P(G),H} : \mathrm{Bun}_{P(G)\times H} \to \mathrm{Bun}_{G\times H}$ *is the projection.*

As in the previous section, ([16], Proposition 1) implies the following.

**Proposition 11.** *There exists an isomorphism*

$$\mathrm{Four}_\psi(\pi_{\mathcal{L}})_! \,\mathrm{glob}_{x,H}(I_0) \widetilde{\to} \nu^*_{P(G),H} \mathrm{Aut}_{G,H}[\dim.\mathrm{rel}(\nu_{P(G),H})] \tag{63}$$

*Once* $\sqrt{-1} \in k$ *is chosen, this isomorphism is well-defined up to a sign.*

### 8.3.3 Relating the two models

Our immediate purpose is the following result. Write $\mathrm{Bun}_{P(G)\times P(H)} \stackrel{\nu_{P(H)}}{\to} \mathrm{Bun}_{P(G)\times H}$ and $\mathrm{Bun}_{P(G)\times P(H)} \stackrel{\nu_{P(G)}}{\to} \mathrm{Bun}_{G\times P(H)}$ for the projections.

**Proposition 12.** *The two functors* $\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}(\mathrm{Bun}_{P(G)\times P(H)})$ *given by*

$$\mathrm{Funct}_H(K) := \nu^*_{P(H)} \mathrm{Four}_\psi(\pi_{\mathcal{L}})_! \,\mathrm{glob}_{x,H}(K)[\dim.\mathrm{rel}(\nu_{P(H)})]$$

*and*

$$\mathrm{Funct}_G(K) := \nu^*_{P(G)} \mathrm{Four}_\psi(\pi_{\mathcal{U}})_! \,\mathrm{glob}_{x,G}(K)[\dim.\mathrm{rel}(\nu_{P(G)})]$$

*are canonically isomorphic. That is, the following diagram is canonically 2-commutative*

$$
\begin{array}{lllll}
\mathrm{D}({}_{x,\infty}\mathcal{V}_{n,H}) & \stackrel{(\pi_{\mathcal{L}})_!}{\to} \mathrm{D}(\mathcal{Y}_{P(G)} \times \mathrm{Bun}_H) & \stackrel{\mathrm{Four}_\psi}{\to} & \mathrm{D}(\mathrm{Bun}_{P(G)\times H}) & \\
\uparrow \mathrm{glob}_{x,H} & & & & \searrow \nu^*_{P(H)}[\dim.\mathrm{rel}(\nu_{P(H)})] \\
\mathcal{W}\mathrm{eil}^{ss}_{G,H} & & & & \qquad \mathrm{D}(\mathrm{Bun}_{P(G)\times P(H)}) \\
\downarrow \mathrm{glob}_{x,G} & & & & \nearrow \nu^*_{P(G)}[\dim.\mathrm{rel}(\nu_{P(G)})] \\
\mathrm{D}({}_{x,\infty}\mathcal{V}_{m,G}) & \stackrel{(\pi_{\mathcal{U}})_!}{\to} \mathrm{D}(\mathcal{Y}_{P(H)} \times \mathrm{Bun}_G) & \stackrel{\mathrm{Four}_\psi}{\to} & \mathrm{D}(\mathrm{Bun}_{P(H)\times G}) &
\end{array}
$$

Our proof of Proposition 12 is inspired, on one hand, by the explicit formulas of Section 3.2 at the level of functions and, on the other hand, by the argument used by Drinfeld, Gaitsgory and Braverman in the proof of the functional equation for geometric Eisenstein series ([4], Lemma 7.3.6). However, instead of abstract resolutions as in *loc.cit*, we use some explicit ones.

8.3.4 Let $\mathcal{W}^x$ denote the stack over $\mathrm{Bun}_{P(G)\times P(H)}$ whose fibre over $(L \subset M) \in \mathrm{Bun}_{P(G)}, (U \subset V) \in \mathrm{Bun}_{P(H)}$ is

$$
\{\bar{v} \in \mathrm{H}^0(X - x, U^* \otimes M), \bar{r} \in L^* \otimes \Omega \otimes V(F_x) \mid \text{ the images of } \bar{r}, \bar{v} \text{ in } U^* \otimes L^* \otimes \Omega(F_x) \text{ are the same, and } s_{\mathcal{L}}(\bar{r}) : \mathrm{Sym}^2 L(\mathcal{O}_x) \to \Omega^2(F_x) \text{ factors through } \Omega^2(\mathcal{O}_x)\}
$$

Given a point $(L \subset M, U \subset V) \in \mathrm{Bun}_{P(G)\times P(H)}$ consider the set

$$
S_{\mathrm{reg}} := \{r \in M \otimes V(F_x) \mid s_{\mathcal{L}}(\bar{r}) : \mathrm{Sym}^2 L(\mathcal{O}_x) \to \Omega^2(\mathcal{O}_x) \text{ is regular}\}, \tag{64}
$$

where $\bar{r}$ is the image of $r$ in $L^* \otimes \Omega \otimes V(F_x)$.

**Lemma 19.** *There is a natural map $\epsilon : S_{\mathrm{reg}} \to \Omega(F_x)/\Omega(\mathcal{O}_x)$ with the following properties.*
*i) The group $L \otimes V(F_x)$ acts on $S_{\mathrm{reg}}$ by translations, and*

$$
\epsilon(r + r_0) = \epsilon(r) + \frac{1}{2}\langle \bar{r}, r_0\rangle
$$

*for $r_0 \in L \otimes V(F_x), r \in S_{\mathrm{reg}}$.*
*ii) Let $t_x \in \mathcal{O}_x$ be a uniformizor. For $r \in S_{reg} \cap (M \otimes V)(t_x^{-d}\mathcal{O}_x)$ and $s \in M \otimes V(t_x^d\mathcal{O}_x)$ one has $r + s \in S_{\mathrm{reg}}$ and $\epsilon(r + s) = \epsilon(r)$.*

*Proof* Pick any splitting over $\mathrm{Spec}\, \mathcal{O}_x$ of the exact sequence

$$
0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0 \tag{65}
$$

defining the $P$-torsor $(L \subset M) \in \mathrm{Bun}_{P(G)}$ on $X$. Then we may write $r = r_1 + \bar{r}$ with $r_1 \in L \otimes V(F_x)$, $\bar{r} \in L^* \otimes \Omega \otimes V(F_x)$. Set

$$
\epsilon(r) = \frac{1}{2}\langle \bar{r}, r_1\rangle
$$

The group $\Omega^{-1}\otimes \mathrm{Sym}^2 L(\mathcal{O}_x)$ acts transitively on the set of splittings of (65) over $\mathrm{Spec}\,\mathcal{O}_x$. Acting by $b\in \Omega^{-1}\otimes \mathrm{Sym}^2 L(\mathcal{O}_x)$ on our splitting, the pair $(r_1,\bar r)$ gets replaced by the pair $(r_1+b\bar r,,\bar r)$, so that $\frac{1}{2}\langle \bar r, r_1\rangle$ gets replaced by

$$\frac{1}{2}\langle \bar r, r_1+b\bar r\rangle=\frac{1}{2}\langle \bar r, r_1\rangle$$

Indeed, $\langle \bar r, b\bar r\rangle=\langle b, s_{\mathcal{L}}(\bar r)\rangle\in\Omega(\mathcal{O}_x)$. The properies $i), ii)$ are immediate. □

Let us construct a morphism of stacks

$$ev:\mathcal{W}^x\to\mathbb{A}^1,$$

here $ev$ stands for 'evaluation'. Given a point $(\bar r,\bar v)\in\mathcal{W}^x$ pick any $r\in M\otimes V(F_x)$ over $\bar r$, and any $v\in \mathrm{H}^0(X-x, V\otimes M)$ over $\bar v$. Since $v-r\in L\otimes V(F_x)+U\otimes M(F_x)$, we may pick $l\in V\otimes L(F_x)$ and $u\in U\otimes M(F_x)$ with $v=r+l+u$. Set

$$ev(\bar r,\bar v)=\delta(r,l,u)+\epsilon(r)\in\Omega(F_x)/\Omega(\mathcal{O}_x)+\mathrm{H}^0(X-x,\Omega), \tag{66}$$

where

$$\delta(r,l,u):=\frac{1}{2}\langle r,l\rangle+\frac{1}{2}\langle r,u\rangle+\frac{1}{2}\langle l,u\rangle\in\Omega(F_x)/\Omega(\mathcal{O}_x)+\mathrm{H}^0(X-x,\Omega)$$

Here $\langle\cdot,\cdot\rangle:\wedge^2(V\otimes M)\to\Omega$ denotes the symplectic form on $V\otimes M$ constructed as the tensor product of the symmetric form on $V$ and the symplectic form on $M$. We have used the following.

*Remark* 11. For any vector bundle $\mathcal{E}$ on $X$ we have canonically

$$\mathrm{H}^1(X,\mathcal{E})\,\tilde{\to}\,\mathcal{E}(F_x)/\mathcal{E}(\mathcal{O}_x)+\mathrm{H}^0(X-x;\mathcal{E}) \tag{67}$$

**Lemma 20.** *The value (66) depends only on the point $(\bar r,\bar v)\in\mathcal{W}^x$.*

*Proof* Let $(r,l,u)$ be chosen as above for $(\bar r,\bar v)\in\mathcal{W}^x$. Clearly, $\delta(r,l,u)=\delta(r,l+c,u-c)$ for any $c\in L\otimes U(F_x)$. For $r_1\in L\otimes V(F_x)$ we get

$$\delta(r+r_1,l-r_1,u)=\delta(r,l,u)-\frac{1}{2}\langle r,r_1\rangle$$

and $\epsilon(r+r_1)=\epsilon(r)+\frac{1}{2}\langle r,r_1\rangle$. So, $\delta(r,l,u)+\epsilon(r)$ does not change when $(r,l,u)$ is replaced by $(r+r_1,l-r_1,u)$ with $r_1\in L\otimes V(F_x)$.

Finally, for $u_1\in\mathrm{H}^0(X-x,U\otimes M)$ we have $\delta(r,l,u+u_1)=\delta(r,l,u)$. We are done. □

*Remark* 12. Another way to think about $ev:\mathcal{W}^x\to\mathbb{A}^1$ is as follows. Given a point of $\mathcal{W}^x$, pick a splitting over $X-x$ of the exact sequence

$$0\to\wedge^2 U\to ?\to\mathcal{O}\to 0 \tag{68}$$

defining the underlying $P(H)$-torsor $(U\subset V)$. Pick also a splitting of (65) over $\mathrm{Spec}\,\mathcal{O}_x$. Then $\bar v\in\mathrm{H}^0(X-x,U^*\otimes M)\subset\mathrm{H}^0(X-x,V\otimes M)$ and $\bar r\in L^*\otimes\Omega\otimes V(F_x)\subset M\otimes V(F_x)$. Note that over $\operatorname{Spec} F_x$ we get the splittings of both (68) and (65). Then we may pick $l \in U^* \otimes L(F_x)$, $u \in L^* \otimes \Omega \otimes U(F_x)$ such that $v = r + l + u$. Then

$$ev(\bar{r}, \bar{v}) = \operatorname{Res}_x \langle l, u \rangle$$

Our splittings yield a projection $L^* \otimes \Omega \otimes V(F_x) \to L^* \otimes \Omega \otimes U(F_x)$, the image of $\bar{r}$ in under this map equals $-u$. Our splittings yield also a projection $\mathrm{H}^0(X - x, U^* \otimes M) \to U^* \otimes L(F_x)$, and the image of $\bar{v}$ under this map equals $l$.

8.3.5 Informal explanation Let $\hat{\mathcal{W}}_H$ be the stack over $\operatorname{Bun}_{P(H) \times P(G)}$ whose fibre over

$$(L \subset M) \in \operatorname{Bun}_{P(G)}, (U \subset V) \in \operatorname{Bun}_{P(H)} \tag{69}$$

is

$$\{\bar{r} \in L^* \otimes \Omega \otimes V(F_x) \mid \ s_{\mathcal{L}}(\bar{r}) : \operatorname{Sym}^2 L(\mathcal{O}_x) \to \Omega^2(F_x) \text{ factors through } \Omega^2(\mathcal{O}_x)\}$$

Let $i_H : \mathcal{W}_H \hookrightarrow \hat{\mathcal{W}}_H$ be the substack given by the condition that

$$\bar{r} \in \mathrm{H}^0(X - x, L^* \otimes \Omega \otimes V) \quad \text{and} \quad s_{\mathcal{L}}(\bar{r}) : \operatorname{Sym}^2 L \to \Omega^2 \text{ is regular over } X$$

Let $ev_H : \mathcal{W}_H \to \mathbb{A}^1$ be the map sending a point of $\mathcal{W}_H$ to the pairing of $s_{\mathcal{L}}(\bar{r})$ with the exact sequence $0 \to \operatorname{Sym}^2 L \to ? \to \Omega \to 0$ corresponding to the underlying $P(G)$-torsor on $X$.

Let $\mathcal{W}_G^{ex}$ be the stack over $\operatorname{Bun}_{P(H) \times P(G)}$ whose fibre over (69) is $\mathrm{H}^0(X - x, U^* \otimes M)$. Let

$$i_G : \mathcal{W}_G \hookrightarrow \mathcal{W}_G^{ex}$$

be the substack given by the condition that for $\bar{v} \in \mathrm{H}^0(X - x, U^* \otimes M)$ the map $s_{\mathcal{U}}(\bar{v}) : \wedge^2 U \to \Omega$ is regular over $X$.

Let $ev_G : \mathcal{W}_G \to \mathbb{A}^1$ be the map sending a point of $\mathcal{W}_G$ to the pairing of $s_{\mathcal{U}}(\bar{v})$ with the exact sequence $0 \to \wedge^2 U \to ? \to \mathcal{O} \to 0$ corresponding to the underlying $P(H)$-torsor on $X$.

We get a diagram, where $p$ and $q$ are the natural projections

$$\begin{array}{ccccccc} & & & \mathcal{W}^x & \stackrel{p_{\mathcal{W},G}}{\to} & \mathcal{W}_G^{ex} & \stackrel{i_G}{\hookleftarrow} \mathcal{W}_G \stackrel{p_G}{\to} {}_{x,\infty}\mathcal{V}_{m,G} \\ & & & \downarrow p_{\mathcal{W},H} & & \downarrow q_{\mathcal{W},G} & \\ {}_{x,\infty}\mathcal{V}_{n,H} & \stackrel{p_H}{\leftarrow} & \mathcal{W}_H \stackrel{i_H}{\hookrightarrow} & \hat{\mathcal{W}}_H & \stackrel{q_{\mathcal{W},H}}{\to} & \operatorname{Bun}_{P(H) \times P(G)} & \end{array} \tag{70}$$

Informally speaking, we will have a globalization functor $\operatorname{glob}_{\hat{\mathcal{W}},H} : \mathcal{W}\mathrm{eil}_{G,H}^{ss} \to \mathrm{D}(\hat{\mathcal{W}}_H)$. Then the idea is to consider for $K \in \mathcal{W}\mathrm{eil}_{G,H}^{ss}$ the complex

$$\mathcal{P}(K) := p_{\mathcal{W},H}^* \operatorname{glob}_{\hat{\mathcal{W}},H}(K) \otimes ev^* \mathcal{L}_\psi \in \mathrm{D}(\mathcal{W}^x)$$

and to calculate its direct image with compact support on $\operatorname{Bun}_{P(H) \times P(G)}$ by two different ways suggested by the diagram (70). Informally, we will get the isomorphisms (up to a shift)

$$(p_{\mathcal{W},H})_! \mathcal{P}(K) \widetilde{\to} (i_H)_! (p_H^* \operatorname{glob}_{x,H}(K) \otimes ev_H^* \mathcal{L}_\psi)$$

$$(p_{\mathcal{W},G})_!\mathcal{P}(K) \widetilde{\to} (i_G)_!(p_G^* \operatorname{glob}_{x,G}(K) \otimes ev_G^*\mathcal{L}_\psi)$$

functorial in $K \in \mathcal{W}\mathrm{eil}^{ss}_{G,H}$. The desired isomorphism of Proposition 12 will follow.

However, the stacks in (70) are not in the class of algebraic stacks locally of finite type over $\operatorname{Spec} k$. To give an exact meaning to the above, over a given open substack $U$ of $\operatorname{Bun}_{P(G)\times P(H)}$ we will approximate (70) by a diagram of nice algebraic stacks in such a way that these approximations for $U \subset U'$ will be compatible, then we will go to the limit as $U$ becomes larger and larger.

8.3.6 For a vector bundle $\mathcal{E}$ on $X$ write ${}_{d,d}\mathcal{E} = \mathcal{E}(t_x^{-d}\mathcal{O}_x)/\mathcal{E}(t_x^d\mathcal{O}_x)$, where $t_x \in \mathcal{O}_x$ is a uniformizor.

For $d \geq 0$ denote by ${}_d\operatorname{Bun}_{P(G)\times P(H)} \subset \operatorname{Bun}_{P(G)\times P(H)}$ the open substack classifying collections (69) such that

$(*)$ $\mathrm{H}^1(X, U \otimes L(dx)) = 0$, $\mathrm{H}^1(X, U \otimes L^* \otimes \Omega(dx)) = 0$, $\mathrm{H}^1(L \otimes U^*(dx)) = 0$

Let ${}_d\mathcal{W}^x$ be the stack over ${}_d\operatorname{Bun}_{P(G)\times P(H)}$, whose fibre over (69) is

$$\{\bar r \in {}_{d,d}(L^* \otimes \Omega \otimes V)_{reg},\ \bar v \in \mathrm{H}^0(X, U^* \otimes M(dx)) \mid \text{ the images of } \bar r, \bar v \text{ in } {}_{d,d}L^* \otimes \Omega \otimes U^* \text{ are the same}\}$$

Here we have set

$${}_{d,d}(L^* \otimes \Omega \otimes V)_{reg} := \{\bar r \in {}_{d,d}L^* \otimes \Omega \otimes V \mid s_{\mathcal{L}}(\bar r) \text{ is regular}\},$$

the regularity of $s_{\mathcal{L}}(\bar r)$ means that for any lifting $r \in M \otimes V(F_x)$ of $\bar r$ one has $r \in S_{reg}$, this property does not depend on a lifting of $\bar r$.

Note that for $(\bar r, \bar v) \in {}_d\mathcal{W}^x$ the images of $\bar r, \bar v$ in ${}_{d,d}L^* \otimes \Omega \otimes U^*$ lie actually in

$$\mathrm{H}^0(X, U^* \otimes L^* \otimes \Omega(dx)) \hookrightarrow {}_{d,d}L^* \otimes \Omega \otimes U^*$$

Define the map $ev : {}_d\mathcal{W}^x \to \mathbb{A}^1$ as above. Namely, given $(\bar r, \bar v) \in {}_d\mathcal{W}^x$ pick any $r \in M \otimes V(t_x^{-d}\mathcal{O}_x)$ over $\bar r$, and any $v \in \mathrm{H}^0(X, V \otimes M(dx))$ over $\bar v$. Since

$$v - r \in L \otimes V(t_x^{-d}\mathcal{O}_x) + U \otimes M(t_x^{-d}\mathcal{O}_x),$$

we may pick $l \in V \otimes L(t_x^{-d}\mathcal{O}_x)$ and $u \in U \otimes M(t_x^{-d}\mathcal{O}_x)$ with $v = r + l + u$. Set

$$ev(\bar r, \bar v) = \delta(r, l, u) + \epsilon(r) \in \Omega(F_x)/\Omega(\mathcal{O}_x) + \mathrm{H}^0(X - x, \Omega)$$

One checks that $ev : {}_d\mathcal{W}^x \to \mathbb{A}^1$ is a well-defined morphism of stacks.

Let ${}_d\hat{\mathcal{W}}_H$ be the stack over ${}_d\operatorname{Bun}_{P(H)\times P(G)}$ whose fibre over (69) is ${}_{d,d}(L^* \otimes \Omega \otimes V)_{reg}$. The condition $(*)$ implies that $\mathrm{H}^0(X, L^* \otimes \Omega \otimes V(dx)) \hookrightarrow {}_{d,d}L^* \otimes \Omega \otimes V$ is injective. So, we define the closed substack

$$i_H : {}_d\mathcal{W}_H \hookrightarrow {}_d\hat{\mathcal{W}}_H$$

by the condition that $\bar r \in \mathrm{H}^0(X, L^* \otimes \Omega \otimes V(dx))$ and $s_{\mathcal{L}}(\bar r) : \operatorname{Sym}^2 L \to \Omega^2$ is regular over $X$.

Denote by ${}_d\mathcal{W}^{ex}_G$ the stack over ${}_d\operatorname{Bun}_{P(H)\times P(G)}$ whose fibre over (69) is $\mathrm{H}^0(X, U^*\otimes M(dx))$. Let

$$i_G : {}_d\mathcal{W}_G \hookrightarrow {}_d\mathcal{W}^{ex}_G$$

be the closed substack given by the condition that $s_{\mathcal{U}}(\bar v) : \wedge^2 U \to \Omega$ is regular over $X$ for $\bar v \in \mathrm{H}^0(X, U^*\otimes M(dx))$.

We approximate (70) by the following diagram, where $p$ and $q$ are the projections

$$\begin{array}{ccccccc} & & {}_d\mathcal{W}^x & \stackrel{p_{\mathcal{W},G}}{\to} & {}_d\mathcal{W}^{ex}_G & \stackrel{i_G}{\leftarrow} {}_d\mathcal{W}_G \stackrel{p_G}{\to} & {}_{x,\le d}\mathcal{V}_{m,G} \\ & & \downarrow p_{\mathcal{W},H} & & \downarrow q_{\mathcal{W},G} & & \\ {}_{x,\le d}\mathcal{V}_{n,H} \stackrel{p_H}{\leftarrow} {}_d\mathcal{W}_H & \stackrel{i_H}{\hookrightarrow} & {}_d\hat{\mathcal{W}}_H & \stackrel{q_{\mathcal{W},H}}{\to} & {}_d\operatorname{Bun}_{P(H)\times P(G)} & & \end{array} \tag{71}$$

Define the maps $ev_H : {}_d\mathcal{W}_H \to \mathbb{A}^1$ and $ev_G : {}_d\mathcal{W}_G \to \mathbb{A}^1$ as in Section 8.3.5.

Assume given an object $K = (\mathcal{F}_1, \mathcal{F}_2, \beta) \in {}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H}$. Remind that

$$\mathcal{F}_1 \in \mathrm{P}_{(Q_G\times H)(\mathcal{O})}({}_{d,d}\Upsilon), \;\; \mathcal{F}_2 \in \mathrm{P}_{(Q_H\times G)(\mathcal{O})}({}_{d,d}\Pi)$$

in the notation of Definition 4. As in Section 4.6, restriction to $\operatorname{Spec}(\mathcal{O}_x/t_x^{2d})$ yields the morphisms of stacks

$${}_d\mathcal{W}_G \to (Q_H\times G)(\mathcal{O}_x/t_x^{2d})\backslash_{d,d}\Pi \qquad \text{and} \qquad {}_d\hat{\mathcal{W}}_H \to (Q_G\times H)(\mathcal{O}_x/t_x^{2d})\backslash_{d,d}\Upsilon$$

and the corresponding globalization functors denoted

$$\mathrm{glob}_{d,G} : {}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_d\mathcal{W}_G) \quad \text{and} \quad \mathrm{glob}_{d,\hat{\mathcal{W}},H} : {}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_d\hat{\mathcal{W}}_H)$$

We also define

$$\mathrm{glob}_{d,H} : {}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H} \to \mathrm{D}({}_d\mathcal{W}_H)$$

by $\mathrm{glob}_{d,H} = i_H^*\,\mathrm{glob}_{d,\hat{\mathcal{W}},H}$. The above functors are normalized by the condition: there is a canonical isomorphism of functors from ${}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ to $\mathrm{D}({}_d\mathcal{W}_H)$

$$\mathrm{glob}_{d,H} \widetilde{\to} p_H^*\,\mathrm{glob}_{x,H}[\dim.\mathrm{rel}],$$

where $\dim.\mathrm{rel} = \dim \operatorname{Bun}_{P(H)\times P(G)} - \dim \operatorname{Bun}_n - \dim \operatorname{Bun}_H$ is a function of a connected component of ${}_d\mathcal{W}_H$. There is a canonical isomorphism of functors from ${}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ to $\mathrm{D}({}_d\mathcal{W}_G)$

$$\mathrm{glob}_{d,G} \widetilde{\to} p_G^*\,\mathrm{glob}_{x,G}[\dim.\mathrm{rel}],$$

where $\dim.\mathrm{rel} = \dim \operatorname{Bun}_{P(H)\times P(G)} - \dim \operatorname{Bun}_m - \dim \operatorname{Bun}_G$ is a function of a connected component of ${}_d\mathcal{W}_G$.

Given an object $K \in {}_d\,\mathcal{W}\mathrm{eil}^{ss}_{G,H}$ set

$${}_d\mathcal{P}(K) = p_{\mathcal{W},H}^*\,\mathrm{glob}_{d,\hat{\mathcal{W}},H}(K) \otimes ev^*\mathcal{L}_\psi \in \mathrm{D}({}_d\mathcal{W}^x)$$

**Proposition 13.** *For the diagram (71) one has canonically*

$$(p_{\mathcal{W},H})_! ev^*\mathcal{L}_\psi \widetilde{\to} (i_H)_! ev_H^*\mathcal{L}_\psi[-2dnm - 2a_{\mathcal{W}}] \tag{72}$$

*and*

$$(p_{\mathcal{W},G})_!({}_d\mathcal{P}(K)) \widetilde{\to} (i_G)_!(\mathrm{glob}_{d,G}(K) \otimes ev_G^*\mathcal{L}_\psi)[-2dnm - 2a_{\mathcal{W}}] \tag{73}$$

*Here $a_{\mathcal{W}}$ is a function of a connected component of* $\mathrm{Bun}_{P(G)\times \mathrm{P}(H)}$ *given by* $\chi(U^* \otimes L)$.

*Proof*

**Step 1**. Consider a $k$-point $\eta$ of ${}_d\hat{\mathcal{W}}_H$ given by $(U \subset V) \in \mathrm{Bun}_{P(H)}, (L \subset M) \in \mathrm{Bun}_{P(G)}$ and $\bar{r} \in {}_{d,d}(L^* \otimes \Omega \otimes V)_{\mathrm{reg}}$. Write $Y_\eta$ for the fibre of $p_{\mathcal{W},H}$ over $\eta$. We assume that the image of $\bar{r}$ in ${}_{d,d}L^* \otimes \Omega \otimes U^*$ lies in $\mathrm{H}^0(X, L^* \otimes \Omega \otimes U^*(dx))$, because $Y_\eta$ is empty otherwise.

The condition $(*)$ implies that $Y_\eta$ is a (nonempty) homogeneous space under the action of $\mathrm{H}^0(X, U^* \otimes L(dx))$. Pick a $k$-point $(\bar{r}, \bar{v}) \in Y_\eta$ over $\bar{r}$.

Pick any $r \in M \otimes V(t_x^{-d}\mathcal{O}_x)$ over $\bar{r}$, any $v \in \mathrm{H}^0(X, V \otimes M(dx))$ over $\bar{v}$, any $l \in V \otimes L(t_x^{-d}\mathcal{O}_x)$ and $u \in U \otimes M(t_x^{-d}\mathcal{O}_x)$ such that

$$v = r + l + u$$

By $(*)$, the map $\mathrm{H}^0(X, V \otimes L(dx)) \to \mathrm{H}^0(X, U^* \otimes L(dx))$ is surjective. Pick $l_1 \in \mathrm{H}^0(X, V \otimes L(dx))$, denote by $\bar{l}_1 \in \mathrm{H}^0(X, U^* \otimes L(dx))$ the image of $l_1$. Then $(\bar{r}, \bar{v} + \bar{l}_1) \in Y_\eta$ and

$$ev(\bar{r}, \bar{v} + \bar{l}_1) = \mathrm{Res}_x(\delta(r, l + l_1, u) + \epsilon(r)) =$$
$$\mathrm{Res}_x(\langle r, l_1\rangle + \delta(r, l, u) + \epsilon(r)) = ev(\bar{r}, \bar{v}) + \mathrm{Res}_x(\langle r, l_1\rangle)$$

The linear functional $f(l_1) := \mathrm{Res}_x(\langle r, l_1\rangle)$ on $\mathrm{H}^0(X, V \otimes L(dx))$ is nothing but the image of $\bar{r}$ under the map ${}_{d,d}(L^* \otimes \Omega \otimes V) \to \mathrm{H}^1(X, V \otimes L^* \otimes \Omega(-dx))$ coming from the exact sequence

$$0 \to L^* \otimes \Omega \otimes V(-dx) \to L^* \otimes \Omega \otimes V(dx) \to {}_{d,d}L^* \otimes \Omega \otimes V \to 0$$

The fibre of $(p_{\mathcal{W},H})_! ev^*\mathcal{L}_\psi$ at $\eta$ vanishes unless $f = 0$, which is equivalent to

$$\bar{r} \in \mathrm{H}^0(X, L^* \otimes \Omega \otimes V(dx)).$$

To get the isomorphism (72) it remains to check the commutativity of the diagram

$$\begin{array}{ccc} {}_d\mathcal{W}_H \times_{{}_d\hat{\mathcal{W}}_H} {}_d\mathcal{W}^x & \xrightarrow{\mathrm{pr}} & {}_d\mathcal{W}^x \\ \downarrow \mathrm{pr} & & \downarrow ev \\ {}_d\mathcal{W}_H & \xrightarrow{ev_H} & \mathbb{A}^1 \end{array}$$

Assuming $(\bar{r}, \bar{v}) \in {}_d\mathcal{W}_H \times_{{}_d\hat{\mathcal{W}}_H} {}_d\mathcal{W}^x$ pick $r \in \mathrm{H}^0(X, M \otimes V(dx))$ over $\bar{r}$, pick $v \in \mathrm{H}^0(X, M \otimes V(dx))$ over $\bar{v}$. Consider the exact sequence

$$0 \to L \otimes U(dx) \to L \otimes V(dx) \oplus U \otimes M(dx) \to (L \otimes V + U \otimes M)(dx) \to 0$$

Since $\mathrm{H}^1(X, L\otimes U(dx))=0$, we may further pick $l\in \mathrm{H}^0(X, L\otimes V(dx))$, $u\in \mathrm{H}^0(X, U\otimes M(dx))$ such that $v=r+l+u$. It follows that $ev(\bar r,\bar v)=\mathrm{Res}_x(\epsilon(r))$.

Apply Remark 11 to the vector bundle $\Omega^{-1}\otimes \mathrm{Sym}^2 L$ on $X$. Pick splittings of the exact sequence (65) over $X-x$ and over $\mathrm{Spec}\,\mathcal{O}_x$, they yield an element $b\in \Omega^{-1}\otimes\mathrm{Sym}^2 L(F_x)$. The splitting over $X-x$ allows to write $r=\tilde r_1+\bar r$ with $\tilde r_1\in \mathrm{H}^0(X-x, L\otimes V)$. The splitting over $\mathrm{Spec}\,\mathcal{O}_x$ allows to write $r=r_1+\bar r$ with $r_1\in L\otimes V(t_x^{-d}\mathcal{O}_x)$. Our choice of the isomorphism (67) for $\mathcal{E}=\Omega^{-1}\otimes\mathrm{Sym}^2 L$ is such that $r_1=\tilde r_1+b\bar r$. We obtain

$$\mathrm{Res}_x(\epsilon(r))=\mathrm{Res}_x(\frac{1}{2}\langle \bar r,\tilde r_1+b\bar r\rangle)=\mathrm{Res}_x(\frac{1}{2}\langle\bar r, b\bar r\rangle)=ev_H(\bar r),$$

because $\langle \bar r,\tilde r_1\rangle\in \mathrm{H}^0(X-x,\Omega)$ has no residue at $x$.

**Step 2**. For $c\geq 0$ write

$$_{d,c}\mathrm{Bun}_{P(G)\times P(H)}\subset {}_d\mathrm{Bun}_{P(G)\times P(H)}$$

for the open substack given by

$$\mathrm{H}^1(X,\wedge^2 U(cx))=0 \tag{74}$$

Let $_{d,c}\mathcal{Y}$ be the stack over $_{d,c}\mathrm{Bun}_{P(H)\times P(G)}$ whose fibre over (69) is a triple: splittings of the exact sequences (65) and (68) over $\mathrm{Spec}\,\mathcal{O}_x$, a splitting $s$ of (68) over $X-x$ which extends to a regular section $s\in\mathrm{H}^0(X, U\otimes V(cx))$. The existence of such $s$ follows from (74).

Define the stacks $_{d,c}\mathcal{Y}^x$, $_{d,c}\mathcal{Y}^{ex}_G$, $_{d,c}\mathcal{W}^x$ and $_{d,c}\mathcal{W}^{ex}_G$ by the cartesian square

$$\begin{array}{ccccc}
_{d,c}\mathcal{Y}^x & \to & _{d,c}\mathcal{Y}^{ex}_G & \to & _{d,c}\mathcal{Y}\\
\downarrow & & \downarrow & & \downarrow\\
_{d,c}\mathcal{W}^x & \to & _{d,c}\mathcal{W}^{ex}_G & \to & _{d,c}\mathrm{Bun}_{P(G)\times P(H)}\\
\downarrow & & \downarrow & & \downarrow\\
_{d}\mathcal{W}^x & \stackrel{p_{\mathcal{W},G}}{\to} & _{d}\mathcal{W}^{ex}_G & \stackrel{q_{\mathcal{W},G}}{\to} & _{d}\mathrm{Bun}_{P(G)\times P(H)}
\end{array}$$

The idea is to establish the isomorphism (73) after restriction to $_{d,c}\mathcal{Y}^{ex}_G$ and then show that it descends to an isomorphism over $_{d,c}\mathcal{W}^{ex}_G$. The isomorphisms obtained in this way will be compatible for different $c$, so they will give rise to the desired isomorphism (73) over

$$_d\mathcal{W}^{ex}_G=\bigcup_{c\geq 0}\ {}_{d,c}\mathcal{W}^{ex}_G$$

Consider a point of $_{d,c}\mathcal{Y}^x$ over $(\bar r,\bar v)\in {}_{d,c}\mathcal{W}^x$. By Remark 11, the two splittings of (68) over $\mathrm{Spec}\, F_x$ yield $b\in\wedge^2 U(F_x)$. Pick any $r\in L^*\otimes\Omega\otimes V(t_x^{-d}\mathcal{O}_x)$ over $\bar r\in {}_{d,d}(L^*\otimes\Omega\otimes V)$. We will always write the elements of $M\otimes V(F_x)$ as sums in

$$L\otimes U(F_x)\oplus L\otimes U^*(F_x)\oplus L^*\otimes\Omega\otimes U(F_x)\oplus L^*\otimes\Omega\otimes U^*(F_x)$$

using the splittings of (65) and (68) over $\mathrm{Spec}\,\mathcal{O}_x$. In particular,

$$r=r_1+z\quad\text{and}\quad \bar v=\bar v_1+z$$

for unique $r_1 \in L^* \otimes \Omega \otimes U(t_x^{-d}\mathcal{O}_x)$, $z \in L^* \otimes \Omega \otimes U^*(t_x^{-d}\mathcal{O}_x)$ and $\bar{v}_1 \in L \otimes U^*(t_x^{-d}\mathcal{O}_x)$. The splitting $s$ can be viewed as a map $s : U^* \to V(cx)$. For $s \otimes \mathrm{id} : U^* \otimes M \to V \otimes M(cx)$ we get

$$(s \otimes \mathrm{id})(\bar{v}) = b\bar{v} + \bar{v} \in U \otimes M(F_x) \oplus U^* \otimes M(t_x^{-d}\mathcal{O}_x)$$

where $b$ is viewed as a map $b : U^*(F_x) \to U(F_x)$. We get $(b\bar{v} + \bar{v}) = r + l + u$ for

$$l = b\bar{v}_1 + \bar{v}_1 \quad \text{and} \quad u = bz - r_1$$

So,

$$ev(\bar{r}, \bar{v}) = \delta(r, l, u) = \langle r_1, \bar{v}_1 \rangle + A, \tag{75}$$

where we have set $A = \frac{1}{2}\langle z, b\bar{v}_1 \rangle + \frac{1}{2}\langle \bar{v}_1, bz \rangle$. Actually, $A$ can be seen as a morphism of stacks

$$A : {}_{d,c}\mathcal{Y}_G^{ex} \to \mathbb{A}^1$$

We claim that the LHS of (73) is the extension by zero from ${}_d\mathcal{W}_G$. Indeed, from (75) we learn that $(p_{\mathcal{W},G})_!({}_d\mathcal{P}(K)) \mid_{{}_{d,c}\mathcal{Y}_G^{ex}}$ identifies (up to a shift) with $\mathrm{glob}_{d,G}(K) \mid_{{}_{d,c}\mathcal{Y}_G^{ex}}$ tensored by $A^*\mathcal{L}_\psi$. From Theorem 7 we know that $\mathrm{glob}_{d,G}(K)$ is the extension by zero under $i_G : {}_d\mathcal{W}_G \to {}_d\mathcal{W}_G^{ex}$.

Let ${}_{d,c}\mathcal{Y}_G$ be the preimage of ${}_d\mathcal{W}_G$ under the projection ${}_{d,c}\mathcal{Y}_G^{ex} \to {}_{d,c}\mathcal{W}_G^{ex}$. Let us show that the diagram commutes

$$\begin{array}{ccc} {}_{d,c}\mathcal{Y}_G & \hookrightarrow & {}_{d,c}\mathcal{Y}_G^{ex} \\ \downarrow & & \downarrow A \\ {}_d\mathcal{W}_G & \stackrel{ev_G}{\to} & \mathbb{A}^1 \end{array}$$

Indeed, for a point of ${}_{d,c}\mathcal{Y}_G^{ex}$ chosen as above we get $ev_G(\bar{v}) = \mathrm{Res}_x(\frac{1}{2}\langle \bar{v}, b\bar{v} \rangle)$. Since

$$\frac{1}{2}\langle \bar{v}, b\bar{v} \rangle = \frac{1}{2}\langle \bar{v}_1 + z, b\bar{v}_1 + bz \rangle = A,$$

our assertion follows.

For the convenience of the reader we remind that $a_G = \chi(U^* \otimes M)$, $a_H = \chi(L^* \otimes \Omega \otimes V)$, and $a_{\mathcal{W}} = \chi(U^* \otimes L)$. The shift in (73) is correct due to the formula $a_G - 2a_{\mathcal{W}} = a_H$. $\square$

*Proof of Proposition 12*

Assume $K \in {}_{d'}\mathcal{W}\mathrm{eil}_{G,H}^{ss}$ for some $d' \geq 0$. Then for $d \geq d'$ consider the complex ${}_d\mathcal{P}(K)$ on ${}_d\mathcal{W}^x$ and calculate its direct image with compact support to ${}_d\mathrm{Bun}_{P(H)\times P(G)}$ using the diagram (71). By Proposition 13, we get a canonical isomorphism over ${}_d\mathrm{Bun}_{P(H)\times P(G)}$

$$(q_{\mathcal{W},H})_!(i_H)_!(\mathrm{glob}_{d,H}(K) \otimes ev_H^*\mathcal{L}_\psi) \widetilde{\to} (q_{\mathcal{W},G})_!(i_G)_!(\mathrm{glob}_{d,G}(K) \otimes ev_G^*\mathcal{L}_\psi)$$

By definition, the LHS (resp., the RHS) identifies with $\mathrm{Funct}_H(K) \mid_{{}_d\mathrm{Bun}_{P(H)\times P(G)}}$ (resp., with $\mathrm{Funct}_G(K) \mid_{{}_d\mathrm{Bun}_{P(H)\times P(G)}}$). The isomorphisms

$$\mathrm{Funct}_H(K) \mid_{{}_d\mathrm{Bun}_{P(H)\times P(G)}} \widetilde{\to} \mathrm{Funct}_G(K) \mid_{{}_d\mathrm{Bun}_{P(H)\times P(G)}}$$

thus obtained are compatible for different $d$, we are done. $\square$

**Proposition 14.** *The isomorphisms (8) and (9) of Theorem 4 hold after restriction with respect to* $\nu_{P(G)\times P(H)} : \mathrm{Bun}_{P(G)\times P(H)} \to \mathrm{Bun}_G \times \mathrm{Bun}_H$.

*Proof* Choose an isomorphism (63) from Proposition 11. The isomorphisms we are going to construct may depend on this choice. By Proposition 12, there is a uniquely defined isomorphism (60) such that the restrictions of (60) and of (63) to $\mathrm{Bun}_{P(H)\times P(G)}$ are compatible.

Remind that we have fixed $x \in X$. By Lemma 17 and Proposition 10, for $\mathcal{S} \in \mathrm{Sph}_G$ we get

$$\nu^*_{P(G)\times P(H)} \mathrm{H}^{\leftarrow}_G(\mathcal{S}, \mathrm{Aut}_{G,H})[\dim.\mathrm{rel}(\nu_{P(G)\times P(H)}] \widetilde{\to} \mathrm{Funct}_G(\mathrm{H}^{\leftarrow}_G(\mathcal{S}, I_0))$$

By Lemma 18 and Proposition 11, for $\mathcal{S} \in \mathrm{Sph}_H$ we get

$$\nu^*_{P(G)\times P(H)} \mathrm{H}^{\leftarrow}_H(\mathcal{S}, \mathrm{Aut}_{G,H})[\dim.\mathrm{rel}(\nu_{P(G)\times P(H)}] \widetilde{\to} \mathrm{Funct}_H(\mathrm{H}^{\leftarrow}_H(\mathcal{S}, I_0))$$

The desired isomorphisms over $\mathrm{Bun}_{P(G)\times P(H)}$ follow now from Theorem 7. □

8.3.7 In this subsection we establish some additional properties of $\mathrm{Aut}_{G,H}$. Write $S_i$ for the stratum of $\mathrm{Bun}_{G,H}$ given by $\dim \mathrm{H}^0(X, M \otimes V) = i$ for $M \in \mathrm{Bun}_G, V \in \mathrm{Bun}_H$.

**Proposition 15.** *i) The complex* $\mathrm{Aut}_{G,H}$ *is placed in perverse degrees* $\geq 0$.
*ii) For any* $n, m$ *the complex* $\mathrm{Aut}_{G,H}$ *has nontrivial perverse cohomologies in degrees larger than any given integer.*

*Proof* i) Let ${}^0\mathrm{Bun}_{P(G)\times H} \subset \mathrm{Bun}_{P(G)\times H}$ be the open substack given by

$$\mathrm{H}^0(X, \mathrm{Sym}^2 L) = 0 \quad \text{and} \quad \mathrm{H}^1(X, L^* \otimes \Omega \otimes V) = 0 \tag{76}$$

for $V \in \mathrm{Bun}_H$, $(L \subset M) \in \mathrm{Bun}_{P(G)}$.

Remind that $\mathcal{Y}_{P(G)}$ is the stack classifying $L \in \mathrm{Bun}_n$ with a section $\mathrm{Sym}^2 L \to \Omega^2$. Denote by ${}^0(\mathcal{Y}_{P(G)} \times \mathrm{Bun}_H) \subset \mathcal{Y}_{P(G)} \times \mathrm{Bun}_H$ the open substack given by (76). We have the Fourier transform

$$\mathrm{Four}_\psi : \mathrm{D}({}^0(\mathcal{Y}_{P(G)} \times \mathrm{Bun}_H)) \to \mathrm{D}({}^0\mathrm{Bun}_{P(G)\times H})$$

The projection ${}^0\nu : {}^0\mathrm{Bun}_{P(G)\times H} \to \mathrm{Bun}_{G\times H}$ is smooth and surjective.

Remind that $\mathcal{V}_{n,H}$ classifies $L \in \mathrm{Bun}_n$, $V \in \mathrm{Bun}_H$ and $v : L \to V \otimes \Omega$. Let ${}^0\mathcal{V}_{n,H} \subset \mathcal{V}_{n,H}$ be the open substack given by (76). We have an affine map ${}^0\pi_{\mathcal{L}} : {}^0\mathcal{V}_{n,H} \to {}^0(\mathcal{Y}_{P(G)} \times \mathrm{Bun}_H)$ sending $v$ to the composition

$$\mathrm{Sym}^2 L \stackrel{v\otimes v}{\to} \mathrm{Sym}^2(V \otimes \Omega) \to \Omega^2$$

The complex $\mathrm{glob}_{x,H}(I_0)$ introduced in Section 8.3.2 is perverse over ${}^0\mathcal{V}_{n,H}$ and coincides with $\bar{\mathbb{Q}}_\ell[\dim({}^0\mathcal{V}_{n,H})]$. Since ${}^0\pi_{\mathcal{L}}$ is affine, $({}^0\pi_{\mathcal{L}})_!$ is left exact for the perverse t-structure. Now from Proposition 11 it follows that

$${}^0\nu^* \mathrm{Aut}_{G,H}[\dim.\mathrm{rel}({}^0\nu)]$$

is placed in perverse degrees $\geq 0$.

ii) Consider two cases.

CASE $2n > m-1$. Pick a $k$-point $M$ of $\mathrm{Bun}_G$ and a rank $m$ vector bundle $E$ on $X$ of degree zero. Let $\mathcal{A}$ be a line bundle on $X$ of degree $a > 0$ large enough in the sense below. Set $U = E \otimes \mathcal{A}$ and $V = U \oplus U^*$ with the symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}_X$ as in Section 3.2. The pair $(M, V)$ can be viewed as a closed substack $Y := B(\mathrm{Aut}(M) \times \mathrm{Aut}(V))$ of $\mathrm{Bun}_{G,H}$. We will let $a$ go to infinity, below we write *const* for quantities independent of $a$.

The dimension of the group of automorphisms $\mathrm{Aut}(M)$ is constant, whence $\dim \mathrm{Aut}(V) = m(m-1)a + \mathit{const}$. Since $a$ is large enough, $Y \subset S_i$ for $i = 2nma + \mathit{const}$. Assuming that $\mathrm{Aut}_{G,H}$ is placed in perverse degrees $\le C$ for some $C > 0$, we get

$$\mathit{const} - m(m-1)a = \dim Y \le \dim S_i \le \dim \mathrm{Bun}_{G,H} - i + C = \mathit{const} - 2nma$$

Since $2n > m-1$, this is a contradiction.

CASE $2m > n+1$. Pick a $k$-point $V$ of $\mathrm{Bun}_H$ and a rank $n$ vector bundle $E$ on $X$ of degree zero. Let $\mathcal{A}$ be a line bundle on $X$ of degree $a > 0$ large enough. Set $L = E \otimes \mathcal{A}$ and $M = L \oplus L^* \otimes \Omega$ with symplectic form $\wedge^2 M \to \Omega$ as in Section 6.1. As above, we get a closed substack $Y = B(\mathrm{Aut}(M) \times \mathrm{Aut}(V))$ of $\mathrm{Bun}_{G,H}$. Write *const* for quantities independent of $a$.

In this case $\dim \mathrm{Aut}(M) = n(n+1)a + \mathit{const}$. For $a$ large enough we have $Y \subset S_i$ for $i = 2nma + \mathit{const}$. If $\mathrm{Aut}_{G,H}$ is placed in perverse degrees $\le C$ then

$$\mathit{const} - n(n+1)a = \dim Y \le \dim S_i \le \dim \mathrm{Bun}_{G,H} - i + C = \mathit{const} - 2nma$$

Since $2m > n+1$, this is a contradiction.

Any pair of integers $n, m > 0$ satisfies one of the above inequalities. We are done. $\square$

*Remark* 13. The complex $\mathrm{Aut}_{G,H}$ on $\mathrm{Bun}_{G,H}$ is not pure in general. Let us show that for $m = 1$ and any $n$ the complex $\mathrm{Aut}_{G,H}$ is not pure.

Remind that $\mathrm{Bun}_G$ is irreducible. We have $\mathrm{Bun}_H = \mathrm{Pic}\, X$. Consider the connected component $(\mathrm{Pic}^0 X) \times \mathrm{Bun}_G$, the intersection $\mathcal{U}$ of this component with $S_0$ is nonempty. Indeed, take take $V = \mathcal{O}_X^2$, the corresponding point of $\mathrm{Pic}^0 X$ is $\mathcal{O}_X$. There is $M \in \mathrm{Bun}_G$ with $\mathrm{H}^0(M) = 0$.

Over $\mathcal{U}$ the complex $\mathrm{Aut}_{G,H}$ identifies with the constant perverse sheaf $\bar{\mathbb{Q}}_\ell[\dim \mathcal{U}]$. Its intermediate extension to $(\mathrm{Pic}^0 X) \times \mathrm{Bun}_G$ is the constant perverse sheaf. If $\mathrm{Aut}_{G,H}$ was pure, this constant perverse sheaf would be its direct summand. However, $(\mathrm{Pic}^0 X) \times \mathrm{Bun}_G$ contains points of $S_i$ for some $i > 0$. The $*$-fibre of $\mathrm{Aut}_{G,H}$ at such point is the trivial one-dimensional space placed in usual degree $i - \dim \mathcal{U}$. It can not contain $\bar{\mathbb{Q}}_\ell[\dim \mathcal{U}]$ as a direct summand.

# References


[1] J. Adams, L-functoriality for dual pairs, Soc. Math. de France, Astérique 171-172 (1989), p. 85-129

[2] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Asterisque 100 (1982)

[3] T. Braden, Hyperbolic localization of intersection cohomology, Transform. Groups 8 (2003), no. 3, 209–216

[4] A. Braverman, D. Gaitsgory, Geometric Eisenstein series, Inv. Math. 150 (2002), 287-384

[5] V. Drinfeld, C. Simpson, $B$-structures on $G$-bundles and local triviality, Math. Res. Letters 2, 823-829 (1995)

[6] E. Frenkel, D. Gaitsgory, Whittaker patterns in the geometry of moduli spaces of bundles on curves, Ann. of Math. 153 (2001), 699-748

[7] E. Frenkel, D. Gaitsgory, K. Vilonen, On the geometric Langlands conjecture. J. Amer. Math. Soc. 15 (2002), no. 2, 367–417

[8] D. Gaitsgory, On a vanishing conjecture appearing in the geometric Langlands correspondence, Ann. of Math. 160, no. 2 (2004), 617-682

[9] D. Gaitsgory, The notion of a category over an algebraic stack, math.AG/0507192

[10] D. Gaitsgory, On De Jong's conjecture, math.AG/0402184

[11] D. Gaitsgory, D. Nadler, Loops and quasimaps into spherical varieties, in preparation (draft 12/3/04)

[12] R. Howe, Another look at the local theta correspondence for an unramified dual pair, Festschrift in Honor of Piatetaski-Shapiro, vol. 2, Israel Math. Conf. Proc., 1990, p. 93-124

[13] M. Kapranov, E. Vasserot, Vertex algebras and the formal loop space, math.AG/0107143

[14] S. Kudla, Notes on the local theta correspondence, Notes from 10 lectures given at the European School on Group Theory in September 1996, downloadable at http://www.math.umd.edu/∼ssk/castle.pdf

[15] S. Kudla, On the local theta-correspondence, Invent. Math. 83, 229-255 (1986)

[16] S. Lysenko, Geometric Waldspurger periods, math.AG/0510110, to appear in Compositio Math.

[17] S. Lysenko, Moduli of metaplectic bundles on curves and Theta-sheaves, Ann. Scient. Éc. Norm. Sup. 4 série, t.39 (2006), 415-466

[18] V. Lafforgue, S. Lysenko, Geometric Weil representation: local field case, arXiv:0705.4213

[19] I. Mirković, K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, math.RT/0401222 (to appear in Ann. of Math.)

[20] C. Moeglin, M.-F. Vigneras, J. L. Waldspurger, Correspondence de Howe sur un corps $p$-adique, Lecture Notes in Math. 1291 (1987)

[21] D. Prasad, On the local Howe duality correspondence, IMRN, no. 11 (1993), p. 279 - 287

[22] Rallis, Langlands functoriality and the Weil representation, Amer. J. Math., vol.104, no. 3, p. 469-515 (1982)