Series Representation and Simulation of Multifractional Lévy Motions

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Abstract

This paper introduces a method of generating Real Harmonizable Multifractional Lévy Motions, in short RHMLMs. The simulation of these fields is closely related to that of infinitely divisible laws or Lévy processes. In the case where the control measure of the RHMLM is finite, one uses generalized shot noises series. An estimation of the error is also given. Otherwise the RHMLM X_h is split into two independent RHMLMs $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$. More precisely, $X_{\varepsilon,2}$ is a RHMLM whose control measure is finite. Then it can be rewritten as a generalized shot noise series. The asymptotic behaviour of $X_{\varepsilon,1}$ as $\varepsilon \to 0_+$ is further elaborated. Sufficient conditions to approximate $X_{\varepsilon,1}$ by a Multifractional Brownian Motion are given. The error rate in term of Berry-Esseen bounds is then discussed. Finally some examples of simulation are given.

Keywords: Functional Central Limit Theorem, Generalized Shot Noise Series, Infinitely Divisible Distribution, Multifractional Brownian Motion, Simulation.

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1 Introduction

The Multifractional Brownian Motion, in short MBM, has been introduced independently in [15] and [4] as a generalization of the Fractional Brownian Motion, in short FBM, defined in [14]. The FBM and the MBM provide very powerful models in applied mathematics. Whereas the pointwise Hölder exponent of the FBM is almost surely equal to a constant, the MBM one is allowed to vary along the trajectory. However these fields are Gaussian and thus can not model non-Gaussian phenomena. It is a serious drawback. For instance, examples of non-Gaussian phenomena can be found in [13] and in [21] for image modeling.

Real Harmonizable Fractional Lévy Motions, in short RHFLMs, have been defined in [3] in order to obtain non-Gaussian fields which share many properties with the FBM. In particular, their sample paths are locally Hölder and their pointwise Hölder exponent is almost surely equal to a constant. Then in order to allow the pointwise Hölder exponent to vary along the trajectory, the class of Real Harmonizable Multifractional Lévy Motions, in short RHMLMs, has been introduced in [11]. RHMLMs makes up a class of locally asymptotically self-similar fields which includes RHFLMs and the MBM.

The main aim of this paper is to describe a method for generating the sample paths of RHMLMs and thereby to give an account of the sample paths theoretical roughness.

Let us recall that a RHMLM Z_h with multifractional function h is defined as the stochastic integral:

$$Z_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} L(d\xi),$$

where $\|\xi\|$ is the Euclidean norm of ξ and $L(d\xi)$ is a Lévy random measure. In fact,

$$Z_h = aB_h + bX_h,$$

where $(a,b) \in \mathbb{R}^2$, B_h is a Multifractional Brownian Motion and

$$X_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} \, M(d\xi),$$

with $M(d\xi)$ a Lévy random measure without Brownian component. Let us notice that B_h and X_h are two independent fields.

In this paper, only the generation of the non-Brownian part X_h is discussed. As for the FBM, many authors have already studied the simulation of its sample paths, see for example [23] or [1]. Implementations of severals methods of simulation of the FBM can be found in [8]. On the other hand, methods of generating sample paths of the MBM are given in [15] and in [7].

Many authors have already been interested in the simulation of non-Gaussian fields defined as stochastic integrals. As an example, in [19], simulation of solutions of stochastic differential equations driven by Lévy processes are studied. Furthermore, simulation of stochastic integrals with respect to Lévy processes are discussed in [22]. However in the case of RHMLMs, the stochastic integral is on whole $\mathbb{R}^d$ and not only on a compact interval of $\mathbb{R}$. In [9], a method for generating symmetric α-stable processes based on a multiresolution analysis is proposed. Nevertheless, the stochastic integral is truncated. On the other hand, X_h is an infinitely divisible field and infinitely divisible laws can be represented as generalized shot noise series. An overview of these representations is given in [18]. Moreover the simulation implementation is discussed in [6]. Generalized shot noise series allow us to represent X_h as series without truncating the stochastic integral. Nevertheless, it only works if the control measure $\nu(dz)$ of the Lévy random measure $M(d\xi)$ has finite mass. In this case, the generalized shot noise series

$$Y_h(x) = 2\sum_{n=1}^{+\infty} \Re\left\{ f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_n \right\},$$

for a suitable choice of f and of random variables (T_n, Z_n, U_n), converges almost surely. Moreover for this choice,

$$\{X_h(x) : x \in \mathbb{R}^d\} \stackrel{d}{=} \{Y_h(x) : x \in \mathbb{R}^d\},$$

where $\stackrel{d}{=}$ denotes equality in distribution. In practice, one then simulates the sample paths of Y_h which is equal in law to X_h. When ν is not a finite measure, the approximation of X_h is closely related to those of Lévy processes with infinite Lévy measure given in [2]. In this case, X_h is split into two independent RHMLMs

$$X_h = X_{\varepsilon,1} + X_{\varepsilon,2}$$

where $X_{\varepsilon,2}$ is associated with a finite control measure $\nu_{\varepsilon,2}(dz)$. Thus $X_{\varepsilon,2}$ can be approximated by a generalized shot noise series. It remains to generate $X_{\varepsilon,1}$. As it is done in [2] in the case of Lévy processes, a functional Central Limit Theorem leads to a normal approximation of $X_{\varepsilon,1}$.

In the next section, the construction and some properties of RHMLMs are recalled. Then section 3 is devoted to the case of a finite control measure. The rate of convergence of the shot noise series is studied. Sufficient conditions to establish a Central Limit Theorem are discussed in section 4. Finally some simulation examples are given in the one dimensional case.

2 Preliminaries

Let us precise the construction of a RHMLM. Let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$. Suppose its mean measure is $n(d\xi, dz) = d\xi\,\nu(dz)$ with $\nu(dz)$ a non-vanishing measure such that $\nu(\{0\}) = 0$ and

$$\forall p \geq 2, \int_{\mathbb{C}} |z|^p \,\nu(dz) < +\infty. \tag{1}$$

Let $M(d\xi)$ be a Lévy random measure in sense of [3] associated with the Poisson random measure $N(d\xi, dz)$. Let us recall that M is defined by

$$\int_{\mathbb{R}^d} g(\xi)\, M(d\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} [g(\xi)z + g(-\xi)\overline{z}]\, \widetilde{N}(d\xi, dz), \tag{2}$$

where $\widetilde{N} = N - n$ and $g \in L^2(\mathbb{R}^d)$. Moreover if

$$\forall \xi \in \mathbb{R}^d, \quad g(-\xi) = \overline{g(\xi)} \tag{3}$$

the stochastic integral $\int g\, dM$ is a real-valued random variable and its characteristic function is

$$\mathbb{E}\left(\mathrm{e}^{iu\int g\, dM}\right) = \exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} \left(\mathrm{e}^{2iu\Re(g(\xi)z)} - 1 - 2iu\Re(g(\xi)z)\right) d\xi\, \nu(dz)\right],\ u \in \mathbb{R}, \tag{4}$$

where $\Re(z)$ is the real part of the complex z.

As it is done in [3], the control measure $\nu(dz)$ is assumed to be rotationally invariant. More precisely let $P\left(\rho \mathrm{e}^{i\theta}\right) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_*^+$. Then

$$P(\nu(dz)) = d\theta\, \nu_\rho(d\rho), \tag{5}$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$. Therefore when g satisfies (3),

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} g(\xi)\, M(d\xi)\right|^2\right] = 4\pi \|g\|^2_{L^2(\mathbb{R}^d)} \int_0^{+\infty} \rho^2\, \nu_\rho(d\rho). \tag{6}$$

A RHMLM $(Z_h(x))$ associated with the Lévy random measure $M(d\xi)$ with multifractional function $h : \mathbb{R}^d \longrightarrow]0, 1[$ is the sum of two independent fields

$$Z_h = aB_h + bX_h, \tag{7}$$

where $(a, b) \in \mathbb{R}^2$, B_h is a Multifractional Brownian Motion with multifractional function h and

$$X_h(x) = \int_{\mathbb{R}^d} f(x, \xi)\, M(d\xi), \tag{8}$$

with

$$f(x, \xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}}. \tag{9}$$

Notice that X_h is a real-valued field. The function h is supposed to be locally β-Hölderian.

Methods of generating sample paths of the MBM have already been studied. In the following, only the generation of X_h is treated. The next section is devoted to the case where ν is a finite measure.

3 Generalized shot noise series

In this section, ν is supposed to be a finite measure. The random field X_h can be then represented as a generalized shot noise series.

Let us first introduce some notations that will be used throughout the paper.

Notation Let $(Z_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ be independent sequences of random variables.

- $(Z_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with common law

$$\mathcal{L}(Z_n) = \frac{\nu(dz)}{\nu(\mathbb{C})}.$$

- $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that U_1 is uniformly distributed on the unit sphere S^{d-1} of the Euclidean space $\mathbb{R}^d$. Let σ_{d-1} be the uniform measure on S^{d-1}. Then

$$\mathcal{L}(U_n) = \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

  Let us recall that

$$\sigma_{d-1}\left(S^{d-1}\right) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

  where Γ is the Gamma-function. Moreover let us introduce

$$c_d = \frac{\sigma_{d-1}\left(S^{d-1}\right)}{d}$$

  the volume of the unit ball of $\mathbb{R}^d$.

- T_n is the nth arrival time of a Poisson process with intensity 1.
- For $t \in \mathbb{R}$, $[t]$ denote the integer part of t.
- Let $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

Remark that when $d = 1$, U_n is a symmetric Bernoulli random variable.

Proposition 3.1. *Let $K \subset \mathbb{R}^d$ be a compact set. Then almost surely, the series*

$$Y_h(\cdot) = 2\sum_{n=1}^{+\infty} \Re\left\{ f\left(\cdot, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_n \right\} \tag{10}$$

converges uniformly on K and

$$\{X_h(x) : x \in K\} \stackrel{d}{=} \{Y_h(x) : x \in K\}.$$

Therefore, since Y_h is equal in law to X_h, in practice one simulates the sample paths of Y_h. These sample paths can be approximated on K owing to (10). For the sake of simplicity, in dimension $d = 1$, they are simulated on a compact interval K.

Proof. The series (10) can be rewritten as the generalized shot noise series

$$\sum_{n=1}^{+\infty} H(T_n, V_n)(x),$$

where $V_n = (U_n, Z_n)$ and

$$H(r, v)(x) = 2\Re\left\{ f\left(x, \left(\frac{r}{c_d\,\nu(\mathbb{C})}\right)^{1/d} u\right) z \right\}, \quad r > 0 \quad v = (u, z).$$

$(V_n)_{n\geq 1}$ is a sequence of i.i.d. random variables which is independent of $(T_n)_{n\geq 1}$. In order to obtain the convergence of (10), one shall verify the conditions of Theorem 2.4 in [17]. This theorem implies the convergence in E_K the space of real-valued continuous functions on K endowed with the uniform norm $\|\cdot\|_K$: for every $g \in E_K$, $\|g\|_K = \sup_{x\in K} |g(x)|$. Endowed with this norm, E_K is a separable Banach space. Moreover

$$\begin{array}{rcl} H \,:\, \mathbb{R}_*^+ \times \mathcal{D} & \longrightarrow & E_K \\ (r, v) & \longmapsto & H(r, v), \end{array}$$

where $\mathcal{D} = S^{d-1} \times \mathbb{C}$, is a Borel measurable map. Define a measure F_K on the Borel σ-ring $\mathcal{B}_{E_K}$ of E_K by

$$F_K(A) = \int_0^{+\infty}\!\!\int_{\mathcal{D}} \mathbf{1}_{A\backslash\{0\}}(H(r, v))\, \lambda(dv)\, dr,$$

where λ is the law of V_1. Since ν is a symmetric measure, so is F_K. First one proves that F_K is a Lévy measure. Let E'_K be the dual of E_K and for $y' \in E'_K$, $y \in E_K$ denote $< y', y >= y'(y)$. Let us recall, see [17], that F_K is a Lévy measure if for every $y' \in E'_K$,

$$\int_{E_K} \left(< y', y >^2 \wedge 1\right) F_K(dy) < +\infty \tag{11}$$

and if

$$\begin{array}{rcl} \Phi_K \,:\, E'_K & \longrightarrow & \mathbb{C} \\ y' & \longmapsto & \exp\left\{ \displaystyle\int_{E_K} \left(\mathrm{e}^{i<y',y>} - 1 - i < y', y > \mathbf{1}_{\|y\|_K \leq 1} \right) F_K(dy) \right\} \end{array}$$

is the characteristic function of a probability on E_K.

Notice that

$$\int_{E_K} \|y\|_K^2 \, F_K(dy) = \int_{]0,+\infty[\times\mathcal{D}} \|H(r, v)\|_K^2 \, dr\, \lambda(dv)$$

Therefore by applying the change of variable $\rho = (r/(c_d\,\nu(\mathbb{C})))^{1/d}$,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) = 4\int_{\mathbb{C}}\int_{S^{d-1}}\int_0^{+\infty} g^2(\rho u, z)\,\rho^{d-1} d\rho\,\sigma_{d-1}(du)\,\nu(dz)$$

where $g(\xi, z) = \sup_{x\in K} |\Re(f(x,\xi)z)|$. Hence using polar coordinates,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) = 4\int_{\mathbb{R}^d\times\mathbb{C}} g^2(\xi, z)\, d\xi\,\nu(dz).$$

As a result,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) \le 4\int_{\mathbb{C}} |z|^2\,\nu(dz)\int_{\mathbb{R}^d} \tilde{g}^2(\xi)\, d\xi$$

with $\tilde{g}(\xi) = \sup_{x\in K} |f(x,\xi)|$. Furthermore,

$$|\tilde{g}(\xi)| \le \frac{C_K}{\|\xi\|^{M_K-1+d/2}}\mathbf{1}_{\|\xi\|\le 1} + \frac{2}{\|\xi\|^{m_K+d/2}}\mathbf{1}_{\|\xi\|>1},$$

where $C_K = \max_{u\in K}\|u\|$, $m_K = \min_K h$ and $M_K = \max_K h$. Therefore, since $0 < m_K < 1$ and $0 < M_K < 1$, $\tilde{g} \in L^2(\mathbb{R}^d)$. Then by (1),

$$\int_{E_K} \|y\|_K^2\, F_K(dy) < +\infty, \tag{12}$$

which implies (11). Moreover since the integral in (12) is finite,

$$\int_{\|y\|_K\ge 1} < y', y > \; F_K(dy)$$

is well defined and is equal to 0 by symmetry of F_K. Hence

$$\Phi_K(y') = \exp\left\{\int_{E_K}\left(\mathrm{e}^{i<y',y>} - 1 - i < y', y >\right) F_K(dy)\right\}.$$

Let us recall, see proposition 3.4 in [11], that the sample paths of X_h are almost surely continuous. Thus, we can consider $< y', X_{h|_K} >$. Let us first extend y' to the space of continuous functions on K with values in $\mathbb{C}$. For every continuous function $g : K \longrightarrow \mathbb{C}$, let $< y', g >=< y', \Re(g) > +i < y', \Im(g) >$. In particular, by definition, $\Re(< y', g >) =< y', \Re(g) >$ and $\Im(< y', g >) =< y', \Im(g) >$. Then, since

$$< y', X_{h|_K} >= \int_{\mathbb{R}^d} < y', f(\cdot,\xi)_{|_K} > \; M(d\xi), \tag{13}$$

by (4),

$$\mathbb{E}\left(\mathrm{e}^{i<y',X_{h|_K}>}\right) = \exp\left\{\int_{\mathbb{R}^d\times\mathbb{C}}\left(\mathrm{e}^{2i\Re\left(<y' f(\cdot,\xi)_{|_K}>z\right)} - 1 - 2i\Re\left(< y', f(\cdot,\xi)_{|_K} > z\right)\right) d\xi\,\nu(dz)\right\}.$$

Since $2\Re\left(< y', f(\cdot,\xi)_{|_K} > z\right) =< y', 2\Re\left(f(\cdot,\xi)_{|_K} z\right) >$, then by definition of F_K,

$$\Phi_K(y') = \mathbb{E}\left(\mathrm{e}^{i<y',X_{h|_K}>}\right). \tag{14}$$

As a consequence, Φ_K is the characteristic function of $X_{h|_K}$. Then F_K is a Lévy measure on E_K. According to Theorem 2.4 in [17],

$$\sum_{j=1}^n H(T_j, V_j) - A(T_n),$$

where for $s \ge 0$

$$A(s) = \int_0^s\int_{\mathcal{D}} H(r,v)\mathbf{1}_{\|H(r,v)\|_K\le 1}\,\lambda(dv)\,dr,$$

is convergent in E_K. Since ν is a symmetric measure, for every $s \ge 0$, $A(s) = 0$, which gives the convergence of (10) in E_K. Then let Y_h be its limit. In view of Theorem 2.4 in [17], the characteristic function of $(Y_h(x))_{x\in K}$ is Φ_K. Therefore

$$\{X_h(x),\, x\in K\} \stackrel{d}{=} \{Y_h(x),\, x\in K\}$$

follows from (14), which concludes the proof. $\square$

Remark 3.1. *In view of* (12) *and corollary 2.5 in [17] applied with* $p = 2$*, the series* (10) *converges also in* $\mathrm{L}^2(E_K)$*, i.e.*

$$\lim_{N\to+\infty} \mathbb{E}\left\{ \left(\sup_{x\in K} |Y_{h,N}(x) - Y_h(x)| \right)^2 \right\} = 0,$$

where

$$Y_{h,N}(x) = 2\sum_{n=1}^{N} \Re\left\{ f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_n \right\}. \tag{15}$$

In simulation, Y_h is approximated by $Y_{h,N}$. Then one is interested in the rate of convergence of (10). We first consider one-dimensional distribution. The next proposition studies the error in L^q. From this, an almost sure error is deduced and stated in corollary 3.1. Then a functional result is established.

Proposition 3.2. *Let* $q > 0$ *and* $x \in \mathbb{R}^d$*. Then for every* $N \in \mathbb{N}^*$ *such that* $N > q/2 + qh(x)/d - 1$*,* $Y_h(x) - Y_{h,N}(x) \in L^q$ *and*

$$\mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q) \leq C_{q,x}\, \frac{D_{N,q}(h(x))}{N^{qh(x)/d}}, \tag{16}$$

where $C_{q,x}$ *does not depend on* N *and for* $0 < s < 1$ *and* $n > q/2 + qs/d$

$$D_{n,q}(s) = \frac{\Gamma(n+1-q/2-qs/d)\, (n+1)^{q/2+qs/d}}{\Gamma(n+1)}. \tag{17}$$

Remark 3.2. *By the Stirling formula,* $\lim_{N\to+\infty} D_{N,q}(s) = 1$*. Therefore thanks to* (16)*, the rate of convergence of* $Y_{h,N}(x)$ *in* L^q *is at least* $N^{h(x)/d}$*. Actually, we will prove that the rate of convergence in* (16) *is optimal, see proposition 3.3.*

For simulation reasons, it can be useful to explicitly provide a constant $C_{q,x}$ *for which* (16) *holds. In fact one can take*

$$C_{q,x} = 2^{1+3q/2}\, B_q^q \left(\frac{d}{h(x)} \right)^{q/2} \mathbb{E}(|\Re(Z_1)|^q)\, (c_d \nu(\mathbb{C}))^{q/2+qh(x)/d}, \tag{18}$$

where

$$B_q = \begin{cases} 1 & \text{when } 0 < q \leq 2, \\[2ex] \sqrt{2} \left(\dfrac{\Gamma((q+1)/2)}{\sqrt{\pi}} \right)^{1/q} & \text{when } q > 2. \end{cases} \tag{19}$$

Proof. Let

$$\xi_n(x) = 2\Re\left\{ f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_n \right\},\ n \in \mathbb{N}^*,$$

and

$$R_{N,P}(x) = Y_{h,P}(x) - Y_{h,N}(x) = \sum_{n=N+1}^{P} \xi_n(x),\ \ 1 \leq N < P.$$

Then $\lim_{P\to+\infty} R_{N,P}(x) = Y_h(x) - Y_{h,N}(x)$ almost surely. In fact as $P \to +\infty$, $R_{N,P}$ converges in E_K for any compact subset K of $\mathbb{R}^d$.

As $(\xi_n(x))_{n\geq 1}$ is a symmetric sequence of random variables, by proposition 2.3 pages 47-48 in [12],

$$\mathbb{E}\left(\max_{N+1\leq M\leq P} |R_{N,M}(x)|^q \right) \leq 2\mathbb{E}(|R_{N,P}(x)|^q).$$

Moreover since T_n, U_n and Z_n are independent random variables and since the law of Z_n is rotationally invariant,

$$\xi_n(x) \stackrel{d}{=} \widetilde{\xi_n}(x)$$

where

$$\widetilde{\xi_n}(x) = 2\left| f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) \right| \Re(Z_n).$$

Furthermore for given values of $(U_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$, $(\xi_n(x))_{n\geq 1}$ and $\left(\widetilde{\xi_n}(x)\right)_{n\geq 1}$ are sequences of independent random variables. Therefore

$$(\xi_n(x))_{n\geq 1} \overset{d}{=} \left(\widetilde{\xi_n}(x)\right)_{n\geq 1}.$$

As a consequence,

$$\mathbb{E}(|R_{N,P}(x)|^q) = \mathbb{E}\left\{\left(\sum_{n=N+1}^{P} \widetilde{\xi_n}(x)\right)^q\right\}.$$

Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of i.i.d. symmetric Bernoulli random variables which is independent of $(Z_n, U_n, T_n)_{n\geq 1}$. Hence by symmetry of the law of Z_n, Z_n can be replaced in $R_{N,P}$ by $\varepsilon_n Z_n$. Then by applying the Khintchine inequality with constant B_q and conditionally with respect to $(Z_n, U_n, T_n)_{n\geq 1}$,

$$\mathbb{E}(|R_{N,P}(x)|^q) \leq B_q^q\, \mathbb{E}\left\{\left(\sum_{n=N+1}^{P} \widetilde{\xi_n}^{\,2}(x)\right)^{q/2}\right\}. \tag{20}$$

Notice that the best possible constant B_q in the Khintchine inequality is known. According to [10], the best constant is

$$B_q = \begin{cases} 1 & \text{when } 0 < q \leq 2, \\ \sqrt{2}\left(\dfrac{\Gamma((q+1)/2)}{\sqrt{\pi}}\right)^{1/q} & \text{when } q > 2. \end{cases}$$

Then by the Minkowski inequality,

$$\mathbb{E}(|R_{N,P}(x)|^q) \leq B_q^q \left\{\sum_{n=N+1}^{P} \mathbb{E}\left(\left|\widetilde{\xi_n}(x)\right|^q\right)^{2/q}\right\}^{q/2}.$$

Let us now evaluate $\mathbb{E}\left(\left|\widetilde{\xi_n}(x)\right|^q\right)$. By independence and since $Z_1 \overset{d}{=} Z_n$,

$$\mathbb{E}\left(\left|\widetilde{\xi_n}(t)\right|^q\right) = 2^q\, \mathbb{E}(|\Re(Z_1)|^q)\mathbb{E}\left\{\left|f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right)\right|^q\right\}$$

Moreover

$$\mathbb{E}\left\{\left|f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right)\right|^q\right\} \leq 2^q\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d} \int_0^{+\infty} \frac{r^{n-1-q/2-qh(x)/d}}{(n-1)!}\, \mathrm{e}^{-r}\, dr.$$

As a result, when $n > qh(x)/d + q/2$, $\widetilde{\xi_n}(x) \in L^q$ and

$$\mathbb{E}\left(\left|\widetilde{\xi_n}(x)\right|^q\right) \leq 2^{2q}\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d}\, \mathbb{E}(|\Re(Z_1)|^q)\, \frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}.$$

Then let $N > qh(x)/d + q/2 - 1$. Therefore

$$\mathbb{E}\left(\max_{N+1\leq M\leq P} |R_{N,M}(x)|^q\right) \leq A_{q,x}\left\{\sum_{n=N+1}^{P}\left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2}, \tag{21}$$

where $A_{q,x} = 2^{2q+1} B_q^q\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d}\, \mathbb{E}(|\Re(Z_1)|^q)$.
Then by the Stirling Formula, as $P \to +\infty$,

$$\sum_{n=N+1}^{P}\left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}\right)^{2/q}$$

converges and

$$\sum_{n=N+1}^{+\infty}\left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}\right)^{2/q} \sim \sum_{n=N+1}^{+\infty} \frac{1}{n^{1+2h(x)/d}} \sim \frac{d}{2h(x)N^{2h(x)/d}}.$$

Owing to (21) and to a monotone convergence argument,

$$\mathbb{E}\left(\max_{M\geq N+1}|R_{N,M}(x)|^q\right)\leq A_{q,x}\sup_{n\geq N+1}D_{n-1,q}(h(x))\left(\sum_{n=N+1}^{+\infty}\frac{1}{n^{1+2h(x)/d}}\right)^{q/2},$$

where for $0<s<1$ and $n>q/2+qs/d-1$,

$$D_{n,q}(s)=\frac{\Gamma(n+1-q/2-qs/d)\,(n+1)^{q/2+qs/d}}{\Gamma(n+1)}.$$

As a consequence,

$$\mathbb{E}\left(\max_{M\geq N+1}|R_{N,M}(x)|^q\right)\leq\frac{C_{q,x}}{N^{qh(x)/d}}\sup_{n\geq N}D_{n,q}(h(x)),$$

where

$$C_{q,x}=\left(\frac{d}{2h(x)}\right)^{q/2}A_{q,x}.$$

Therefore by an argument of dominated convergence,

$$\mathbb{E}(|Y_h(x)-Y_{h,N}(x)|^q)\leq\frac{C_{q,x}}{N^{qh(x)/d}}\sup_{n\geq N}D_{n,q}(h(x)).$$

Moreover

$$\frac{D_{n+1,q}(h(x))}{D_{n,q}(h(x))}=\exp(g(n+1))$$

where for $y>q/2+qh(x)/d$,

$$g(y)=\frac{q(d+2h(x))}{2d}\ln\left(1+\frac{1}{y}\right)+\ln\left(1-\frac{q(d+2h(x))}{2dy}\right).$$

A simple study of g shows that $g(x)<0$. Therefore $(D_{n,q}(h(x)))_{n\geq[q/2+qh(x)/d]+1}$ is a non-increasing sequence. As a result,

$$\sup_{n\geq N}D_{n,q}(h(x))=D_{N,q}(h(x)),$$

which concludes the proof. □

The next proposition studies the asymptotic of the mean square error. From this proposition, we deduce that the rate of convergence in (16) is optimal. For the sake of clearness, the proof of this proposition is given in appendix.

Proposition 3.3. *Let $x\in\mathbb{R}^d\backslash\{0\}$. Then,*

$$\lim_{N\to+\infty}N^{2h(x)/d}\,\mathbb{E}\left(|Y_h(x)-Y_{h,N}(x)|^2\right)=\frac{C_{2,x}}{4},$$

where $C_{2,x}$ is defined by (18).

The following corollary gives a rate of almost sure convergence of $Y_{h,N}$. It is deduced from the proposition 3.2 and the Borel-Cantelli lemma.

Corollary 3.1. *Let $x\in\mathbb{R}^d$ and $H'<h(x)$. Then there exists a finite positive random variable C such that, almost surely,*

$$\forall N\geq 1,\ |Y_h(x)-Y_{h,N}(x)|\leq\frac{C}{N^{H'/d}}.$$

The two previous results study the approximation errors for a one-dimensional distribution. However the sample paths are approximated on the compact set K. Therefore errors in term of norms on E_K are studied.

Let $K\subset\mathbb{R}^d$ be a compact set and let us endow E_K with a L^p-norm:

$$\|g\|_{p,K}=\left\{\int_K|g(\xi)|^p\,d\xi\right\}^{\frac{1}{p}},\tag{22}$$

where $p\geq 1$. By applying the following proposition with $q=2$ and $p=2$, the mean integrated square error is evaluated.

Proposition 3.4. *Let $q \geq p$, $m_K = \min_K h$ and $M_K = \max_K h$. Then for every integer $N \in \mathbb{N}^*$ such that $N \geq q/2 + qM_K/d + 1$,*

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q\Big) \leq C_q \frac{D_{N,q}(m_K)}{N^{qm_K/d}},$$

where C_q does not depend on N and $D_{N,q}$ is defined by (17)*.*

Remark 3.3. *In fact proposition 3.4 holds with*

$$C_q = 2^{1+3q/2}\, B_q^q \left(\frac{d}{m_K}\right)^{q/2} \mathbb{E}(|\Re(Z_1)|^q)\, d_K^{q/p} \max_{x\in K}\Big\{(c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d}\Big\},$$

where B_q is defined by (19) *and d_K is the volume of the compact set K.*

Notice that

$$\max_{x\in K} (c_d\, \nu(\mathbb{C}))^{qh(x)/d} = \begin{cases} (c_d\, \nu(\mathbb{C}))^{qM_K/d} & \text{if } c_d\, \nu(\mathbb{C}) \geq 1 \\ (c_d\, \nu(\mathbb{C}))^{qm_K/d} & \text{if } c_d\, \nu(\mathbb{C}) \leq 1. \end{cases}$$

Proof. Let $N \in \mathbb{N}^*$ such that $N > q/2 + qM_K - 1$. Since $q \geq p$, by the Hölder inequality,

$$\|Y_h - Y_{h,N}\|_{p,K} \leq {d_K}^{1/p-1/q}\, \|Y_h - Y_{h,N}\|_{q,K},$$

where d_K is the volume of K. Therefore

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q\Big) \leq {d_K}^{q/p-1} \int_K \mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q)\, dx.$$

Then proposition 3.2 is not directly applied. However in the sequel the notation is the same as in the proof of proposition 3.2. Let us recall, see inequality (21), that

$$\mathbb{E}\left(\max_{N+1\leq M\leq P} |R_{N,P}(x)|^q\right) \leq A_{q,x} \left\{\sum_{n=N+1}^{P} \left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2},$$

where $A_{q,x} = 2^{2q+1} B_q^q\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d}\, \mathbb{E}(|\Re(Z_1)|^q)$. Therefore since

$$n - q/2 - qm_K/d \geq N + 1 - q/2 - qh(x)/d \geq N + 1 - q/2 - qM_K/d \geq 2,$$

and since Γ is an increasing function on $[2, +\infty[$

$$\mathbb{E}\left(\max_{N+1\leq M\leq P} |R_{N,P}(x)|^q\right) \leq A_{q,x} \left\{\sum_{n=N+1}^{P} \left(\frac{\Gamma(n - q/2 - qm_K/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2}.$$

Furthermore the arguments used in the proof of the proposition 3.2 leads to

$$\mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q) \leq \left(\frac{d}{2m_K}\right)^{q/2} \sup_{x\in K} A_{q,x} \frac{D_{N,q}(m_K)}{N^{qm_K/d}}.$$

As a result,

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q\Big) \leq C_q \frac{D_{N,q}(m_K)}{N^{qm_K/d}},$$

where $C_q = d_K^{q/p} \left(\frac{d}{2m_K}\right)^{q/2} \sup_{x\in K} A_{q,x}$. □

Consequently, the Borel-Cantelli lemma yields the following corollary.

Corollary 3.2. *Let $H' < \min_K h$. Then there exists a finite positive random variable C such that, almost surely,*

$$\forall N \geq 1,\ \|Y_h - Y_{h,N}\|_{p,K} \leq \frac{C}{N^{H'/d}}.$$

In practice, when ν is a finite measure, one simulates the sample paths of Y_h which is equal in law to X_h owing to a generalized shot noise series representation. However when ν is not a finite measure, it becomes more complicated. The field X_h is split into two fields $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$. First $X_{\varepsilon,2}$ can be simulated as a generalized shot noise series. Then, the next section gives conditions to approximate $X_{\varepsilon,1}$ by a Multifractional Brownian Motion.

4 Normal Approximation

In this part ν is not supposed to be finite. Then let $\varepsilon > 0$ and let us split $X_h = X_{\varepsilon,1} + X_{\varepsilon,2}$ into two random fields where

$$X_{\varepsilon,1}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re(f(x,\xi)z) \mathbf{1}_{|z|<\varepsilon} \widetilde{N}(d\xi, dz), \tag{23}$$

and

$$X_{\varepsilon,2}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re(f(x,\xi)z) \mathbf{1}_{|z|\geq\varepsilon} \widetilde{N}(d\xi, dz). \tag{24}$$

Let us consider the two independent Poisson random measures

$$N_{\varepsilon,1}(d\xi, dz) = \mathbf{1}_{|z|<\varepsilon} N(d\xi, dz)$$

and

$$N_{\varepsilon,2}(d\xi, dz) = \mathbf{1}_{|z|\geq\varepsilon} N(d\xi, dz).$$

Let $M_{\varepsilon,i}$ be the Lévy random measure associated with $N_{\varepsilon,i}$ by (2). Remark that $X_{\varepsilon,i}$ is a RHMLM associated with $M_{\varepsilon,i}$. Moreover $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$ are independent. Notice that the control measure $\nu_{\varepsilon,2}(dz) = \mathbf{1}_{|z|\geq\varepsilon} \nu(dz)$ of $M_{\varepsilon,2}$ is finite. Therefore $X_{\varepsilon,2}$ can be simulated as a generalized shot noise series, see section 3.

Now it remains to approximate $X_{\varepsilon,1}$. In [2], it is proposed to simulate the small jump part of a Lévy process by a Brownian Motion. Here proposition 4.1 gives sufficient conditions to approximate $X_{\varepsilon,1}$ by a Multifractional Brownian Motion. These conditions are closely related to those given in [2] for Lévy processes.

Let us introduce $\nu_{\varepsilon,1}(dz) = \mathbf{1}_{|z|<\varepsilon} \nu(dz)$ the control measure of $M_{\varepsilon,1}$ and

$$\sigma(\varepsilon) = \left(\int_{0<\rho<\varepsilon} \rho^2 \nu_\rho(d\rho) \right)^{1/2}. \tag{25}$$

The following proposition can be stated with simpler conditions, see corollary 4.1. However in some simple cases, the assumptions of this proposition are satisfied whereas those of corollary 4.1 are not. An example that comes from [2] will be then given.

Proposition 4.1. *Suppose that for each $\kappa > 0$,*

$$\mathbf{H1} \quad \sigma(\kappa\sigma(\varepsilon) \wedge \varepsilon) \sim \sigma(\varepsilon) \quad \text{as } \varepsilon \to 0_+$$

then

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)} \right)_{x \in \mathbb{R}^d} \stackrel{(d)}{=} (C_h(x) B_h(x))_{x \in \mathbb{R}^d}, \tag{26}$$

where the limit is in distribution for all finite-dimensional margins, B_h is a standard Multifractional Brownian Motion with multifractional function h and

$$C_h(x) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(h(x)+1/2)}{h(x)\,\Gamma(2h(x))\sin(\pi h(x))\,\Gamma(h(x)+d/2)} \right)^{1/2}. \tag{27}$$

Let us recall that h is a locally β-Hölder function. Let $K \subset \mathbb{R}^d$ be a compact set, $m_K = \min_K h$ and $p_K = 1 + \left[\frac{d}{2\min(m_K, \beta)} \right]$. Then if **H1** *and*

$$\mathbf{H2} \quad \exists \varepsilon_0 > 0,\, \exists C \in]0, +\infty[,\, \forall \varepsilon \leq \varepsilon_0,\, \int_{0<\rho<\varepsilon} \rho^{2p_K} \nu_\rho(d\rho) \leq C\sigma^{2p_K}(\varepsilon)$$

are satisfied, the convergence (26) *is a convergence in distribution on the space E_K of continuous functions on K endowed with the topology of uniform convergence.*

For the sake of clearness, the proof of this proposition is given in appendix.

Remark 4.1.

- *Suppose that* **H2** *is satisfied for all closed balls of $\mathbb{R}^d$. Then under* **H1***, the convergence* (26) *is a convergence in distribution on the space of continuous functions endowed with the topology of uniform convergence on compact sets.*

- *When $d = 1$ and $m_K > 1/2$, $p_K = 1$ and* **H2** *is fulfilled.*

As it is done in [2], a simpler convergence condition is given.

Corollary 4.1. *If*

$$\textbf{H3} \quad \lim_{\varepsilon \to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty,$$

then the assumptions **H1** *and* **H2** *are satisfied.*

Proof. The comparison between **H3** and **H1** has already been done in [2]. Moreover for $k \geq 1$,

$$\int_{0<\rho<\varepsilon} \rho^{2k} \, \nu_\rho(d\rho) \leq \varepsilon^{2k-2} \sigma^2(\varepsilon).$$

Thanks to this inequality, **H3** implies **H2**. □

It can be shown by the same way that **H2** is satisfied as soon as

$$\liminf_{\varepsilon \to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} > 0.$$

Let us give an example which satisfies the assumptions **H1** and **H2** of proposition 4.1 whereas it does not satisfy **H3**. In fact it is the Example 2.1 in [2].

Example 4.1. *Let $(a_n)_{n\geq 1}$ be a decreasing sequence such that*

$$\lim_{n\to+\infty} a_n = 0 \text{ and } \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 0.$$

Assume that $a_1 = 1$. Let ν_ρ be a Lévy measure on $]0, +\infty[$ such that

$$\sigma(\varepsilon) = \begin{cases} a_n & \text{for } \varepsilon \in]a_{n+1}, a_n] \\ 1 & \text{for } \varepsilon \geq 1. \end{cases}$$

First, the Lévy measure ν associated to ν_ρ by (5) satisfies (1). Moreover since $\liminf_{\varepsilon\to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = 1$, **H3** *is not fulfilled whereas* **H2** *is. Moreover according to [2],* **H1** *is satisfied. As a result, proposition 4.1 can be applied.*

Let us now discuss the rate of convergence in terms of Berry-Esseen bounds. For the sake of simplicity, $X_{\varepsilon,1}(x)$ is supposed to have a moment of order three, which allows us to apply the classical Berry-Esseen inequality. However a generalization of the classical Berry-Esseen inequality, see Theorem 5.7 in [16], under weaker moment assumption, allows us to extend the following results. Then an estimation in term of Berry-Esseen bounds is deduced from the next lemma which is a consequence of the classical Berry-Esseen inequality.

Lemma 4.1. *Let μ be an infinitely divisible law on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^3 \, \mu(dx) < +\infty$ and $\int_{\mathbb{R}} x \, \mu(dx) = 0$. Then*

$$\sup_{x\in\mathbb{R}} \left| \mu(]-\infty, x[) - \int_{-\infty}^{x/\sigma} \mathrm{e}^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq 0.7975 \, \sigma^{-3} \int_{\mathbb{R}} |x|^3 \, \Lambda(dx),$$

where $\sigma = \left(\int_{\mathbb{R}} |x|^2 \, \mu(dx)\right)^{1/2}$ and Λ is the Lévy measure of μ.

Proof. Let $(Z(t))_{t\geq 0}$ be a Lévy process such that μ is the law of $Z(1)$. Then proceed as in the proof of Theorem 3.1 page 487 in [2], i.e. write

$$Z(1) = \sum_{k=1}^{n} \left(Z\left(\frac{k}{n}\right) - Z\left(\frac{k-1}{n}\right) \right).$$

Hence $Z(1)$ is a sum of i.i.d real-valued random variables with mean zero and variance σ^2/n. Moreover, according to [2],

$$\lim_{n\to+\infty} n\mathbb{E}\left(\left| Z\left(\frac{1}{n}\right) \right|^3 \right) = \int_{\mathbb{R}} |x|^3 \, \Lambda(dx).$$

Therefore the conclusion is given by the classical Berry-Esseen inequality. □

Then an estimation in term of Berry-Esseen bounds of the rate of convergence stated in proposition 4.1 can be given.

Proposition 4.2. *Let $K \subset \mathbb{R}^d$ be a compact set, $y' \in E'_K$ and assume that $\max_K h < 1 - d/6$. Then $\sup_{x\in K} |f(x,\xi)| \in L^3(\mathbb{R}^d)$ and*

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\Big(< y', X_{\varepsilon,1_{|K}} > \leq u \Big) - \mathbb{P}\Big(< y', \sigma(\varepsilon)(C_h B_h)_{|K} > \leq u \Big) \right| \leq C(y') \, \frac{m_3^3(\varepsilon)}{\sigma^3(\varepsilon)}$$

where $m_3^3(\varepsilon) = \int_{0<\rho<\varepsilon} \rho^3 \, \nu_\rho(d\rho)$ and

$$C(y') = \frac{8\,A \displaystyle\int_{\mathbb{R}^d} \left| < y', f(\cdot,\xi)_{|K} > \right|^3 d\xi}{3\left(\pi \displaystyle\int_{\mathbb{R}^d} \left| < y', f(\cdot,\xi)_{|K} > \right|^2 d\xi \right)^{3/2}},$$

with $A = 0.7975$.

Proof. $< y', X_{\varepsilon,1_{|K}} >$ is a real-valued infinitely divisible random variable. Its Lévy measure Λ is the push forward of $\nu_{\varepsilon,1}$ by

$$(\xi, z) \longmapsto 2\Re\Big(< y', f(\cdot,\xi)_{|K} > z \Big).$$

Therefore,

$$\begin{aligned} \mathbb{E}\left(\left| < y', X_{\varepsilon,1_{|K}} > \right|^2 \right) &= 4 \int_{\mathbb{R}^d \times \mathbb{C}} \left| 2\Re\Big(< y', f(\cdot,\xi)_{|K} > z \Big) \right|^2 \mathbf{1}_{|z|<\varepsilon} \, d\xi \, \nu(dz) \\ &= 4\pi\,\sigma^2(\varepsilon) \int_{\mathbb{R}^d} \left| < y', f(\cdot,\xi)_{|K} > \right|^2 d\xi \\ &= \mathrm{Var}\Big(< y', \sigma(\varepsilon)(C_h B_h)_{|K} > \Big). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}} |x|^3 \, \Lambda(dx) &= 4\, m_3^3(\varepsilon) \int_0^{2\pi} |\cos\theta|^3 \, d\theta \int_{\mathbb{R}^d} \left| < y', f(\cdot,\xi)_{|K} > \right|^3 d\xi \\ &= \tfrac{64}{3}\, m_3^3(\varepsilon) \int_{\mathbb{R}^d} \left| < y', f(\cdot,\xi)_{|K} > \right|^3 d\xi. \end{aligned}$$

One concludes by applying the lemma 4.1. ☐

Remark 4.2. *In proposition 4.2, the assumption $\max_K h < 1 - d/6$ means that for every $x \in K$, $X_h(x)$ has moment of order three. However the preceding proposition can be generalized to any RHMLM. In fact there always exists $\delta \in]0,1[$ such that $\max_K h < 1 - d/2 + d/(2+\delta)$. Then for every $x \in K$, $\mathbb{E}\Big(|X_{\varepsilon,1}(x)|^{2+\delta} \Big) < +\infty$. As a consequence, a generalization of proposition 4.2 is obtained owing to Theorem 5.7 in [16].*

Hence, in the case where ν is not a finite measure, X_h can be approximated in law as soon as ν satisfies assumptions of proposition 4.1. In this case, according to propositions 4.1 and 3.1, an approximation is given by

$$Y_{\varepsilon,N}(x) = \sigma(\varepsilon) C(h(x)) B_h(x) + 2 \sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_{\varepsilon,n} \right),$$

where B_h, T_n, U_n and $Z_{\varepsilon,n}$ are independent. T_n and U_n are defined in section 3. We have that $(Z_{\varepsilon,n})_n$ are i.i.d. random variables with common law $\nu_{\varepsilon,2}(dz)/\nu_{\varepsilon,2}(\mathbb{C})$. It is supposed that $\nu_{\varepsilon,2}(\mathbb{C}) \neq 0$, which is the case for ε sufficiently small.

Then the approximations given in section 3 and section 4 are used in the next part to generate sample paths of X_h. First, examples of RHMLMs with finite control measure are given since it is the simplest case of simulation.

5 Simulation

5.1 Case of finite measure

Suppose here that $0 < \nu(\mathbb{C}) < +\infty$. Let us recall that

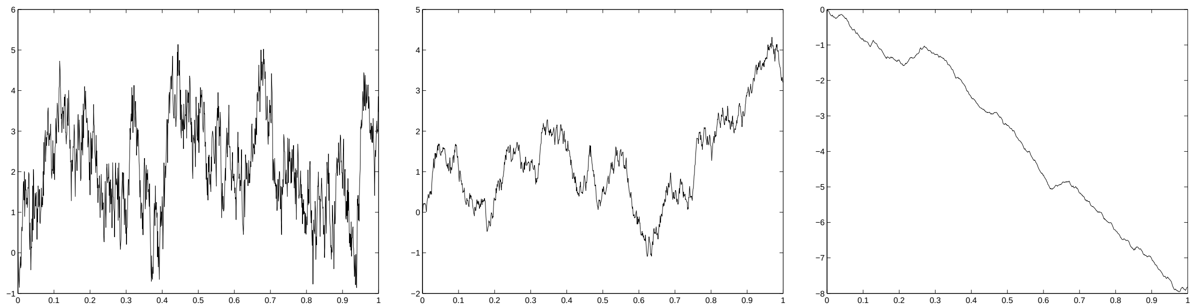
$$Y_{h,N}(x) = 2\sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_n \right).$$

converges almost surely to Y_h which is equal in law to X_h. Therefore the sample paths of Y_h are simulated and are approximated by $Y_{h,N}$.

Let us present some examples. Suppose that ν is the uniform law on the unit circle of $\mathbb{C}$. Then $\nu(\mathbb{C}) = 1$.

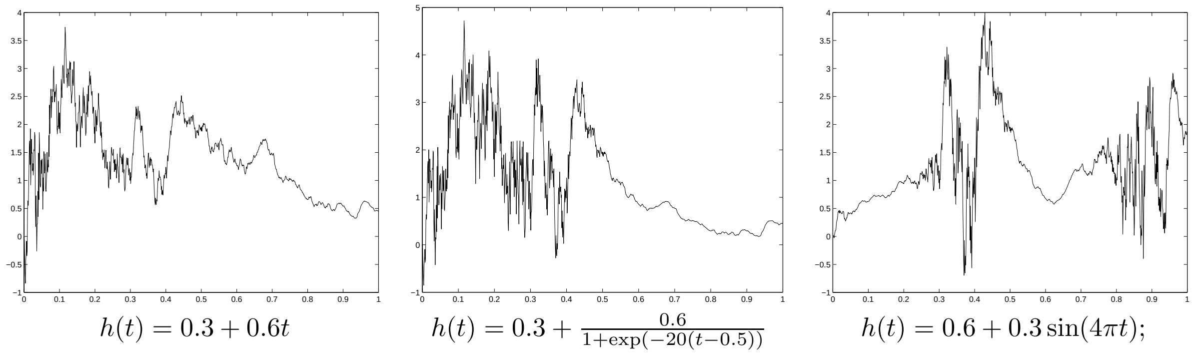
First assume that h is constant to H, which means that X_h is a RHFLM. According to [3], the sample paths of $X_H = X_h$ are locally Hölderian. Furthermore, the pointwise Hölder exponent does not vary along the trajectory and is almost surely equal to H. Figure 1 yields illustrations of these facts.

Let now $h : \mathbb{R}^d \longrightarrow]0,1[$ be a locally β-Hölder function. The pointwise Hölder exponent of a RHFLM is constant but for a general RHMLM it is not. More precisely, if $h(x) < \beta$, the pointwise Hölder exponent at point x of X_h is almost surely equal to $h(x)$. In figure 2, examples of RHMLMs are given. One can observe that the regularity varies along the trajectory as h does. The greater $h(t)$ is, the smoother the trajectories are on a neighbourhood of t.



$H = 0.3$ $\quad$ $H = 0.5$ $\quad$ $H = 0.8$

Figure 1: Examples of RHFLMs



$h(t) = 0.3 + 0.6t$ $\quad$ $h(t) = 0.3 + \frac{0.6}{1+\exp(-20(t-0.5))}$ $\quad$ $h(t) = 0.6 + 0.3\sin(4\pi t);$

Figure 2: Examples of RHMLMs

When ν is a finite measure, simulations exhibit the same smoothness properties as the theoretical model.

Next paragraph is devoted to examples of RHMLMs in the case where ν is not a finite measure.

5.2 Case of infinite measure

Suppose that ν is a measure which satisfies the assumptions of proposition 4.1. Hence X_h is split into two independent RHMLMs $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$, defined by (23) and (24). Then for ε sufficiently small, $0 < \nu_{\varepsilon,2}(\mathbb{C}) < +\infty$.

Let $(Z_{\varepsilon,n})_n$ be a sequence i.i.d. random variables with common law $\nu_{\varepsilon,2}(dz)/\nu_{\varepsilon,2}(\mathbb{C})$ and B_h a standard Multifractional Brownian Motion. Let us recall that $(U_n)_n$ is a sequence of i.i.d uniform random variables on S^{d-1} and T_n is the nth arrival time of a Poisson process with intensity 1. Assume that B_h, $(Z_{\varepsilon,n})_n$, $(U_n)_n$ and $(T_n)_n$ are independent. As a consequence,

$$Y_{\varepsilon,N}(x) = \sigma(\varepsilon)C_h(x)B_h(x) + 2\sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_{\varepsilon,n}\right),$$

approximates a field Y_h which is equal in law to X_h.

In order to simulate B_h, we use J.-F. Coeurjolly's programs which are available on

http://www-lmc.imag.fr/SMS/software.html.

More precisely, the Fractional Brownian Motion (case h constant) is generated by the method of circulant embedding, introduced in [23].

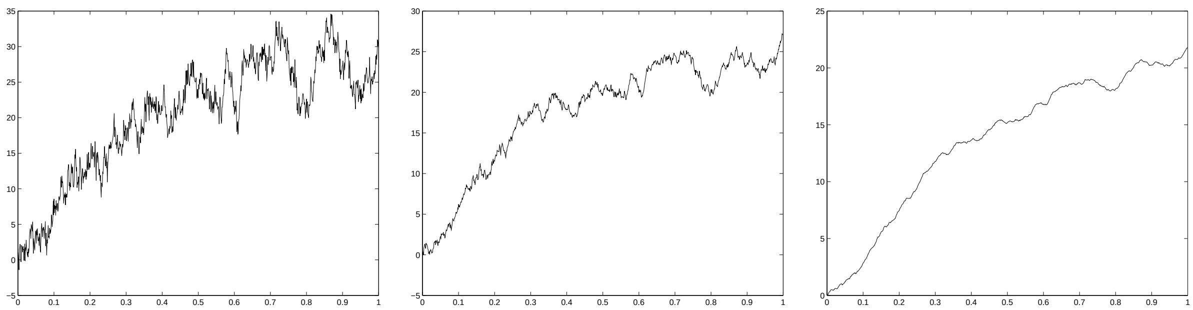
Let us now give some examples. Assume that $\nu_\rho(d\rho)$ is a truncated α-stable measure. More precisely, suppose that

$$\nu_\rho(d\rho) = \mathbf{1}_{0<\rho<1}\,\frac{d\rho}{\rho^{1+\alpha}},$$

with $0 < \alpha < 2$. Let $0 < \varepsilon < 1$. Then

$$\nu_{\varepsilon,2}(\mathbb{C}) = \frac{2\pi(\varepsilon^{-\alpha} - 1)}{\alpha} \quad \text{and} \quad \sigma^2(\varepsilon) = \frac{\varepsilon^{2-\alpha}}{2-\alpha}.$$

Figures 3 and 4 present some examples of RHMLMs with control measure ν. Like in the case of finite measures, the theoretical smoothness of the trajectory is observed.



$H = 0.3$ $\qquad$ $H = 0.5$ $\qquad$ $H = 0.8$

Figure 3: Examples of RHFLMs

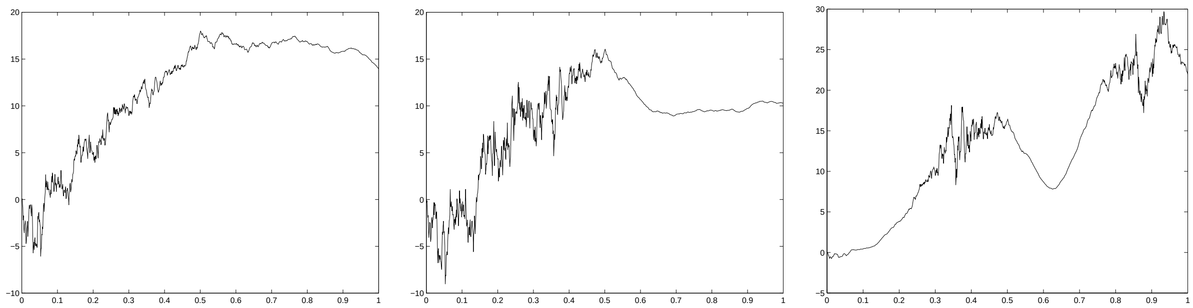
Appendix

A Proof of the proposition 3.3

In the case where $d \geq 2$, the proof of the proposition 3.3 is based on the study of the asymptotic of

$$K_n(y,b) = \int_0^{+\infty} \mathrm{e}^{ir^{1/d}y} r^{n-b} \mathrm{e}^{-r}\, dr, \tag{28}$$

where $y \in \mathbb{R}$ and $b > 0$. Else, if $d = 1$, the proof of the proposition 3.3 is simpler. The next lemma gives the asymptotic of K_n.



$h(t) = 0.3 + 0.6t$ $\qquad$ $h(t) = 0.3 + \frac{0.6}{1+\exp(-20(t-0.5))}$ $\qquad$ $h(t) = 0.6 + 0.3\sin(4\pi t)$;

Figure 4: Examples of RHMLMs

Lemma A.1. *Let $d \geq 2$, $y \in \mathbb{R}$ and $b > 0$. Then*

$$\lim_{n\to+\infty} \frac{K_n(y,b)}{n^{n-b+1/2}\,\mathrm{e}^{-n}\mathrm{e}^{in^{1/d}y}} = \begin{cases} \sqrt{2\pi}\,\mathrm{e}^{-y^2/8} & \text{if } d = 2, \\ \sqrt{2\pi} & \text{if } d \geq 3. \end{cases}$$

Proof. Let

$$\widetilde{K_n}(y,b) = \frac{K_n(y,b)}{n^{n-b+1/2}\,\mathrm{e}^{-n}\mathrm{e}^{in^{1/d}y}}.$$

By applying the change of variables $s = \frac{r-n}{\sqrt{n}}$,

$$\widetilde{K_n}(y,b) = \mathrm{e}^{-in^{1/d}y} \int_{-\sqrt{n}}^{+\infty} \mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} \left(1 + \frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}}\,ds.$$

A Taylor expansion leads to

$$\lim_{n\to+\infty} \left(1 + \frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}} = \mathrm{e}^{-s^2/2}.$$

Moreover, using a Taylor expansion,

$$\lim_{n\to+\infty} \mathrm{e}^{-in^{1/d}y}\mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} = \begin{cases} \mathrm{e}^{isy/2} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$$

Consequently,

$$\lim_{n\to+\infty} \mathrm{e}^{-in^{1/d}y}\mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} \left(1 + \frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}} = \begin{cases} \mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2} & \text{if } d = 2, \\ \mathrm{e}^{-s^2/2} & \text{if } d \geq 3. \end{cases}$$

We do not directly apply a dominated convergence argument. Let us first write

$$\widetilde{K_n}(y,b) = \widetilde{K_{n,1}}(y,b) + \widetilde{K_{n,2}}(y,b),$$

where

$$\widetilde{K_{n,1}}(y,b) = \int_{\sqrt{n}}^{+\infty} g_n(y,b,s)\,ds$$

and

$$\widetilde{K_{n,2}}(y,b) = \int_{-\sqrt{n}}^{\sqrt{n}} g_n(y,b,s)\,ds$$

with $g_n(y,b,s) = \mathrm{e}^{-in^{1/d}y}\,\mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} \left(1 + \frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}}$.

Study of $\widetilde{K_{n,1}}(y,b)$ Remark that

$$\begin{aligned} \left|\widetilde{K_{n,1}}(y,b)\right| &\leq \int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n-b}\mathrm{e}^{-s\sqrt{n}}\,ds \\ &\leq \int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n}\mathrm{e}^{-s\sqrt{n}}\,ds, \end{aligned}$$

since $b>0$ and $1+s/\sqrt{n}\geq 1$. Furthermore, integrating by parts leads to

$$\int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n}\mathrm{e}^{-s\sqrt{n}}\,ds \leq \left(2^{n+1}-1\right)\frac{\mathrm{e}^{-n}}{\sqrt{n}}.$$

As a consequence,

$$\lim_{n\to+\infty}\widetilde{K_{n,1}}(y,b)=0. \tag{29}$$

Study of $\widetilde{K_{n,2}}(y,b)$ Let us recall that

$$\lim_{n\to+\infty} g_n(y,b,s)=\begin{cases} \mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2} & \text{if } d=2, \\ \mathrm{e}^{-s^2/2} & \text{if } d\geq 3. \end{cases}$$

Let $n>b$. Let us notice that for every $|x|<1$, $\ln(1+x)\leq x-x^2/6$. Then,

$$\begin{aligned} |g_n(y,b,s)|\mathbf{1}_{|s|<\sqrt{n}} &\leq \mathrm{e}^{-\frac{bs}{\sqrt{n}}}\mathrm{e}^{-\frac{n-b}{6n}s^2} \\ &\leq \mathrm{e}^{b}\mathrm{e}^{-\frac{n-b}{6n}s^2} \end{aligned}$$

since $\left|\frac{bs}{\sqrt{n}}\right|\leq b$. Then for n sufficiently large,

$$|g_n(y,b,s)|\mathbf{1}_{|s|<\sqrt{n}}\leq \mathrm{e}^{b}\mathrm{e}^{-\frac{s^2}{12}},$$

for every $s\in\mathbb{R}$. Therefore, using a dominated convergence argument, one concludes that

$$\lim_{n\to+\infty}\widetilde{K_{n,2}}(y,b)=\begin{cases} \displaystyle\int_{\mathbb{R}}\mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2}\,ds & \text{if } d=2, \\ \displaystyle\int_{\mathbb{R}}\mathrm{e}^{-s^2/2}\,ds & \text{if } d\geq 3. \end{cases}$$

As a consequence,

$$\lim_{n\to+\infty}\widetilde{K_{n,2}}(y,b)=\begin{cases} \sqrt{2\pi}\,\mathrm{e}^{-y^2/8} & \text{if } d=2, \\ \sqrt{2\pi} & \text{if } d\geq 3. \end{cases} \tag{30}$$

The conclusion is then given by (29) and (30). □

Let us now prove the proposition 3.3.

Proof. Let $x\in\mathbb{R}^d\backslash\{0\}$. In the following, the notation is the same as in the proof of proposition 3.2. Let us first recall that

$$\mathbb{E}\left(|R_{N,P}(x)|^2\right)=\mathbb{E}\left\{\left(\sum_{n=N+1}^{P}\widetilde{\xi_n}(x)\right)^2\right\},$$

where $\widetilde{\xi_n}(x)=2\left|f\left(x,\left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d}U_n\right)\right|\Re(Z_n)$.

Therefore, since $(\Re(Z_n))_n$ is a sequence of i.i.d symmetric random variables and since this sequence is independent of $(U_n, T_n)_n$,

$$\mathbb{E}\Big(|R_{N,P}(x)|^2\Big) = \sum_{n=N+1}^{P} \mathbb{E}\Big(\widetilde{\xi_n}^2(x)\Big).$$

Moreover, $R_{N,P}(x)$ converges to $Y_h(x) - Y_{h,N}(x)$ in quadratic mean as $P \to +\infty$. Therefore,

$$\begin{aligned}
\mathbb{E}\Big(|Y_h(x) - Y_{h,N}(x)|^2\Big) &= \sum_{n=N+1}^{+\infty} \mathbb{E}\Big(\widetilde{\xi_n}^2(x)\Big) \\
&= 4\,\mathbb{E}\big(\Re^2(Z_1)\big) \sum_{n=N+1}^{+\infty} \mathbb{E}\left\{\left| f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right)\right|^2\right\}.
\end{aligned}$$

Then, by definition of the sequence $(T_n, U_n)_{n\geq 1}$,

$$\mathbb{E}\Big(|Y_h(x) - Y_{h,N}(x)|^2\Big) = 4(c_d\nu(\mathbb{C}))^{1+2h(x)/d}\,\mathbb{E}\big(\Re^2(Z_1)\big) \sum_{n=N+1}^{+\infty} I_n\left(\frac{x}{(c_d\nu(\mathbb{C}))^{1/d}}\right), \tag{31}$$

where for every $v \in \mathbb{R}^d$,

$$I_n(v) = \int_{S^{d-1}}\int_0^{+\infty} \left|\mathrm{e}^{-ir^{1/d}v\cdot u} - 1\right|^2 r^{n-2-2h(x)/d}\,\mathrm{e}^{-r}\,\frac{dr}{(n-1)!}\,\frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let $n > 1 + 2h(x)$. Let us remark that

$$I_n(v) = \frac{2\Gamma(n-1-2h(x)/d)}{\Gamma(n)} - \frac{2}{\Gamma(n)}\Re(J_n(v))$$

with

$$J_n(v) = \int_{S^{d-1}}\int_0^{+\infty} r^{n-2-2h(x)/d}\,\mathrm{e}^{ir^{1/d}v\cdot u - r}\,dr\,\frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let us assume in the following that $v \neq 0$.

Step 1 Case $\mathbf{d = 1}$ Using the characteristic function of a Gamma-distribution, we obtain that

$$J_n(v) = \left(1+v^2\right)^{h(x)/d+(1-n)/2} \cos\left((n-1-2h(x)/d)\arctan v\right)\Gamma(n-1-2h(x)/d).$$

Hence, since $v \neq 0$,

$$I_n(v) \sim \frac{2\Gamma(n-1-2h(x)/d)}{\Gamma(n)}. \tag{32}$$

as $n \to +\infty$.

Step 2 Case $\mathbf{d \geq 2}$ We prove that the equivalence (32) remains true for $d \geq 2$. It is then sufficient to prove that

$$\lim_{n\to+\infty} \frac{J_n(v)}{\Gamma(n-1-2h(x)/d)} = 0. \tag{33}$$

Remark that

$$J_n(v) = \int_{S^{d-1}} K_n(v\cdot u, 2+2h(x)/d)\,\frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})},$$

where K_n is defined by (28).
Since

$$\Gamma(n-1-2h(x)/d) \sim \sqrt{2\pi}n^{n-3/2-2h(x)/d}\mathrm{e}^{-n},$$

by applying the lemma A.1,

$$\lim_{n\to+\infty} \frac{K_n(v\cdot u, 2+2h(x)/d)}{\Gamma(n-1-2h(x)/d)}\mathrm{e}^{-in^{1/d}v\cdot u} = l_d(v\cdot u),$$

where $l_d(y) = \left\{ \begin{array}{cl} \mathrm{e}^{-y^2/8} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{array} \right.$

Furthermore,

$$\frac{J_n(v)}{\Gamma(n-1-2h(x)/d)} = J_{n,1}(v) + J_{n,2}(v),$$

where

$$J_{n,1}(v) = \int_{S^{d-1}} \mathrm{e}^{in^{1/d} v \cdot u}\, l_d(v \cdot u)\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}$$

and

$$J_{n,2}(v) = \int_{S^{d-1}} \bigg(\frac{K_n(v \cdot u, 2 + 2h(x)/d)}{\Gamma(n-1-2h(x)/d)} \mathrm{e}^{-in^{1/d} v \cdot u} - l_d(v \cdot u) \bigg) \mathrm{e}^{in^{1/d} v \cdot u}\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let us first remark that by definition,

$$0 \leq \frac{|K_n(v \cdot u, 2 + 2h(x)/d)|}{\Gamma(n-1-2h(x)/d)} \leq 1.$$

Then, a dominated convergence argument leads to

$$\lim_{n \to +\infty} J_{n,2}(y) = 0.$$

Furthermore, by rotational invariance of the measure $\sigma_{d-1}(du)$,

$$J_{n,1}(v) = \int_{S^{d-1}} \mathrm{e}^{in^{1/d} \|v\| e_1 \cdot u}\, l_d(\|v\| e_1 \cdot u)\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}$$

with $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. As a consequence,

$$J_{n,1}(v) = \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_{-1}^{1} \mathrm{e}^{in^{1/d} \|v\| s}\, l_d(\|v\| s) \big(1 - s^2\big)^{(d-3)/2}\, ds.$$

Let us recall that $v \neq 0$. Therefore, by the Riemann-Lebesgue lemma,

$$\lim_{n \to +\infty} J_{n,1}(y) = 0.$$

As a result,

$$\lim_{n \to +\infty} \frac{J_n(v)}{\Gamma(n-1-2h(x)/d)} = 0,$$

which implies (32).

Step 3 Conclusion By applying (32) and the Stirling formula,

$$I_n(v) \sim \frac{2}{n^{1+2h(x)/d}}.$$

Then, because of (31), as $N \to +\infty$,

$$\mathbb{E}\Big(|Y_h(x) - Y_{h,N}(x)|^2 \Big) \sim 4 (c_d \nu(\mathbb{C}))^{1+2h(x)/d}\, \mathbb{E}\big(\Re^2(Z_1)\big) \sum_{n=N+1}^{+\infty} \frac{2}{n^{1+2h(x)/d}},$$

which gives the conclusion. □

B Proof of the proposition 4.1

This part is devoted to the proof of the proposition 4.1.

Proof of proposition 4.1. Let $Y_\varepsilon(x) = \frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)}$.

Convergence of the finite dimensional margins

Let $r \in \mathbb{N}^*$, $u = (u_1, \cdots, u_r) \in \left(\mathbb{R}^d\right)^r$ and $v = (v_1, \cdots, v_r) \in \mathbb{R}^r$. Then

$$\sum_{k=1}^{r} v_k Y_\varepsilon(u_k) = \int_{\mathbb{R}^d} \frac{g(\xi, u, v)}{\sigma(\varepsilon)} M_{\varepsilon,1}(d\xi),$$

where

$$g(\xi, u, v) = \sum_{k=1}^{r} v_k f(u_k, \xi)$$

Thus by (4),

$$\mathbb{E}\left(\sum_{k=1}^{r} v_k Y_\varepsilon(u_k)\right) = \exp\left(\Psi_\varepsilon(u, v)\right)$$

with

$$\Psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times \mathbb{C}} \left(\exp\left(\frac{2i\Re(g(\xi, u, v)z)}{\sigma(\varepsilon)}\right) - 1 - \frac{2i\Re(g(\xi, u, v)z)}{\sigma(\varepsilon)}\right) d\xi\, \nu_{\varepsilon,1}(dz).$$

Then by rotational invariance of ν,

$$\Psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times [0,2\pi]} I_\varepsilon(2|g(\xi, u, v)| \cos(\theta))\, d\xi\, d\theta,$$

where for every $y \in \mathbb{R}$,

$$I_\varepsilon(y) = \int_0^{+\infty} \left(\mathrm{e}^{i\frac{y\rho}{\sigma(\varepsilon)}} - 1 - i\frac{y\rho}{\sigma(\varepsilon)}\right) \mathbf{1}_{0<\rho<\varepsilon}\, \nu_\rho(d\rho).$$

Under the assumption **H1**, see [2],

$$\lim_{\varepsilon \to 0_+} I_\varepsilon(y) = \frac{-y^2}{2}.$$

Moreover, since

$$\forall y \in \mathbb{R},\ |I_\varepsilon(y)| \leq \frac{y^2}{2},$$

a dominated convergence argument yields

$$\begin{aligned}
\lim_{\varepsilon \to 0_+} \Psi_\varepsilon(u, v) &= -2\pi \int_{\mathbb{R}^d} \left|\sum_{k=1}^{r} \frac{v_k\left(\mathrm{e}^{-iu_k\cdot\xi} - 1\right)}{\|\xi\|^{h(u_k)+d/2}}\right|^2 d\xi \\
&= -\tfrac{1}{2}\mathrm{Var}\left(\sum_{k=1}^{r} v_k C_h(u_k) B_h(u_k)\right),
\end{aligned}$$

where B_h is a standard Multifractional Brownian Motion and where for every $x \in \mathbb{R}^d$,

$$C_h(x) = \left(4\pi \int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2h(x)}} d\xi\right)^{1/2},$$

with $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.

Let us now prove the formula (27) i.e.

$$C_h(x) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(h(x) + 1/2)}{h(x)\,\Gamma(2h(x))\sin(\pi h(x))\,\Gamma(h(x) + d/2)}\right)^{1/2},$$

which will conclude the proof of the convergence of the finite dimensional margins. In fact, when $d = 1$, this formula is already known, see pages 328-329 in [20]. Suppose that $d \geq 2$. Then, thanks to polar coordinates,

$$C_h^2(x) = \sigma_{d-2}\left(S^{d-2}\right) \int_{\mathbb{R}} \frac{\left|\mathrm{e}^{-ir} - 1\right|^2}{r^{1+2h(x)}} dr \int_0^{\pi/2} (\cos\theta)^{2h(x)} (\sin\theta)^{d-2}\, d\theta.$$

By applying the formula (27) for $d=1$,

$$C_h^2(x) = \frac{\pi^{(d+3)/2} B(H+1/2,(d-1)/2)}{h(x)\Gamma(2h(x))\sin(\pi h(x))\Gamma((d-1)/2)},$$

where B is the Beta-function. Then, the formula (27) is deduced from the relations between the Beta-function and the Gamma-function.

Tightness Now assume that **H1** and **H2** are satisfied. Let us prove that the family $\left\{(Y_\varepsilon(x))_{x\in K},\ \varepsilon>0\right\}$ is tight in E_K. The field Y_ε is split into two fields $Y_\varepsilon = Y_\varepsilon^+ + Y_\varepsilon^-$ where

$$Y_\varepsilon^+(x) = \frac{1}{\sigma(\varepsilon)}\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1 - P_n(-ix\cdot\xi)\mathbf{1}_{\|\xi\|\le 1}}{\|\xi\|^{h(x)+d/2}}\, M_{\varepsilon,1}(d\xi)$$

and

$$Y_\varepsilon^-(x) = \frac{1}{\sigma(\varepsilon)}\int_{\|\xi\|\le 1} \frac{P_n(-ix\cdot\xi)}{\|\xi\|^{h(x)+d/2}}\, M_{\varepsilon,1}(d\xi)$$

with $n\in\mathbb{N}^*$ such that $n>d/2$ and

$$P_n(t) = \sum_{k=1}^{n} \frac{t^k}{k!}.$$

According to [11], the sample paths of Y_ε^+ and Y_ε^- are continuous. Therefore to prove the tightness of $(Y_\varepsilon)_\varepsilon$ in E_K, it is sufficient to prove the tightness of $(Y_\varepsilon^+)_\varepsilon$ and $(Y_\varepsilon^-)_\varepsilon$. Since Y_ε^+ has moments of every order, the tightness of $(Y_\varepsilon^+)_{\varepsilon>0}$ is shown owing to the Kolmogorov criterion.

Step 1: Tightness of $(Y_\varepsilon^+)_\varepsilon$
Let $x_0\in K$. Notice that

$$\mathbb{E}\left(\left|Y_\varepsilon^+(x_0)\right|^2\right) = \mathbb{E}\left(\left|Y_1^+(x_0)\right|^2\right) < +\infty.$$

As a result, $(Y_\varepsilon^+(x_0))_{\varepsilon>0}$ is a tight family of random variables.

Moreover by proposition 2.2 in [3],

$$\left\|Y_\varepsilon^+(u) - Y_\varepsilon^+(v)\right\|_{2p_K}^{2p_K} = \sum_{m=1}^{p_K} (2\pi)^m \sum_{L_m}\prod_{q=1}^{m} \frac{(2l_q)!\,\|g_n(u,v,\cdot)\|_{2l_q}^{2l_q}\, m_{2l_q}^{2l_q}(\varepsilon)}{\sigma^{2l_q}(\varepsilon) l_q!}, \tag{34}$$

where $m_{2l_q}^{2l_q}(\varepsilon) = \int_{0<\rho<\varepsilon}\rho^{2l_q}\,\nu_\rho(d\rho)$, $g_n(u,v,\xi) = g_n^+(u,\xi) - g_n^+(v,\xi)$ with

$$g_n^+(y,\xi) = \frac{\mathrm{e}^{-iy\cdot\xi} - 1 - P_n(-iy\cdot\xi)\mathbf{1}_{\|\xi\|\le 1}}{\|\xi\|^{h(y)+d/2}},$$

and where $\sum_{L_m}$ stands for the sum over the set of partitions L_m of $\{1,\dots,2p_K\}$ in m subsets K_q such that the cardinality of K_q is $2l_q$ with $l_q\ge 1$.
Moreover, by lemmas 3.2 and 3.3 in [11], there exists $C>0$ such that

$$\forall (u,v)\in K^2,\ \|g_n(u,v,\cdot)\|_{2l_q}^{2l_q} \le C\|u-v\|^{2l_q\min(m_K,\beta)}. \tag{35}$$

Let us now study $\frac{m_{2l_q}^{2l_q}(\varepsilon)}{\sigma^{2l_q}(\varepsilon)}$ for $1\le l_q\le p_K$. Since

$$\frac{1}{2p_K} \le \frac{1}{2l_q} \le \frac{1}{2},$$

there exists $\theta\in[0,1]$ such that

$$\frac{1}{2l_q} = \frac{\theta}{2} + \frac{1-\theta}{2p_K}.$$

Therefore by the Hölder inequality,

$$\left(\int_{0<\rho<\varepsilon}\rho^{2l_q}\,\nu_\rho(d\rho)\right)^{\frac{1}{2l_q}} \le \left(\int_{0<\rho<\varepsilon}\rho^{2}\,\nu_\rho(d\rho)\right)^{\frac{\theta}{2}}\left(\int_{0<\rho<\varepsilon}\rho^{2p_K}\,\nu_\rho(d\rho)\right)^{\frac{1-\theta}{2p_K}}.$$

Consequently by **H2**,

$$m_{2l_q}(\varepsilon) \le C^{1-\theta}\sigma(\varepsilon). \tag{36}$$

Then owing to (34), (35) and (36), there exists $C > 0$ such that

$$\forall \varepsilon \le \varepsilon_0,\, \forall (u,v) \in K^2,\, \left\| Y_\varepsilon^+(u) - Y_\varepsilon^+(v) \right\|_{2p_K}^{2p_K} \le C\|u-v\|^{2p_K \min(m_K,\beta)}.$$

As $2p_K \min(m_K,\beta) > d$, $(Y_\varepsilon^+)_{\varepsilon \le \varepsilon_0}$ is a tight family in E_K.

Step 2: Tightness of $(Y_\varepsilon^-)_\varepsilon$

Let $x_0 \in K$. $\mathbb{E}\Big(|Y_\varepsilon^-(x_0)|^2\Big)$ does not depend on ε and is finite, which gives the tightness of $(Y_\varepsilon^-(x_0))_{\varepsilon>0}$. For the sake of clearness, for $\alpha = (\alpha_1 \cdots, \alpha_d) \in \mathbb{N}^d$ and $z = (z_1 \cdots, z_d) \in \mathbb{C}^d$, let

$$|\alpha| = \sum_{j=1}^d \alpha_j \text{ and } z^\alpha = \prod_{j=1}^d z_j{}^{\alpha_j}.$$

For $u \in \mathbb{R}^d$ and $0 < y < 1$,

$$Z_{\varepsilon,n}(u,y) = \frac{1}{\sigma(\varepsilon)} \int_{\|\xi\|\le 1} \frac{P_n(-iu\cdot\xi)}{\|\xi\|^{y+d/2}}\, M_{\varepsilon,1}(d\xi).$$

Therefore $Y_\varepsilon^-(u) = Z_{\varepsilon,n}(u,h(u))$. On the other hand,

$$Z_{\varepsilon,n}(u,y) = \sum_{1\le|\alpha|\le n} C_\alpha u^\alpha\, Y_\varepsilon^\alpha(y).$$

where for $\alpha = (\alpha_1 \cdots, \alpha_d) \in \mathbb{N}^d$,

$$Y_\varepsilon^\alpha(y) = \frac{1}{\sigma(\varepsilon)} \int_{\|\xi\|\le 1} \frac{(-i\xi)^\alpha}{\|\xi\|^{y+d/2}}\, M_{\varepsilon,1}(d\xi).$$

Then

$$\begin{aligned} |Y_\varepsilon^-(u) - Y_\varepsilon^-(v)| &\le |Z_{\varepsilon,n}(u,h(u)) - Z_{\varepsilon,n}(v,h(u))| + |Z_{\varepsilon,n}(v,h(u)) - Z_{\varepsilon,n}(v,h(v))| \\ &\le \sum_{1\le|\alpha|\le n} |C_\alpha\, Y_\varepsilon^\alpha(h(u))||u^\alpha - v^\alpha| + \sum_{1\le|\alpha|\le n} |C_\alpha\, v^\alpha||Y_\varepsilon^\alpha(h(u)) - Y_\varepsilon^\alpha(h(v))|. \end{aligned}$$

Since there exists $C > 0$ such that for every $(u,v) \in K^2$, $|u^\alpha - v^\alpha| \le C\|u-v\|$ and $|v^\alpha| \le C$, then

$$\left|Y_\varepsilon^-(u) - Y_\varepsilon^-(v)\right| \le C\left(\|u-v\| \sum_{1\le|\alpha|\le n} |Y_\varepsilon^\alpha(h(u))| + \sum_{1\le|\alpha|\le n} |Y_\varepsilon^\alpha(h(u)) - Y_\varepsilon^\alpha(h(v))| \right)$$

Therefore by continuity of h, to obtain the tightness of $(Y_\varepsilon^-)_\varepsilon$, it is sufficient to show that for each $\alpha \in \mathbb{N}^d$ such that $1 \le |\alpha| \le n$,

1. $(Y_\varepsilon^\alpha)_\varepsilon$ is tight in the space of the continuous functions on $[m_K, M_K]$, where $M_K = \max_K h$.
2. the family $\left(\sup_{[m_K,M_K]} |Y_\varepsilon^\alpha(y)|\right)_\varepsilon$ is tight.

According to [5], 2 is a consequence of 1. Moreover,

$$\mathbb{E}\Big(|Y_\varepsilon^\alpha(m_K)|^2\Big) = \mathbb{E}\Big(|Y_1^\alpha(m_K)|^2\Big) < +\infty$$

and by a Taylor expansion,

$$\forall \varepsilon > 0,\, \forall (y,y') \in [m_K, M_K]^2,\, \mathbb{E}\Big(|Y_\varepsilon^\alpha(y) - Y_\varepsilon^\alpha(y')|^2\Big) \le C|y-y'|^2,$$

which by the Kolmogorov criterion gives 1 and concludes the proof. □

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