Fields with exceptional tangent fields

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Abstract

The asymptotic self-similarity property describes the local structure of a random field. In this paper, we introduce a locally asymptotically self-similar second order field $X_{H,\beta}$ whose local structures at $x = 0$ and at $x \neq 0$ are very far from each other. More precisely, whereas its tangent field at $x \neq 0$ is a Fractional Brownian Motion, its tangent field at $x = 0$ is a Fractional Stable Motion. In addition, $X_{H,\beta}$ is asymptotically self-similar at infinity with a Gaussian field, which is not a Fractional Brownian Motion, as tangent field. Then, its trajectories regularity is studied. Finally, the Hausdorff dimension of its graphs is given.

Key-words: Local Asymptotic Self-Similarity, Tangent fields, Infinitely Divisible Distributions.

1 Introduction

The Fractional Brownian Motion, in short FBM, is probably the most famous self-similar field. Furthermore, the FBM, introduced in [16, 13], provides a very powerful model in applied mathematics. Actually, it allows to model some phenomena in finance, hydrology, economics or telecommunications for example. In addition, several self-similar random fields, such as harmonizable fractional stable motions, has been studied in [18]. Since the self-similarity property is a global property of invariance with respect to change of scales, a local version has been defined in [8]. Then, this local property extends the range of applications. More precisely, a field $(Y(x))_{x\in\mathbb{R}^d}$ is

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locally asymptotically self-similar, in short lass, at point x_0 with index H if

$$\lim_{\varepsilon\to 0_+}\left(\frac{Y(x_0+\varepsilon u)-Y(x_0)}{\varepsilon^H}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}(T_{x_0}(u))_{u\in\mathbb{R}^d},$$

where T_{x_0} is a non-degenerate field called the tangent field at point x_0. Moreover, T_{x_0} is necessarily a self-similar process with index H. Furthermore, general properties of tangent fields are given in [11] in the framework of continuous fields and in [12] in the case of processes with jumps. In addition, the most famous lass field is certainly the Multifractional Brownian Motion, in short MBM, introduced independently in [8] and in [17]. However, many other examples have been studied, see for instance [1, 2, 3, 7, 4, 15]. For all these fields, at each point x, the tangent field is a FBM. In addition, the lass Gaussian fields introduced in [9] do not have FBM as tangent field; however, these fields have stationary increments and then the same Gaussian field as tangent field at each point. Furthermore, [6] has defined some lass fields, called Moving Average Fractional Lévy Motions (in short MAFLMs), whose tangent fields are not Gaussian. Nevertheless, the tangent field of a MAFLM at point x does not depend on x: it is the same Fractional Stable Motion. Hence, in all encountered examples of lass fields, tangent fields are the same at each point with the same index of self-similarity or are at least of the same kind. Then, in term of tangent fields, the local behaviour of these lass fields does not depend on the position. Hence, one can wonder if the local behaviour at point x of a lass field may vary with x. In particular, does it exist some lass fields with radical changes of the kind of the tangent fields ?

This paper introduces a second-order lass field $X_{H,\beta}$ whose tangent field at point x is not the same whether $x=0$ or not. Actually, its tangent field at $x=0$ is a Real Harmonizable Fractional Stable Motion (in short RHFSM) with index $\widetilde{H}$ whereas at $x\neq 0$ it is a FBM with index H. Hence, its local behaviour at $x=0$ and $x\neq 0$ are very different from each other since the tangent field at $x=0$ is not Gaussian. In addition, $X_{H,\beta}$ is a stochastic integral process, i.e. $X_{H,\beta}(x)$ can be defined as a stochastic integral under a random measure $\widetilde{N}$ which does not depend on x. Actually,

$$X_{H,\beta}(x)=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\right)\psi\Big(\|x\|^{\beta}|z|\Big)\,\widetilde{N}(d\xi,dz),$$

where $\|\cdot\|$ denotes the Euclidean norm and $\widetilde{N}=N-n$ with $N(d\xi,dz)$ a Poisson random measure whose intensity is $n(d\xi,dz)=d\xi\nu(dz)$. Remark that the definition of $X_{H,\beta}$ and of RHFLMs are similar. However, in the case of RHFLMs, $\nu(dz)$ has a second order moment, which won't be fulfilled in this paper. In addition, $X_{H,\beta}$ satisfies an asymptotic self-similarity property at infinity. Actually, when the increments are taken at large scales, the limit field is Gaussian but it is not a FBM. Moreover, the index of self-similarity at $x = 0$, $x \neq 0$ and infinity are different from each other. Then, $X_{H,\beta}$ exhibits different behaviours at low and large scales. Furthermore, its sample paths are locally hölderian on $\mathbb{R}^d \backslash \{0\}$. In spite of its special behaviour at $x = 0$, the Hausdorff dimension of its graphs can be computed and is the same as the FBM B_H one.

The next section is devoted to the construction of $X_{H,\beta}$. Then, in section 3, the lass property is established. Moreover, an asymptotic self-similarity property at infinity is stated. In the last section, the trajectories regularity and the Hausdorff dimension of the graph are studied.

2 Definition

Let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with mean measure $n(d\xi, dz) = \mathbb{E}(N(d\xi, dz)) = d\xi\, \nu(dz)$. Throughout the following, $\nu(dz)$ is a rotationally invariant measure associated with a symmetric α-stable measure. More precisely, let P be the map $P\left(\rho e^{i\theta}\right) = (\theta, \rho) \in [0, 2\pi) \times (0, +\infty)$. Then, in the following,

$$P(\nu(dz)) = d\theta\, \frac{d\rho}{\rho^{1+\alpha}},$$

where $d\theta$ is the Lebesgue measure on $[0, 2\pi)$, $d\rho$ is the Lebesgue measure on $(0, +\infty)$ and $0 < \alpha < 2$.

Let $\widetilde{N} = N - n$ and $\varphi \in L^2\left(\mathbb{R}^d \times \mathbb{C}\right)$ for the measure $n(d\xi, dz)$. It is then classical to define the stochastic integral $\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\, \widetilde{N}(d\xi, dz)$. Then, the map $\varphi \mapsto \int \varphi\, d\widetilde{N}$ is an isometry, which means that

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\, \widetilde{N}(d\xi, dz)\right|^2\right] = \int_{\mathbb{R}^d \times \mathbb{C}} |\varphi(\xi, z)|^2\, n(d\xi, dz), \tag{2.1}$$

Furthermore, if φ is a real-valued function, then $\int \varphi\, d\widetilde{N}$ is a real-valued random variable and for every $u \in \mathbb{R}$,

$$\mathbb{E}\left[e^{iu \int \varphi\, d\widetilde{N}}\right] = \exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} [\exp\left(iu\varphi(\xi, z)\right) - 1 - iu\varphi(\xi, z)]\, d\xi\, \nu(dz)\right]. \tag{2.2}$$

Notation Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$-function such that

$$\psi(u) = \begin{cases} 1 & \text{if } |u| \leq 1/2 \\ 0 & \text{if } |u| \geq 1. \end{cases}$$

Then, ψ and its derivatives are with compact support and then bounded.

Let us now define the field $X_{H,\beta}$ which is a centered real-valued field.

Definition 1. *Let $\beta \in \mathbb{R}$. Then, for every $x \in \mathbb{R}^d$,*

$$X_{H,\beta}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} z \right) \psi\Big(\|x\|^\beta |z| \Big)\, \widetilde{N}(d\xi, dz),$$

with convention $\|x\|^0 = 1$ and if $\beta < 0$ and $x = 0$, $\psi\Big(\|x\|^\beta |z|\Big) = 0$.

Since the map $\varphi \mapsto \int \varphi\, d\widetilde{N}$ is an isometry from $L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$, for the measure $n(d\xi, dz)$, onto $L^2(\Omega)$, $X_{H,\beta}$ is a field with finite second order moments and one can evaluate its covariance function.

Proposition 1. *Let γ be the covariance function of $X_{H,\beta}$, i.e.*

$$\gamma(x, y) = \mathbb{E}(X_{H,\beta}(x) X_{H,\beta}(y)),$$

for every $(x, y) \in \big(\mathbb{R}^d\big)^2$. Then, if $x \neq 0$ and $y \neq 0$, $\gamma(x, y)$ is equal to

$$\frac{C_H^2}{2} \Big(\|x\|^{2H} + \|y\|^{2H} - \|x - y\|^{2H} \Big) \int_0^{+\infty} \rho^{1-\alpha} \psi\Big(\|x\|^\beta \rho\Big) \psi\Big(\|y\|^\beta \rho\Big)\, d\rho,$$

where

$$C_H = \left(4\pi \int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\, d\xi \right)^{1/2}, \tag{2.3}$$

with $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.
Otherwise, i.e. if $x = 0$ or $y = 0$, $\gamma(x, y) = 0$.

Especially, for every $x \in \mathbb{R}^d \backslash \{0\}$,

$$\mathbb{E}\big(X_{H,\beta}^2(x)\big) = C_H^2 \int_0^{+\infty} \rho^{1-\alpha} \psi^2(\rho)\, d\rho\, \|x\|^{2H - \beta(2-\alpha)}.$$

Then, if $\beta > 2H/(2-\alpha)$, $X_{H,\beta}$ is not continuous in quadratic mean at 0.

Proof. see [14]. □

In the sequel, $0 < H < 1$. Moreover, $X_{H,\beta}$ is the field associated with (H, β) by the definition 1.

3 Asymptotic Self-Similarity

Like RHFLMs, $X_{H,\beta}$ is locally asymptotically self-similar at each point. However, whereas a RHFLM has the same tangent field at each point, in general, it does not remain true for $X_{H,\beta}$. Furthermore, this section is concluded by the study of the asymptotic of the field when the increments are taken at large scales.

Before we study the field $X_{H,\beta}$, let us first define

$$Y_H(x,y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\right)\psi(y|z|)\,\widetilde{N}(d\xi,dz), \tag{3.1}$$

where $x\in\mathbb{R}^d$ and $y>0$. Furthermore, let $Y_H(0,0)=0$.
Then, for every $x\in\mathbb{R}^d$,

$$X_{H,\beta}(x) = Y_H\Big(x,\|x\|^\beta\Big). \tag{3.2}$$

Then, the asymptotic self-similarity of $X_{H,\beta}$ at $x\neq 0$, see theorem 1, will be proved using (3.2) and the lass property of RHFLMs. Actually, for every $y>0$, $(Y_H(x,y))_{x\in\mathbb{R}^d}$ is a RHFLM with index H, see lemma 1, and then is a lass field according to [4].
Notation For every $y>0$, $\nu(y,dz)$ denotes the push-forward of $\nu(dz)$ by the map $z\longmapsto z\,\psi(y|z|)$. Hence, for every $y>0$, $\nu(y,dz)$ is a rotationally invariant measure such that $\int_{\mathbb{C}}|z|^p\,\nu(y,dz)<+\infty$ for every $p\geq 2$.

Lemma 1. *Let $y>0$. Then, the field $(Y_H(x,y))_{x\in\mathbb{R}^d}$ is a RHFLM with index H associated with a Poisson random measure $N_y(d\xi,dz)$ whose mean measure is $n_y(d\xi,dz)=d\xi\,\nu(y,dz)$.*

Proof. It is deduced from (2.2) and the definition of RHFLMs. □

Let us now state the lass property at point $x\neq 0$.

Theorem 1. *Let $x\in\mathbb{R}^d\backslash\{0\}$. Then, the field $X_{H,\beta}$ is locally asymptotically self-similar with index H and tangent FBM at point x. More precisely,*

$$\lim_{\varepsilon\to 0_+}\left(\frac{X_{H,\beta}(x+\varepsilon u)-X_{H,\beta}(x)}{\varepsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (C(x,H)B_H(u))_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields, B_H is a standard FBM with index H and

$$C(x,H) = \frac{C_H}{\|x\|^{\beta(1-\alpha/2)}}\left(\int_0^{+\infty}\rho^{1-\alpha}\psi^2(\rho)\,d\rho\right)^{1/2} \tag{3.3}$$

where C_H is defined by (2.3)*.*

Proof. Let $V_\varepsilon(u) = \dfrac{X_{H,\beta}(x+\varepsilon u) - X_{H,\beta}(x)}{\varepsilon^H}$.
Since $x \neq 0$, $V_\varepsilon(u)$ can be split into

$$V_{\varepsilon,1}(u) = \frac{Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big) - X_{H,\beta}(x)}{\varepsilon^H}$$

and

$$V_{\varepsilon,2}(u) = \frac{X_{H,\beta}(x+\varepsilon u) - Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big)}{\varepsilon^H}.$$

Then, the asymptotic of $V_{\varepsilon,1}$ is first studied.
Step 1 For every $u \in \mathbb{R}^d$,

$$V_{\varepsilon,1}(u) = \frac{Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big) - Y_H\Big(x, \|x\|^\beta\Big)}{\varepsilon^H}.$$

Since $x \neq 0$, according to lemma 1, $\Big(Y_H\Big(v, \|x\|^\beta\Big)\Big)_{v\in\mathbb{R}^d}$ is a RHFLM associated with a Poisson random measure whose mean measure is $d\xi\, \nu\Big(\|x\|^\beta, dz\Big)$.
Then, by applying proposition 3.1 in [4], one deduces that

$$\lim_{\varepsilon\to 0_+} (V_{\varepsilon,1}(u))_{u\in\mathbb{R}^d} \overset{(d)}{=} (C(x,H)B_H(u))_{u\in\mathbb{R}^d}, \tag{3.4}$$

where B_H is a standard FBM with index H and $C(x,H)$ is defined by (3.3).
Step 2 Now let us prove that

$$\lim_{\varepsilon\to 0_+} (V_{\varepsilon,2}(u))_{u\in\mathbb{R}^d} \overset{(d)}{=} 0. \tag{3.5}$$

Let us fix $u \in \mathbb{R}^d$. Then by (2.1) and by definition of $\nu(dz)$,

$$\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big) = \frac{4\pi\, I_\varepsilon(u)}{\varepsilon^{2H}} \int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-i(x+\varepsilon u)\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\, d\xi,$$

where $I_\varepsilon(u) = \displaystyle\int_0^{+\infty} \rho^{1-\alpha}\Big(\psi\Big(\|x+\varepsilon u\|^\beta \rho\Big) - \psi\Big(\|x\|^\beta\rho\Big)\Big)^2\, d\rho$.
Hence,

$$\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big) = C_H^2\, \|x+\varepsilon u\|^{2H}\, I_\varepsilon(u)\, \varepsilon^{-2H}.$$

Remark that $I_\varepsilon(0) = 0$. Then, suppose that $u \neq 0$. Let us define the compact set $K_\varepsilon = \{y \in \mathbb{R}^d \,/\, \|y - x\| \leq \varepsilon \|u\|\}$ and fix $\varepsilon_0 > 0$ such that $0 \notin K_{\varepsilon_0}$. Also, $m_{K_{\varepsilon_0}} = \min_{v \in K_{\varepsilon_0}} \|v\|^\beta > 0$.
Then, since the support of ψ is included in $[-1, 1]$ and $x + \varepsilon u \in K_{\varepsilon_0}$,

$$\forall \varepsilon \leq \varepsilon_0,\ \forall \rho \geq m_{K_{\varepsilon_0}}^{-1},\ \psi\Big(\|x + \varepsilon u\|^\beta \rho\Big) - \psi\Big(\|x\|^\beta \rho\Big) = 0.$$

Furthermore, owing to a Taylor expansion, one proves that there exists $D \in \mathbb{R}_+$ such that

$$\forall \varepsilon \leq \varepsilon_0,\ \left|\psi\Big(\|x + \varepsilon u\|^\beta \rho\Big) - \psi\Big(\|x\|^\beta \rho\Big)\right| \leq D \rho \varepsilon\, \mathbf{1}_{[0, m_{K_{\varepsilon_0}}^{-1}]}(\rho).$$

Then, since $m_{K_{\varepsilon_0}} > 0$ and since $0 < \alpha < 2$, there exists $D' \in \mathbb{R}_+$ such that

$$\forall \varepsilon \leq \varepsilon_0,\ 0 \leq I_\varepsilon(u) \leq D' \varepsilon^2,$$

which implies (3.5) and concludes the proof in view of (3.4). □

Hence, $X_{H,\beta}$ is lass at each point $x \in \mathbb{R}^d \backslash \{0\}$ with tangent FBM. The next theorem proves that it remains lass at $x = 0$ when $\beta > d/\alpha$. However, the tangent field is not any more a Gaussian field.

Theorem 2. *Let $\beta > d/\alpha$ and $\widetilde{H} = H + d/2 - d/\alpha$. Assume that $\widetilde{H} > 0$. Then, $X_{H,\beta}$ is locally asymptotically self-similar at point $x = 0$ with index $\widetilde{H}$ and tangent field RHFSM, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\beta}(\varepsilon u)}{\varepsilon^{\widetilde{H}}}\right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \big(S_{\widetilde{H}}(u)\big)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{H}}$ is a RHFSM that has the representation

$$S_{\widetilde{H}}(u) = \Re\left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{H} + d/\alpha}}\, M_\alpha(d\xi)\right) \tag{3.6}$$

with M_α a complex isotropic symmetric α-stable random measure.

Let us remark that $\widetilde{H} < H < 1$. Then, since $\widetilde{H} > 0$, by assumption, the RHFSM $S_{\widetilde{H}}$ is well defined.

Proof. Let $p \in \mathbb{N}\backslash\{0\}$, $(v_1, \ldots, v_p) \in \mathbb{R}^p$, $(u_1, \ldots, u_p) \in \left(\mathbb{R}^d\right)^p$. Then,

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{\widetilde{H}}}\right)\right] = \exp\left(\varphi(\varepsilon)\right)$$

where

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d\times\mathbb{C}} [\exp\left(iK_\varepsilon(\xi, z)\right) - 1 - iK_\varepsilon(\xi, z)] d\xi\, \nu(dz)$$

with

$$K_\varepsilon(\xi, z) = 2\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-i\varepsilon u_k\cdot\xi} - 1}{\varepsilon^{\widetilde{H}} \|\xi\|^{H+d/2}} z\psi\left(\varepsilon^\beta \|u_k\|^\beta |z|\right)\right).$$

First, let us remark that

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} \left[\cos\left(K_\varepsilon\left(\xi, \rho\mathrm{e}^{i\theta}\right)\right) - 1\right] \frac{d\rho}{\rho^{1+\alpha}}\, d\theta\, d\xi.$$

Also, by applying the change of variable $\lambda = \varepsilon\xi$ and $r = \varepsilon^{d/\alpha}\rho$,

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} \widetilde{\varphi_\varepsilon}(\lambda, r, \theta)\, dr\, d\theta\, d\lambda,$$

where

$$\widetilde{\varphi_\varepsilon}(\lambda, r, \theta) = \frac{1}{r^{1+\alpha}}\left[\cos\left(\widetilde{K_\varepsilon}(\lambda, r, \theta)\right) - 1\right]$$

with

$$\widetilde{K_\varepsilon}(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta}\psi\left(\varepsilon^{\beta-d/\alpha}\|u_k\|^\beta r\right)\right).$$

Remark that since $\beta > d/\alpha$, $\lim_{\varepsilon\to 0_+} \varepsilon^{\beta-d/\alpha} = 0$. Thus, by continuity of ψ and since $\psi(0) = 1$,

$$\widetilde{K}(\lambda, r, \theta) = \lim_{\varepsilon\to 0_+} \widetilde{K_\varepsilon}(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta}\right).$$

Furthermore,

$$|\widetilde{\varphi_\varepsilon}(\lambda, r, \theta)| \leq \widetilde{\varphi}(\lambda, r, \theta)$$

with

$$\widetilde{\varphi}(\lambda, r, \theta) = F^2(\lambda) r^{1-\alpha}\, \mathbf{1}_{F(\lambda)r\leq 1} + \frac{2}{r^{1+\alpha}}\, \mathbf{1}_{F(\lambda)r>1}$$

where $F(\lambda)=\sum_{k=1}^{p}|v_k|\frac{\left|\mathrm{e}^{-iu_k\cdot\lambda}-1\right|}{\|\lambda\|^{H+d/2}}$.
Using that $0<\widetilde{H}<1$, one easily shows that $\widetilde{\varphi}\in L^1\big(\mathbb{R}^d\times(0,+\infty)\times[0,2\pi]\big)$. Consequently, by a dominated convergence argument,

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varphi(\varepsilon) &= \int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty}\left[\cos\left(K(\lambda,r,\theta)\right)-1\right]\frac{dr}{r^{1+\alpha}}\,d\theta\,d\lambda \\ &= -C(\alpha)\int_0^{2\pi}|2\cos(\theta)|^\alpha\,d\theta\int_{\mathbb{R}^d}\left|\sum_{k=1}^{p}v_k\frac{\mathrm{e}^{-iu_k\cdot\lambda}-1}{\|\lambda\|^{H+d/2}}\right|^\alpha d\lambda\end{aligned}$$

with $C(\alpha)=\displaystyle\int_0^{+\infty}(1-\cos(r))\frac{dr}{r^{1+\alpha}}$.
As a consequence, since $H+d/2=\widetilde{H}+d/\alpha$,

$$\lim_{\varepsilon\to 0_+}\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p}v_k\frac{X_{H,\beta}(u_k)}{\varepsilon^{\widetilde{H}}}\right)\right]=\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p}v_kS_{\widetilde{H}}(u_k)\right)\right],$$

where $S_{\widetilde{H}}$ is defined by (3.6), which concludes the proof. □

Therefore, when $\beta>d/\alpha$ and when $0<\widetilde{H}<1$, $X_{H,\beta}$ is lass with multifractional function

$$\begin{array}{rcl} h: \mathbb{R}^d & \longrightarrow & (0,1) \\ x & \longmapsto & \begin{cases} H & \text{if } x\neq 0 \\ \widetilde{H} & \text{if } x=0. \end{cases}\end{array}$$

More precisely, for every $x\in\mathbb{R}^d$, $X_{H,\beta}$ is lass at point x with index $h(x)$. On the one hand, even if $X_{H,\beta}$ has finite second order moments, the tangent field at $x=0$, which is a RHFSM, does not. On the other hand, the tangent field at $x\neq 0$ is a FBM, which is a Gaussian model. Therefore the behaviours of $X_{H,\beta}$ at $x=0$ and $x\neq 0$ are very far from each other.

As in the case of RHFLMs, the asymptotic self-similarity is now study at large scale. Actually, when the increments are taken at large scale, the limit is a Gaussian model when $\beta>d/\alpha$. However, since its increments are not stationary, it is not a FBM.

Proposition 2. *Let $\beta>d/\alpha$. Then, the field $X_{H,\beta}$ is asymptotically self-similar with index $H-\beta(1-\alpha/2)$, in the sense that*

$$\lim_{R\to+\infty}\left(\frac{X_{H,\beta}(Ru)}{R^{H-\beta(1-\alpha/2)}}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}\left(W_{H,\beta}(u)\right)_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $W_{H,\beta}$ is a centered Gaussian field which has the same covariance as $X_{H,\beta}$, covariance stated in proposition 1.

In particular, $X_{H,\beta}$ is asymptotically self-similar at infinity with index $H-\beta(1-\alpha/2)$, as it would be expected if it were a Gaussian field.

Proof. Let $p \in \mathbb{N}\backslash\{0\}$, $(v_1,\dots,v_p)\in\mathbb{R}^p$, $(u_1,\dots,u_p)\in\left(\mathbb{R}^d\right)^p$.
Then, the change of variables done in the proof of proposition 2 leads to:

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(Ru_k)}{R^{H-\beta(1-\alpha/2)}}\right)\right] = \exp\left(\varphi(R)\right)$$

where $\varphi(R)$ is equal to

$$\int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} R^{-d+\beta\alpha}\left[\exp\left(iR^{\frac{d-\beta\alpha}{2}}K\left(\lambda, r\mathrm{e}^{i\theta}\right)\right)-1-iR^{\frac{d-\beta\alpha}{2}}K\left(\lambda, r\mathrm{e}^{i\theta}\right)\right]\frac{dr}{r^{1+\alpha}}\,d\theta\,d\lambda.$$

with

$$K(\lambda,r,\theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda}-1}{\|\lambda\|^{H+d/2}}\mathrm{e}^{i\theta}\psi\left(\|u_k\|^{\beta} r\right)\right).$$

Remark that $\lim_{R\to+\infty} R^{\frac{d-\beta\alpha}{2}} = 0$ since $\beta > d/\alpha$. Then, using a dominated convergence argument, one easily concludes that

$$\lim_{R\to+\infty}\varphi(R) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} K^2\left(\lambda, r\mathrm{e}^{i\theta}\right)d\lambda\,d\theta\,\frac{dr}{r^{1+\alpha}}.$$

Hence, by definition of $W_{H,\beta}$, it is straightforward that

$$\lim_{R\to+\infty}\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(Ru_k)}{\varepsilon^{H-\beta(1-\alpha/2)}}\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k W_{H,\beta}(u_k)\right)\right],$$

which concludes the proof. □

4 Trajectories Regularity and Hausdorff dimension

In section 4.1, the trajectories regularity of $X_{H,\beta}$ is studied. In addition, [5] has linked the Hausdorff dimension to the lass property and the trajectories regularity. Therefore, section 4.2 is devoted to the study of the Hausdorff dimension of the graphs of $X_{H,\beta}$.

4.1 Trajectories Regularity

In general, in order to study the trajectories regularity of a field $(Y(x))_{x\in\mathbb{R}^d}$, one evaluates the moments of its increments $\mathbb{E}(|Y(x)-Y(y)|^q)$ and applies the Kolmogorov criterion. However, $X_{H,\beta}$ may not have moments of order greater than two. In fact, since the law of $X_{H,\beta}$ is an infinitely divisible law, owing to theorem 25.3 in [19], one easily proves that for every $q\geq 2$ and every $x\in\mathbb{R}^d\backslash\{0\}$,

$$\mathbb{E}(|X_{H,\beta}(x)|^q)<+\infty \quad\Leftrightarrow\quad H<1-\tfrac{d}{2}+\tfrac{d}{q}.$$

Then, following [4], $X_{H,\beta}$ is split into two fields $X^+_{H,\beta}$ and $X^-_{H,\beta}$ where $X^+_{H,\beta}$ has moments of every order and $X^-_{H,\beta}$ has almost surely $\mathcal{C}^1$-sample paths on $\mathbb{R}^d\backslash\{0\}$. Let us quickly define $X^+_{H,\beta}$ and $X^-_{H,\beta}$.

Let $n\in\mathbb{N}$ such that $n\geq d/2$ and $P_n(t)=\sum_{k=1}^n \frac{t^k}{k!}$.
Let us define

$$g_n^+(x,\xi)=\frac{\mathrm{e}^{-ix\cdot\xi}-1-P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{H+d/2}}. \tag{4.1}$$

Then,

$$X^+_{H,\beta}(x)=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\big(g_n^+(x,\xi)z\big)\psi\Big(\|x\|^\beta|z|\Big)\,\widetilde{N}(d\xi,dz),$$

has moments of every order.
Hence, $X_{H,\beta}=X^+_{H,\beta}+X^-_{H,\beta}$ where the field $\Big(X^-_{H,\beta}(x)\Big)_{x\in\mathbb{R}^d}$ is defined by

$$X^-_{H,\beta}(x)=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\left(\frac{P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{H+d/2}}z\right)\psi\Big(\|x\|^\beta|z|\Big)\,\widetilde{N}(d\xi,dz).$$

Let us first study the trajectories regularity of $X^+_{H,\beta}$.

Lemma 2. *For every $H'<H$, there exists a modification of $X^+_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d\backslash\{0\}$.*

Proof. Let $K\subset\mathbb{R}^d$ be a compact set such that $0\notin K$. Then, let $p\in\mathbb{N}\backslash\{0\}$. By differentiation of the characteristic function of $X^+_{H,\beta}(x)-X^+_{H,\beta}(y)$, given by (2.2), one proves that

$$\mathbb{E}\left[\Big(X^+_{H,\beta}(x)-X^+_{H,\beta}(y)\Big)^{2p}\right]=\sum_{q=1}^{p}\sum_{l\in\mathcal{P}_q}\prod_{n=1}^{p}\int_{\mathbb{R}^d\times\mathbb{C}}|\widetilde{g_n}(x,y,\xi,z)|^{2l_n}d\xi\,\nu(dz), \tag{4.2}$$

where $\widetilde{g_n}(x,y,\xi,z) = 2\Re\Big(g_n^+(x,\xi)z\psi\Big(\|x\|^\beta|z|\Big) - g_n^+(y,\xi)z\psi\Big(\|y\|^\beta|z|\Big)\Big)$ and

$$\mathcal{P}_q = \Big\{l = (l_1,\dots,l_q) \in (\mathbb{N}\backslash\{0\})^d,\ l_1 + \cdots + l_q = p\Big\}.$$

Therefore, let us study

$$I_q(x,y) = \int_{\mathbb{R}^d\times\mathbb{C}} |\widetilde{g_n}(x,y,\xi,z)|^{2q}\, d\xi\, \nu(dz)$$

for every $q \in \mathbb{N}\backslash\{0\}$. By the Minkowski inequality,

$$I_q(x,y) \le (A_q(x,y) + B_q(x,y))^q, \tag{4.3}$$

where

$$A_q(x,y) = \left(\int_{\mathbb{R}^d\times\mathbb{C}} \Big|2\Re\big(g_n^+(x,\xi)z - g_n^+(y,\xi)z\big)\Big|^{2q} \psi^{2q}\Big(\|x\|^\beta z\Big)\, d\xi\, \nu(dz)\right)^{1/q}$$

and

$$B_q(x,y) = \left(\int_{\mathbb{R}^d\times\mathbb{C}} \Big|2\Re\big(g_n^+(y,\xi)z\big)\Big|^{2q} \Big(\psi\Big(\|x\|^\beta z\Big) - \psi\Big(\|y\|^\beta z\Big)\Big)^{2q}\, d\xi\, \nu(dz)\right)^{1/q}.$$

Step 1 Since $\nu(dz)$ is rotationally invariant,

$$A_q^q(x,y) = 2^{2q}\widetilde{I}_q(x,y)\int_{\mathbb{C}} \Re^{2q}(z)\psi^{2q}\Big(\|x\|^\beta|z|\Big)\,\nu(dz),$$

where

$$\widetilde{I}_q(x,y) = \int_{\mathbb{R}^d} \frac{\Big|\mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi} + [P(-iy\cdot\xi) - P(-ix\cdot\xi)]\mathbf{1}_{\|\xi\|\le 1}\Big|^{2q}}{\|\xi\|^{dq+2qH}}\, d\xi.$$

Moreover, there exists a constant $D \in \mathbb{R}_+$, see lemma 3.2 in [15], such that

$$\forall (x,y) \in K^2,\ \widetilde{I}_q(x,y) \le D\|x-y\|^{2qH}.$$

In addition, if $x \neq 0$,

$$\int_{\mathbb{C}} \Re^{2q}(z)\psi^{2q}\Big(\|x\|^\beta|z|\Big)\,\nu(dz) = \frac{1}{\|x\|^{\beta(2q-\alpha)}}\int_0^{2\pi}\cos^{2q}(\theta)\,d\theta\int_0^{+\infty}\rho^{2q-\alpha-1}\psi^{2q}(\rho)\,d\rho.$$

Then, since K is a compact set such that $0 \notin K$, there exists a constant $D \in \mathbb{R}_+$ such that

$$\forall (x,y) \in K^2,\ A_q^q(x,y) \le D\|x-y\|^{2qH}. \tag{4.4}$$

Step 2 By definition of $\nu(dz)$,

$$B_q^q(x,y) = J(y) \int_0^{+\infty} \rho^{2q-\alpha-1} \Big(\psi\Big(\|x\|^\beta \rho\Big) - \psi\Big(\|y\|^\beta \rho\Big)\Big)^{2q} d\rho,$$

where $J(y) = 2^{2q} \int_0^{2\pi} \cos^{2q}(\theta)\, d\theta \int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-iy\cdot\xi} - 1 - P(-iy\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}\right|^{2q}}{\|\xi\|^{dq+2qH}} d\xi$. It is straightforward to prove that $\sup_{v\in K} J(v) < +\infty$.
Then, one proceeds as in step 2 of the proof of theorem 1. In particular, one proves that there exists a constant $D \in \mathbb{R}_+$, such that for every $(x,y) \in K^2$,

$$\Big|\psi\Big(\|x\|^\beta \rho\Big) - \psi\Big(\|y\|^\beta \rho\Big)\Big| \leq D\, \rho \|x-y\|\, \mathbf{1}_{[0, m_K^{-1}]}(\rho).$$

Since $m_K > 0$ and $0 < \alpha < 2$, there exists a constant $D \in \mathbb{R}_+$ such that

$$\forall (x,y) \in K^2,\ B_q^q(x,y) \leq D\|x-y\|^{2q}. \tag{4.5}$$

Step 3 In view of (4.2), (4.3), (4.4) and (4.5), for every $(x,y) \in K^2$,

$$\mathbb{E}\left[\Big(X_{H,\beta}^+(x) - X_{H,\beta}^+(y)\Big)^{2p}\right] \leq D\|x-y\|^{2pH},$$

where the constant $D \in \mathbb{R}_+$ only depends on p and K. Since p can be chosen such that $2pH > d$, the Kolmogorov criterion gives the conclusion. ☐

Lemma 3. *There exists a modification of $X_{H,\beta}^-$ which has $\mathcal{C}^1$-sample paths on $\mathbb{R}^d\backslash\{0\}$.*

Proof. This proof is based on an analogous scheme as the proof of lemma 3.5 in [15].

Notice that for every $x \in \mathbb{R}^d\backslash\{0\}$

$$X_{H,\beta}^-(x) = Y_{H,\beta}^-\Big(x, \|x\|^\beta\Big),$$

where the field $\Big(Y_{H,\beta}^-(x,y)\Big)_{x\in\mathbb{R}^d,\, y>0}$ is defined as follows:

$$Y_{H,\beta}^-(x,y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{P_n(-ix\cdot\xi)}{\|\xi\|^{H+d/2}} z\right) \psi(y|z|)\, \mathbf{1}_{\|\xi\|\leq 1}\, \widetilde{N}(d\xi, dz). \tag{4.6}$$

Then, for every $y > 0$, let

$$Z_\gamma(y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{i^{|\gamma|}\prod_{j=1}^d \xi_j^{\gamma_j}}{\|\xi\|^{H+d/2}} z\right) \psi(y|z|)\, \mathbf{1}_{\|\xi\|\leq 1}\, \widetilde{N}(d\xi, dz),$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$ is such that $1 \leq |\gamma| = \sum_{j=1}^n \gamma_j \leq n$.
Hence, thanks to corollary page 69 in [10], one proves that there exists a modification of Z_γ which has almost surely $\mathcal{C}^1$-sample paths on $(0, +\infty)$, which concludes the proof since P_n is a polynomial. □

The next proposition is an immediate consequence of lemmas 2 and 3.

Proposition 3. *For every $H' < H$, there exists a modification of $X_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d \backslash \{0\}$.*

From proposition 3 and the lass property, the pointwise Hölder exponent of $X_{H,\beta}$ at point $x \neq 0$ can be given. Firstly, let us recall that the pointwise Hölder exponent of a function $f : \mathbb{R}^d \longrightarrow (0,1)$ at point x is

$$H_f(x) = \sup \left\{ \gamma > 0,\ \lim_{y \to 0} \frac{f(x+y) - f(x)}{\|y\|^\gamma} = 0 \right\}.$$

Corollary 1. *At every point $x \neq 0$, the pointwise Hölder exponent $H_{X_{H,\beta}}(x)$ of $X_{H,\beta}$ is almost surely equal to H.*

Proof. see proof of proposition 2.3 in [4]. □

Remark When $\beta > d/\alpha$, from the lass property at $x = 0$, one can deduce that, almost surely, $H_{X_{H,\beta}}(0) \leq \widetilde{H}$. Hence, $H_{X_{H,\beta}}(0) < H$. In particular, for every H' such that $\widetilde{H} < H' < H$, the trajectories of $X_{H,\beta}$ are not locally H'-Hölder on whole $\mathbb{R}^d$.

4.2 Hausdorff dimension

Let $U \subset \mathbb{R}^d$ be a compact set. Then, the graph of $X_{H,\beta}$ on U is

$$\operatorname{graph}\left({X_{H,\beta}}_{|U} \right) = \{(x, X_{H,\beta}(x)),\ x \in U\}.$$

The Hausdorff dimension of $\operatorname{graph}\left({X_{H,\beta}}_{|U} \right)$, denoted by $\dim_{\mathcal{H}} {X_{H,\beta}}_{|U}$, can be computed owing to [5].

Proposition 4. *Let $K = \prod_{i=1}^d [a_i, b_i]$, $(a_i, b_i) \in \mathbb{R}^2$ and $a_i < b_i$. Then the Hausdorff dimension of the graph of $X_{H,\beta}$ on K is almost surely equal to $d + 1 - H$.*

Proof. Let us remark that

$$\dim_{\mathcal{H}} X_{H,\beta_{|K}} = \dim_{\mathcal{H}} \{(x, X_{H,\beta}(x)),\ x \in K\backslash\{0\}\} = \dim_{\mathcal{H}} X_{H,\beta_{|K^*}},$$

where $K^* = K\backslash\{0\}$. Furthermore, when $0 \in K$, K^* can be written as a countable union of blocks which do not contain 0. Hence, it is sufficient to prove the proposition in the case where $0 \notin K$.

Therefore, let us now assume that $0 \notin K$. Then, we will prove that $X_{H,\beta}$ satisfies the assumptions of theorem 2.1 in [5].

Let $(x, y) \in (\mathbb{R}^d)^2$ and $v \in \mathbb{R}$. Then, since $\nu(dz)$ is a rotationally invariant measure, in view of (2.2),

$$\mathbb{E}\left[\exp\left(iv\frac{X_{H,\beta}(x) - X_{H,\beta}(y)}{\|x-y\|^H}\right)\right] = \exp\left(\varphi(x,y,v)\right),$$

where

$$\varphi(x,y,v) = \int_{\mathbb{R}^d\times\mathbb{C}} \left[\cos\left(vG(x,y,\xi,z)\right) - 1\right] d\xi\, \nu(dz)$$

with

$$G(x,y,\xi,z) = \frac{2}{\|x-y\|^H}\,\Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\,\psi\left(\|x\|^\beta |z|\right) - \frac{\mathrm{e}^{-iy\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\,\psi\left(\|y\|^\beta |z|\right)\right).$$

Put $M_K = \max_K \|u\|^\beta$ and let $T = M_K^{-1}/2$. Then,

$$\varphi(x,y,v) \le \int_{\mathbb{R}^d\times\mathbb{C}} \left[\cos\left(vK(x,y,\xi,z)\right) - 1\right] \mathbf{1}_{[0,T]}(|z|)\, d\xi\, \nu(dz).$$

Therefore, by definition of ψ, the last inequality can be rewritten as follows:

$$0 \le \mathbb{E}\left[\exp\left(iv\frac{X_{H,\beta}(x) - X_{H,\beta}(y)}{\|x-y\|^H}\right)\right] \le \mathbb{E}\left[\exp\left(iv\frac{\widetilde{X}_H(x) - \widetilde{X}_H(y)}{\|x-y\|^H}\right)\right],$$

where $\widetilde{X}_H$ is a RHFLM with control measure the push-forward of $\nu(dz)$ by the map $z \longmapsto z\,\mathbf{1}_{[0,T]}(|z|)$. Furthermore, according to [5], $\widetilde{X}_H$ satisfies the assumptions of theorem 2.1 in [5]. Hence, there exists a L^1-function Φ and $\delta_0 > 0$ such for all $u \in \mathbb{R}$ and every $(x,y) \in K^2$ such that $\|x-y\| \le \delta_0$,

$$\left|\mathbb{E}\left(\mathrm{e}^{iv\frac{X_{H,\beta}(x)-X_{H,\beta}(y)}{\|x-y\|^H}}\right)\right| \le \Phi(v).$$

Moreover, since $0 \notin K$, by proposition 3, the sample paths of $X_{H,\beta}$ are $(H-\varepsilon)$-Hölder continuous on K for every $\varepsilon \in (0,H)$. Finally, theorem 2.1 in [5] can be applied and gives the conclusion. □

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