A general framework for simulation of fractional fields

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18th October 2006

Abstract

Besides fractional Brownian motion most non-Gaussian fractional fields are obtained by integration of deterministic kernels with respect to a random infinitely divisible measure. In this paper, generalized shot noise series are used to obtain approximations of most of these fractional fields, including linear and harmonizable fractional stable fields. Almost sure and L^r-norm rates of convergence, relying on asymptotic developments of the deterministic kernels, are presented as a consequence of an approximation result concerning series of symmetric random variables. When the control measure is infinite, normal approximation has to be used as a complement. The general framework is illustrated by simulations of classical fractional fields.

1 Introduction

Irregular phenomena appear in various fields of scientific research: fluid mechanics, image processing and financial mathematics for example. Experts in those fields often ask mathematicians to develop models both easy to use and relevant for their applications. In this perspective, fractional fields are very often used to model irregular phenomena. Among the huge literature devoted to the topic, one can refer the reader to [6] for a recent overview of fractional fields for applications.

One of the simplest model is the fractional Brownian motion introduced in [9] and further developed in [13]. Simulation of fractional Brownian motion is now both theoretically and practically well understood (see [2] for a survey on this problem). Many other fractionals fields with heavy tailed marginals have been proposed for applications, see Chapter 7 in [21] for an introduction to fractional stable processes. More recently other processes that are neither Gaussian nor stable have been proposed to model Internet traffic (cf. [27, 5]). The common feature for many of these fields, see also [3, 4, 10], is the fact that they are obtained by a stochastic integration of a deterministic kernel with respect to some random measure. In terms of models, we can think that the probabilistic structure of the irregular phenomena (light or heavy tails for instance) is implemented in the random measure and the correlation structure is built in the deterministic kernel. Engineers will have to try many kernels and random measures before finding the more appropriate one for their applications. Therefore, they need a common framework to simulate fractional fields to make many attempts.

In the literature, there exist articles for simulation of the fractional fields that are non Gaussian. In [7] a wavelet expansion is used to approximate harmonizable and well-balanced type of fractional stable processes. For the linear fractional stable processes the fast Fourier transform is the main tool for simulation in [23, 28]. One can also quote a recent work [14], where another integral representation of the linear fractional stable processes is used to obtain simulation of the sample paths. Even though,

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all these processes are stable, they have different distributions and for each one a specific method is used. Concerning non stable processes, generalized shot noise series introduced for simulation of Lévy processes in [18, 19, 20] were used for simulation of the sample paths of real harmonizable multifractional fields in [11]. One of the advantages of this method is the fact that it can be applied to fractional fields that are neither with stationary increments nor self-similar. Moreover, it is straightforward to apply this technique to the simulation of fields indexed by multidimensional spaces. In this article, our main goal is to show how this method can be applied to most of the fractional fields.

Let us describe how one can obtain an algorithm of simulation when an integral representation of the fractional field is known. In particular, symmetric α-stable random fields can be represented as stochastic integrals (see [21]). We will be interested in the simulation of stochastic integrals of the form

$$X^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda(d\xi), \quad x \in \mathbb{R}^d,$$

with Λ an infinitely divisible random measure.

Basically, one has to transform the random measure Λ by a sum of weighted Dirac masses at random points at the arrival times of a standard Poisson process. After the transformation, the integrals are series which may be simulated by properly truncating the number of terms.

We also would like to stress that we have obtained rates of convergence for the truncating series. More precisely, almost sure rates of convergence are given both for each marginal of the field, and uniformly if the field is simulated on a compact set. The almost sure convergence is related to asymptotic developments of the deterministic kernel in the integral representation of the field. Let us also emphasize Theorem 2.1 which is an important tool to reach rates of convergence for series of symmetric random variables under moment assumptions. This theorem may have interest of its own and is needed in the heavy tail cases. Rates of convergence in L^r-norm with explicit constant are further obtained.

When the control measure of Λ has infinite mass, a technical complication arises. Following [1, 11], one part of X^f will then be approximated by a Gaussian field and the error due to this approximation will be given in terms of Berry-Esseen bounds. The other part will be represented as a series.

In Section 2, rates of almost sure convergence for shot noise series are studied. Section 3 is devoted to some basic facts concerning stochastic integrals with respect to random measures. Then, convergence and rates of convergence of the generalized shot noise series are given in Section 4. Section 5 gives an approximation of the stochastic integrals when the control measure has infinite mass and establishes Berry-Esseen bounds. Examples, that include most of the classical fractional fields, are given in Section 6, illustrated by simulations. Section 7 is devoted to the case of complex random measures, which are important for harmonizable fields. The proofs of Theorems 2.1 and 2.2 is postponed to the Appendix.

2 Rate of almost sure convergence for shot noise series

In this section, we first establish the main tools to reach rates of convergence of the approximation proposed in Section 4. The two following theorems study rates of convergence for series of symmetric random variables. In particular, they can be applied to

$$S_N^\gamma = \sum_{n=1}^{N} T_n^{-1/\gamma} X_n, \tag{1}$$

where $0 < \gamma < 2$ and T_n is the nth arrival time of a Poisson process with intensity 1. Let us recall that if $(X_n)_{n\geq 1}$ is independent of $(T_n)_{n\geq 1}$, the shot noise series (1) converges almost surely to a stable random variable with index γ as soon as (X_n), $n \geq 1$, are independent and identically distributed (i.i.d) L^γ-symmetric random variables, see for instance [12, 21]. Under a stronger integrability assumption, a rate of almost sure convergence is given by Theorem 2.1. Theorem 2.2 gives a rate of absolute almost sure convergence.

Theorem 2.1. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables. Assume that $(X_n)_{n\geq 1}$ is independent of $(T_n)_{n\geq 1}$ and of a sequence $(Y_n)_{n\geq 1}$ which satisfies*

$$|Y_n| \leq C T_n^{-1/\gamma} \tag{2}$$

for some finite constants $C > 0$ and $\gamma \in (0, 2)$. Furthermore, assume $\mathbb{E}(|X_n|^r) < +\infty$ for some $r > \gamma$. Then, for every $\varepsilon \in (0, 1/\gamma - 1/(r \wedge 2))$, almost surely,

$$\sup_{N\geq 1} N^\varepsilon \left| \sum_{n=N+1}^{+\infty} Y_n X_n \right| < +\infty.$$

Proof. See the Appendix. □

The Theorem 2.1 will give us a rate of almost sure convergence of our approximation by generalized shot noise series (see Section 4). In this paper, we are also interested in the uniform convergence of our approximation when the field X^f is simulated on a compact set. The next theorem will be the main tool to establish this uniform convergence and obtain a rate of uniform convergence.

Theorem 2.2. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables and $\gamma \in (0, 1)$. Assume that $(X_n)_{n\geq 1}$ is independent of $(T_n)_{n\geq 1}$ and that $\mathbb{E}(|X_n|^r) < +\infty$ for some $r > \gamma$. Then, for every $\varepsilon \in (0, 1/\gamma - 1/(r \wedge 1))$, almost surely,*

$$\sup_{N\geq 1} N^\varepsilon \sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |X_n| < +\infty.$$

Proof. See the Appendix. □

3 Stochastic integrals with respect to Poisson random measure

In this section, we first recall some classical facts concerning stochastic integrals with respect to Poisson random measures (see [17] for more details). Let $N(d\xi, dv)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}$ with intensity $n(d\xi, dv) = d\xi\nu(dv)$. Assume that the non-vanishing σ-finite measure $\nu(dv)$ is a symmetric measure such that

$$\int_{\mathbb{R}} \left(|v|^2 \wedge 1 \right) \nu(dv) < +\infty, \tag{3}$$

where $a \wedge b = \min(a, b)$. In particular, $\nu(dv)$ may not have a finite second order moment. Under the assumption (3), which is weaker than the assumptions done in [4], we can study in the same framework fractional stable fields and the fields introduced in [4] (see Examples 3.1 and 3.2). Similarly, in Section 7, the control measure satisfies a weaker assumption than the one done in [3, 10, 11], which introduces a common framework for harmonizable fractional stable fields and harmonizable multifractional Lévy motions.

The stochastic integral

$$\int_{\mathbb{R}^d\times\mathbb{R}} \varphi(\xi, v) \left[N(d\xi, dv) - (1 \vee |\varphi(\xi, v)|)^{-1} n(d\xi, dv) \right],$$

where $a \vee b = \max(a, b)$, is defined if and only if $\int_{\mathbb{R}^d\times\mathbb{R}} \left(|\varphi(\xi, v)|^2 \wedge 1 \right) n(d\xi, dv) < +\infty$, see for instance Lemma 12.13 page 236 in [8].

Then, we can consider a random measure $\Lambda(d\xi)$ on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} g(\xi)\,\Lambda(d\xi) = \int_{\mathbb{R}^d\times\mathbb{R}} g(\xi)v\left(N(d\xi,dv) - (|g(\xi)v|\vee 1)^{-1}\,n(d\xi,dv)\right) \tag{4}$$

for every $g:\mathbb{R}^d\to\mathbb{R}$ such that $\int_{\mathbb{R}^d\times\mathbb{R}} |g(\xi)v|^2\wedge 1\,n(d\xi,dv) < +\infty$. We have that

$$\mathbb{E}\left[\exp\left(i\int_{\mathbb{R}^d} g(\xi)\,\Lambda(d\xi)\right)\right] = \exp\left[\int_{\mathbb{R}^d\times\mathbb{R}} \left[\exp(ig(\xi)v) - 1 - ig(\xi)v\mathbf{1}_{|g(\xi)v|\le 1}\right] d\xi\,\nu(dv)\right], \tag{5}$$

see for instance [8]. Therefore Λ is an infinitely divisible random measure.

As explained below (see Examples 3.1 and 3.2), Lévy random measures and stable random measures are examples of such infinitely divisible random measures represented by a Poisson random measure owing to (4). Here are some illustrations.

Example 3.1. Let $\nu(dv)$ be a symmetric measure such that

$$\int_{\mathbb{R}^d} |v|^2\,\nu(dv) < +\infty.$$

Then, for every $g\in L^2\left(\mathbb{R}^d\right)$, (4) can be rewritten as

$$\int_{\mathbb{R}^d} g(\xi)\,\Lambda(d\xi) \;=\; \int_{\mathbb{R}^d\times\mathbb{R}} g(\xi)v\,(N(d\xi,dv) - n(d\xi,dv)).$$

If the symmetric measure $\nu(dv)$ satisfies the assumptions done in [4], i.e. if

$$\forall p\ge 2,\ \int_{\mathbb{R}} |v|^p\,\nu(dv) < +\infty,$$

$\Lambda(d\xi)$ is a Lévy random measure, without Brownian component, represented by the Poisson random measure $N(d\xi,dv)$ in the sense of [4]. Under the above assumptions, the field $(X_H(x))_{x\in\mathbb{R}^d}$, defined by

$$X_H(x) = \int_{\mathbb{R}^d}\left(\|x-\xi\|^{H-d/2} - \|\xi\|^{H-d/2}\right)\Lambda(d\xi)$$

is a moving average fractional Lévy motion, in short MAFLM, with index H ($0<H<1$, $H\neq d/2$).

Example 3.2. In the case where

$$\nu(dv) = \frac{dv}{|v|^{1+\alpha}}$$

with $0<\alpha<2$, the random measure $\Lambda(d\xi)$, defined by (4), is a symmetric α-stable random measure in the sense of [21]. Then, for instance,

$$X_H(x) = D(\alpha)^{-1/\alpha}\int_{\mathbb{R}^d}\left(\|x-\xi\|^{H-d/\alpha} - \|\xi\|^{H-d/\alpha}\right)\Lambda(d\xi),\quad x\in\mathbb{R}^d,$$

with

$$D(\alpha) = \int_{\mathbb{R}} \frac{1-\cos(r)}{|r|^{1+\alpha}}\,dr, \tag{6}$$

is a moving average fractional stable motion, in short MAFSM, with index H ($0<H<1$, $H\neq d/\alpha$).

In the following, we will be interested in the simulation of stochastic integrals of the form

$$X^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda(d\xi), \quad x \in \mathbb{R}^d, \tag{7}$$

where $\Lambda(d\xi)$ is defined by (4) and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that for every $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d \times \mathbb{R}} \Big(|f(x,\xi)v|^2 \wedge 1 \Big)\, n(d\xi, dv) < +\infty. \tag{8}$$

To analyze these stochastic integrals, we represent them as series (known as shot noise series) for which we carefully study the rates of convergence.

4 Generalized Shot Noise Series

An overview of representations of infinitely divisible laws as series is given in [20, 19] and the field X^f is an infinitely divisible field. Such representation of RHMLMs, fields introduced in [10], has been studied in [11]. As in the case of RHMLMs, the infinitely divisible field X^f can be represented as a generalized shot noise series as soon as the control measure $\nu(dv)$ has finite mass. Hence, in this section,

$$\nu(\mathbb{R}) < +\infty. \tag{9}$$

Let us recall that $\nu(dv)$ is a non-vanishing measure, i.e. $\nu(\mathbb{R}) \neq 0$.

Let us now introduce some notation that will be used throughout the paper.

Notation Let $(V_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ be independent sequences of random variables. We assume that $(U_n, V_n)_{n\geq 1}$ is independent of $(T_n)_{n\geq 1}$.

- $(V_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with common law $\nu(dv)/\nu(\mathbb{R})$.
- $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that U_1 is uniformly distributed on the unit sphere S^{d-1} of the Euclidean space $\mathbb{R}^d$.
- c_d is the volume of the unit ball of $\mathbb{R}^d$.

The following statement is the main series representation we will be using in our investigation.

Proposition 4.1. *Assume that* (8) *is fulfilled. Then, for every $x \in \mathbb{R}^d$, the series*

$$Y^f(x) = \sum_{n=1}^{+\infty} f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{R})}\right)^{1/d} U_n\right) V_n \tag{10}$$

converges almost surely. Furthermore, $\{X^f(x) : x \in \mathbb{R}^d\} \stackrel{(d)}{=} \{Y^f(x) : x \in \mathbb{R}^d\}$.

Remark 4.2. In the framework of RHMLMs, [11] directly establishes the almost convergence of the shot noise series in the space of continuous functions endowed with the topology of uniform convergence on compact sets. Such result assumes the continuity of the deterministic kernel f and in our framework, this kernel function may be discontinuous. Nevertheless, under assumptions on the asymptotics expansion of f as $\|\xi\|$ tends to infinity, (10) also converges almost surely on each compact set. Such result, stated in Proposition 4.6, will be deduced from the Theorem 2.2. Note that we will also give a rate of uniform convergence.

Proof. Let p be an integer, $p \geq 1$, $(u_1, \cdots, u_p) \in \mathbb{R}^p$ and $(x_1, \cdots, x_p) \in (\mathbb{R}^d)^p$. We consider the Borel measurable map

$$\begin{array}{rcl} H : \;]0,+\infty[\times \mathcal{D} & \longrightarrow & \mathbb{R} \\ (r, \widetilde{v}) & \longmapsto & \displaystyle\sum_{i=1}^{p} u_j f\left(x_j, \left(\frac{r}{c_d \nu(\mathbb{R})} \right)^{1/d} u \right) v, \end{array}$$

Then, define a measure Q on the Borel σ-field $\mathcal{B}(\mathbb{R})$ by

$$\forall B \in \mathcal{B}(\mathbb{R}),\; Q(B) = \int_0^{+\infty} \int_{\mathcal{D}} \mathbf{1}_{B\setminus\{0\}}(H(r,\widetilde{v}))\, \lambda(d\widetilde{v})\, dr,$$

where λ is the law of $\widetilde{V_n} = (U_n, V_n)$. Hence, Q is the push-forward of $\lambda(d\widetilde{v})dr$ by H and

$$\int_{\mathbb{R}} |y|^2 \wedge 1\, Q(dy) = \int_{]0,+\infty[\times\mathcal{D}} H^2(r,\widetilde{v}) \wedge 1\, dr\, \lambda(d\widetilde{v}).$$

Then, proceeding as in the proof of Proposition 3.1 in [11], i.e. using the change of variable $\rho = (r/(c_d\nu(\mathbb{R})))^{1/d}$ and polar coordinates, one obtains that

$$\int_{\mathbb{R}} |y|^2 \wedge 1\, Q(dy) = \int_{\mathbb{R}^d\times\mathbb{R}} \left[\left(\sum_{i=1}^{p} u_j f^2(x_j,\xi)\, v^2 \right) \wedge 1 \right] n(d\xi, dv) < +\infty.$$

Then, Q is a Lévy measure on $\mathbb{R}$. Therefore, according to Theorem 2.4 in [19], the sequence

$$\sum_{n=1}^{N} H\left(T_n, \widetilde{V_n}\right) - A(T_N),$$

where for $s \geq 0$,

$$A(s) = \int_0^s \int_{\mathcal{D}} H(r,\widetilde{v}) \mathbf{1}_{|H(r,\widetilde{v})|\leq 1}\, \lambda(d\widetilde{v})\, dr,$$

converges almost surely as $N \to +\infty$. Moreover, since ν is a finite and symmetric measure, by definition of H and of the measure $\lambda(d\widetilde{v})$, $A(s) = 0$ for every $s \geq 0$. Therefore, (taking $p = 1$), for every x,

$$Y^f(x) = \sum_{n=1}^{+\infty} f\left(x, \left(\frac{T_n}{c_d\nu(\mathbb{R})} \right)^{1/d} U_n \right) V_n$$

converges almost surely. Furthermore, due to Theorem 2.4 in [19], we have that

$$\mathbb{E}\left[\exp\left(i \sum_{i=1}^{p} u_i Y^f(x_i) \right) \right] = \exp\left[\int_{\mathbb{R}} \left(\exp(iy) - 1 - iy\mathbf{1}_{|y|\leq 1} \right) Q(dy) \right]$$

By definition of Q and symmetry of $\nu(dv)$, one easily sees that $\{X^f(x) : x \in \mathbb{R}^d\} \overset{(d)}{=} \{Y^f(x) : x \in \mathbb{R}^d\}$. The proof of Proposition 4.1 is then complete.

□

On the basis of Proposition 4.1, Y^f, which is equal in law to X^f, is approximated by

$$Y_N^f(x) = \sum_{n=1}^{N} f\left(x, \left(\frac{T_n}{c_d\nu(\mathbb{R})} \right)^{1/d} U_n \right) V_n, \quad x \in \mathbb{R}^d. \tag{11}$$

We now explain in a few words how the rate of convergence of Y_N^f to Y^f can be studied. Firstly, let us recall the following classical result for Poisson arrival times:

$$\lim_{n\to+\infty} \frac{T_n}{n} = 1 \text{ almost surely.} \tag{12}$$

Hence, the asymptotics of (11) depends on $(V_n)_{n\geq 1}$ and on the asymptotics of $f(x,\xi)$ as $\|\xi\|$ tends to infinity. Under an assumption on this asymptotics, the rate of convergence of Y_N^f will be deduced from the rate of convergence of some series of the kind of S_N^γ defined by (1).

Let us first study the almost sure and L^r errors for each fix x.

Theorem 4.3. *Let $x \in \mathbb{R}^d$. Assume that*

$$\forall \xi \neq 0,\ |f(x,\xi)| \leq \frac{C}{\|\xi\|^\beta}, \tag{13}$$

where $\beta > d/2$ and $C > 0$. Furthermore, assume there exists $r \in (d/\beta, 2]$ such that $\mathbb{E}(|V_1|^r) < +\infty$

1. *Then, for every $\varepsilon \in (0, \beta/d - 1/r)$, almost surely,*

$$\sup_{N\geq 1} N^\varepsilon \left| Y^f(x) - Y_N^f(x) \right| < +\infty.$$

2. *Moreover, for every integer $N > r\beta/d$,*

$$\mathbb{E}\left(\left| Y_N^f(x) - Y^f(x) \right|^r \right) \leq C(r,\beta)\, \frac{D(N,r,\beta)}{N^{r\beta/d-1}}, \tag{14}$$

where

$$D(N,r,\beta) = \frac{\Gamma(N+1-r\beta/d)\,(N+1)^{r\beta/d}}{\Gamma(N+1)} \tag{15}$$

and

$$C(r,\beta) = \frac{dC^r (c_d \nu(\mathbb{R}))^{r\beta/d} \mathbb{E}(|V_1|^r)}{r\beta - d}. \tag{16}$$

Remark 4.4. *Remark that $\lim_{N\to+\infty} D(N,r,\beta) = 1$ by the Stirling formula. Hence, Proposition 4.3 gives a rate of convergence in L^r for the series Y_N^f. Furthermore,* (14) *allows us to control the error of approximation in simulation.*

Remark 4.5. *Assume that* (13) *is only fulfilled for $\|\xi\| \geq A$. Then, let*

$$g(x,\xi) = f(x,\xi)\mathbf{1}_{\|\xi\|\geq A}$$

and remark that

$$Y^f = Y^g + Y^{f-g}, \tag{17}$$

where Y^h is associated with h by (10)*. Hence, since g satisfies the assumptions of Proposition 4.3, an almost sure or L^r error may be obtained. Furthermore, in view of* (12),

$$Y^{f-g}(x) = \sum_{n=1}^{+\infty} (f-g)\left(x, \left(\frac{T_n}{c_d \nu(\mathbb{R})} \right)^{1/d} U_n \right) V_n$$

is, almost surely, a finite sum since for n large enough, $T_n > A^d c_d \nu(\mathbb{R})$. This remark is used for MAFSMs or MAFLMs in Section 6.

Let us now prove Theorem 4.3.

Proof of Theorem 4.3. In the following,

$$\xi_n = \left(\frac{T_n}{c_d\nu(\mathbb{R})}\right)^{1/d} U_n.$$

1. Proof of part 1: Rate of Almost Sure Convergence

In view of (13),

$$|f(x,\xi_n)| \le \frac{C(c_d\nu(\mathbb{R}))^{\beta/d}}{T_n^{\beta/d}}. \tag{18}$$

Then, by applying Theorem 2.1 with $X_n = V_n$ and $Y_n = f(x,\xi_n)$,

$$\sup_{N\ge 1} N^{\varepsilon}\left|Y^f(x) - Y_N^f(x)\right| < +\infty \text{ almost surely}$$

for every $\varepsilon \in (0, \beta/d - 1/r)$.

2. Proof of part 2: L^r-error

Since V_n, $n \in \mathbb{N}\backslash\{0\}$, are i.i.d. symmetric random variables, by the Jensen inequality, applied for $r \in (0,2]$,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \le \mathbb{E}\left[\left(\sum_{n=N+1}^{P} f^2(x,\xi_n)V_n^2\right)^{r/2}\right].$$

Furthermore, since $(a+b)^{r/2} \le a^{r/2} + b^{r/2}$ ($r \in (0,2]$) for every $a, b \ge 0$,

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) &\le \mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\mathbb{E}(|f(x,\xi_n)|^r)\\
&\le C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\mathbb{E}\left(T_n^{-r\beta/d}\right)\\
&\le C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\frac{\Gamma(n-r\beta/d)}{\Gamma(n)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \le C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sup_{n\ge N} D(n,r,\beta)\sum_{n=N+1}^{+\infty}\frac{1}{n^{r\beta/d}}$$

where $D(n,r,\beta)$ is defined by (15). According to the proof of Proposition 3.2 in [11],

$$\sup_{n\ge N} D(n,r,\beta) = D(N,r,\beta)$$

and then

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \le \frac{dC^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)D(N,r,\beta)}{(r\beta-d)N^{r\beta/d-1}}$$

since $r > d/\beta$. Then, by the Fatou lemma,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{+\infty} f(x,\xi_n)V_n\right|^r\right) \le \frac{dC^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)D(N,r,\beta)}{(r\beta-d)N^{r\beta/d-1}}.$$

The proof of Theorem 4.3 is complete. ☐

Actually, if f admits an expansion, roughly speaking uniform in x, as $\|\xi\|$ tends to infinity, the next theorem gives a rate of uniform convergence in x for Y_N^f.

Theorem 4.6. *Let $K \subset \mathbb{R}^d$ be a compact set, $p \geq 1$ and $(\beta_i)_{1\leq i\leq p}$ be a non-decreasing sequence. such that $\beta_1 > d/2$ and $\beta_p > d$. Assume that for every $x \in K$ and $\xi \neq 0$,*

$$\left| f(x,\xi) - \sum_{j=1}^{p-1} \frac{a_j(x) b_j(\xi/\|\xi\|)}{\|\xi\|^{\beta_j}} \right| \leq \frac{b_p(\xi/\|\xi\|)}{\|\xi\|^{\beta_p}} \tag{19}$$

where a_j, $j = 1, \dots, p-1$, are real-valued continuous functions. Furthermore, assume that there exists $r \in (d/\beta_1, 2]$ such that $\mathbb{E}(|V_n|^r) < +\infty$ and $\mathbb{E}(|b_j(U_n)|^r) < +\infty$ for $j = 1, \dots, p$. Then for every $\varepsilon \in (0, \min(\beta_1/d - 1/r, \beta_p/d - 1/(1 \wedge r)))$,

$$\sup_{N\geq 1} N^{\varepsilon} \sup_{x\in K} \left| Y^f(x) - Y_N^f(x) \right| < +\infty \text{ almost surely.}$$

Remark 4.7. *In (19), the non-radial (or anisotropic) part of the asymptotic expansion of f is given by the functions b_j.*

Proof of Theorem 4.6. We have

$$\begin{aligned} \left| Y^f(x) - Y_N^f(x) \right| \;&\leq\; \sum_{j=1}^{p-1} |a_j(x)| \left| \sum_{n=N+1}^{+\infty} \left(\frac{T_n}{c_d \nu(\mathbb{R})} \right)^{-\beta_j/d} b_j(U_n) V_n \right| \\ &\quad + \sum_{n=N+1}^{+\infty} \left(\frac{T_n}{c_d \nu(\mathbb{R})} \right)^{-\beta_p/d} |b_p(U_n) V_n|. \end{aligned}$$

Note that $(b_j(U_n)V_n)_{n\geq 1}$ are i.i.d. symmetric random variables such that $\mathbb{E}(|b_j(U_n)V_n|^r) < +\infty$. Hence, since $0 < d/\beta_j < r \leq 2$, by Theorem 2.1, for every $\varepsilon \in (0, \beta_j/d - 1/r)$,

$$\sup_{N\geq 1} N^{\varepsilon} \left| \sum_{n=N+1}^{+\infty} T_n^{-\beta_j/d} b_j(U_n) V_n \right| < +\infty \text{ almost surely.}$$

In addition, since $\mathbb{E}(|b_p(U_n)V_n|^r) < +\infty$ and $d/\beta_p < 1$, by Theorem 2.2, for every $\varepsilon \in (0, \beta_p/d - 1/(1 \wedge r))$,

$$\sup_{N\geq 1} N^{\varepsilon} \sum_{n=N+1}^{+\infty} T_n^{-\beta_p/d} |b_p(U_n) V_n| < +\infty \text{ almost surely,}$$

which ends the proof since a_j, $j = 1, \dots, p-1$, are continuous and thus bounded on the compact set K. □

5 Normal Approximation

When the assumption (9) is not fulfilled, Section 4 cannot be directly applied. In this case, the simulation of X^f is not only based on a series expansion but also on a normal approximation. Actually, following [1, 11], we will split the field X^f into two fields $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$. It leads to a decomposition of Λ into two random measures $\Lambda_{\varepsilon,1}$ and $\Lambda_{\varepsilon,2}$ such that the control measure of $\Lambda_{\varepsilon,2}$ satisfies the assumption (9). As a consequence of Section 4, $X^f_{\varepsilon,2}$ can be represented as a series. This section is thus devoted to the simulation of the first part $X^f_{\varepsilon,1}$ that will be handled by normal approximation of the Berry-Esseen type.

Suppose now that

$$\nu(\mathbb{R}) = +\infty, \tag{20}$$

which is the case for MAFSMs. Then let $\varepsilon > 0$ and let us split

$$X^f = X^f_{\varepsilon,1} + X^f_{\varepsilon,2}$$

into two random fields where

$$X^f_{\varepsilon,1}(x) = \int_{\mathbb{R}^d\times\mathbb{R}} f(x,\xi)v\mathbf{1}_{|v|<\varepsilon}\Big(N(d\xi,dv) - (|f(x,\xi)v| \vee 1)^{-1}\, n(d\xi,dv)\Big) \tag{21}$$

and

$$X^f_{\varepsilon,2}(x) = \int_{\mathbb{R}^d\times\mathbb{R}} f(x,\xi)v\mathbf{1}_{|v|\geq\varepsilon}\Big(N(d\xi,dv) - (|f(x,\xi)v| \vee 1)^{-1}\, n(d\xi,dv)\Big). \tag{22}$$

Consider the two independent Poisson random measures

$$N_{\varepsilon,1}(d\xi,dv) = \mathbf{1}_{|v|<\varepsilon}\, N(d\xi,dv) \quad\text{and}\quad N_{\varepsilon,2}(d\xi,dv) = \mathbf{1}_{|v|\geq\varepsilon}\, N(d\xi,dv).$$

Let $\Lambda_{\varepsilon,i}$ $(i = 1,2)$ be the infinitely divisible random measure associated with $N_{\varepsilon,i}$ by (4). Remark that $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$ are independent and that

$$X^f_{\varepsilon,i}(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda_{\varepsilon,i}(d\xi), \quad i = 1,2.$$

In addition, the control measure $\nu_{\varepsilon,2}(dv) = \mathbf{1}_{|v|\geq\varepsilon}\,\nu(dv)$ of $\Lambda_{\varepsilon,2}$ is finite and symmetric. Therefore $X^f_{\varepsilon,2}$ can be simulated as a generalized shot noise series (see Section 4). It remains to properly approximate $X^f_{\varepsilon,1}$. To this task, notice that the control measure $\nu_{\varepsilon,1}(dv) = \mathbf{1}_{|v|<\varepsilon}\,\nu(dv)$ of $\Lambda_{\varepsilon,1}$ has moments of every order greater than 2. Hence, $\Lambda_{\varepsilon,1}$ is a Lévy random measure in the sense of [4].

Set

$$\sigma(\varepsilon) = \left(\int_{-\varepsilon}^{\varepsilon} v^2\,\nu(dv)\right)^{1/2}. \tag{23}$$

Proposition 5.1. *Assume that for each $x \in \mathbb{R}^d$, $f(x,\cdot) \in L^2(\mathbb{R}^d)$ and $\lim_{\varepsilon\to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty$. Then*

$$\lim_{\varepsilon\to 0_+} \left(\frac{X^f_{\varepsilon,1}(x)}{\sigma(\varepsilon)}\right)_{x\in\mathbb{R}^d} \overset{(d)}{=} \Big(W^f(x)\Big)_{x\in\mathbb{R}^d}, \tag{24}$$

where, with $W(d\xi)$ a real Brownian random measure,

$$W^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,W(d\xi) \tag{25}$$

and where the limit is understood in the sense of finite dimensional distributions.

Proof. Let $r \geq 1$, $u = (x_1,\ldots,x_r) \in (\mathbb{R}^d)^r$ and $y = (y_1,\ldots,y_r) \in \mathbb{R}^r$. Then

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^r y_k \frac{X^f_{\varepsilon,1}(x_k)}{\sigma(\varepsilon)}\right)\right] = \exp\left(\Psi_\varepsilon(x,y)\right)$$

with

$$\Psi_\varepsilon(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}}\left(\exp\left(\frac{ig(\xi,x,y)v}{\sigma(\varepsilon)}\right) - 1 - \frac{ig(\xi,x,y)v}{\sigma(\varepsilon)}\mathbf{1}_{|g(\xi,x,y)v|\leq\sigma(\varepsilon)}\right) d\xi\,\nu_{\varepsilon,1}(dv)$$

and $g(\xi, x, y) = \sum_{k=1}^{r} y_k f(x_k, \xi)$. Then, by the Fubini theorem,

$$\Psi_\varepsilon(x, y) = \int_{\mathbb{R}^d} I_\varepsilon(g(\xi, x, y))\, d\xi,$$

where for every $a \in \mathbb{R}$ $I_\varepsilon(a) = \int_{\mathbb{R}} \left(\mathrm{e}^{i\frac{av}{\sigma(\varepsilon)}} - 1 - i\frac{av}{\sigma(\varepsilon)} \mathbf{1}_{|av|<\sigma(\varepsilon)} \right) \mathbf{1}_{|v|<\varepsilon}\, \nu(dv)$. Since $\nu(dv)$ is a symmetric Lévy measure,

$$I_\varepsilon(a) = \int_{\mathbb{R}} \left(\mathrm{e}^{i\frac{av}{\sigma(\varepsilon)}} - 1 - i\frac{av}{\sigma(\varepsilon)} \right) \mathbf{1}_{|v|<\varepsilon}\, \nu(dv).$$

As $\lim_{\varepsilon\to 0_+} \sigma(\varepsilon)/\varepsilon = +\infty$, according to [1], $\lim_{\varepsilon\to 0_+} I_\varepsilon(a) = -\frac{a^2}{2}$. Since moreover $|I_\varepsilon(a)| \leq \frac{a^2}{2}$, for every $a \in \mathbb{R}$, a dominated convergence argument yields

$$\lim_{\varepsilon\to 0_+} \Psi_\varepsilon(x, y) = -\frac{1}{2} \int_{\mathbb{R}^d} \left| \sum_{k=1}^{r} y_k f(x_k, \xi) \right|^2 d\xi = -\frac{1}{2} \mathrm{Var}\left(\sum_{k=1}^{r} y_k W^f(x_k) \right).$$

The proof is thus complete. ☐

As in the case of RHMLMs, an estimate in terms of Berry-Esseen bounds on the rate of convergence stated in Proposition 5.1 may be given. The assumption of the following theorem only ensures the existence of the moment of order $(2+\delta)$ for $X^f_{\varepsilon,1}(x)$.

Theorem 5.2. *Let $x \in \mathbb{R}^d$ and assume that*

$$f(x, \cdot) \in L^{2+\delta}\left(\mathbb{R}^d\right) \tag{26}$$

for some $\delta \in (0, 1]$. Then $\mathbb{E}\left(\left| X^f_{\varepsilon,1}(x) \right|^{2+\delta} \right) < +\infty$ and

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\left(X^f_{\varepsilon,1}(x) \leq u \right) - \mathbb{P}\left(\sigma(\varepsilon) W^f(x) \leq u \right) \right| \leq A(x, \delta)\, \frac{m^{2+\delta}_{2+\delta}(\varepsilon)}{\sigma^{2+\delta}(\varepsilon)}$$

where W^f is defined by (25) in Proposition 5.1, $m^{2+\delta}_{2+\delta}(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} |v|^{2+\delta}\, \nu(dv)$ and

$$A(x, \delta) = \frac{A_\delta \int_{\mathbb{R}^d} |f(x, \xi)|^{2+\delta}\, d\xi}{3 \left(\pi \int_{\mathbb{R}^d} |f(x, \xi)|^2\, d\xi \right)^{(2+\delta)/2}}$$

with

$$A_\delta = \begin{cases} 0.7975 & \text{if } \delta = 1 \\ 53.9018 & \text{if } 0 < \delta < 1. \end{cases}$$

Remark 5.3. *Assume that f satisfies assumptions (8) and (26). Then, for every x, $f(x, \cdot) \in L^2(\mathbb{R}^d)$ and $\mathbb{E}\left(X_{\varepsilon,1}(x)^2 \right) < +\infty$.*

Proof. This proof is based on a generalization of Lemma 4.1 in [11].

Let μ be the distribution of the infinitely divisible variable $X^f_{\varepsilon,1}(x)$. The Lévy Q measure of μ is then the push-forward of $n_{\varepsilon,1}(d\xi, dv) = d\xi \nu_{\varepsilon,1}(dv)$ by the map $(\xi, v) \mapsto f(x, \xi)v$. Hence, for every $\gamma > 0$,

$$\int_{\mathbb{R}} |y|^\gamma Q(dy) = m^\gamma_\gamma(\varepsilon) \int_{\mathbb{R}} |f(x, \xi)|^\gamma\, d\xi$$

where $m_\gamma^\gamma(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} |v|^\gamma \, \nu(dv)$. Note that $m_2^2(\varepsilon) = \sigma^2(\varepsilon)$. Then, since $f(x,\cdot) \in L^{2+\delta}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}} |y|^{2+\delta} Q(dy) < +\infty.$$

Therefore, according to Theorem 25.3 in [22],

$$\int_{\mathbb{R}} |y|^{2+\delta} \mu(dy) < +\infty \quad \text{i.e.} \quad \mathbb{E}\left(|X_{\varepsilon,1}(x)|^{2+\delta}\right) < +\infty.$$

As in the proof of Lemma 4.1 in [11], we then consider a Lévy process $(Z(t))_{t\geq 0}$ such that $Z(1) \overset{(d)}{=} X_{\varepsilon,1}(x)$. For each fixed $n \in \mathbb{N}\backslash\{0\}$,

$$Z(1) = \sum_{k=1}^{n} \left(Z\left(\frac{k+1}{n}\right) - Z\left(\frac{k}{n}\right) \right)$$

where $Y_{k,n} = Z\left(\frac{k+1}{n}\right) - Z\left(\frac{k}{n}\right)$, $1 \leq k \leq n$, are i.i.d real-valued centered random variables. Furthermore,

$$\mathbb{E}\left(|Y_{k,n}|^2\right) = \frac{\mathbb{E}\left(|Z(1)|^2\right)}{n} = \frac{\sigma^2(\varepsilon) \int_{\mathbb{R}} |f(x,\xi)|^2 \, d\xi}{n}$$

and since $Z(1) \in L^{2+\delta}$, $Y_{k,n} \in L^{2+\delta}$. Therefore, according to [16], there exists a constant A_δ such that for every $n \in \mathbb{N}\backslash\{0\}$,

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{Z(1)}{\sqrt{\mathbb{E}\left(|Z(1)|^2\right)}} \leq t \right) - \mathbb{P}(W \leq t) \right| \leq \frac{n A_\delta \mathbb{E}\left(\left|Z\left(\frac{1}{n}\right)\right|^{2+\delta}\right)}{\mathbb{E}\left(|Z(1)|^2\right)^{1+\delta/2}}$$

where W is a normal random variable with mean 0 and variance 1. When $\delta = 1$, the preceding inequality is the classical Berry-Esseen inequality and we can take $A_1 = 0.7975$. In [16], on find that $A_\delta = \max(8/3, 64A_1 + 1 + 14/(3\sqrt{2\pi})) = 53.9018$. Furthermore, it is straightforward that

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{Z(1)}{\sqrt{\mathbb{E}\left(|Z(1)|^2\right)}} \leq t \right) - \mathbb{P}(W \leq t) \right| = \sup_{u\in\mathbb{R}} \left| \mathbb{P}(X_{\varepsilon,1}(x) \leq u) - \mathbb{P}\left(\sigma(\varepsilon) W^f(x) \leq u\right) \right|.$$

According to [20],

$$\lim_{n\to+\infty} n\mathbb{E}\left(\left| Z\left(\frac{1}{n}\right) \right|^{2+\delta} \right) = \int_{\mathbb{R}} |y|^{2+\delta} Q(dy),$$

which concludes the proof. □

We now summarize the approximation scheme based on the preceding splitting. First we approximate $X_{\varepsilon,1}^f$ by the Gaussian field $\sigma(\varepsilon)W^f$. According to Section 4, an approximation of $X_{\varepsilon,2}^f$ may be given by

$$Y_{\varepsilon,N,2}^f(x) = \sum_{n=1}^{N} f\left(x, \left(\frac{T_n}{c_d \, \nu_{\varepsilon,2}(\mathbb{R})}\right)^{1/d} U_n\right) V_{\varepsilon,n}, \quad x \in \mathbb{R}^d,$$

where $(V_{\varepsilon,n})_n$ is a sequence of i.i.d. random variables with common law $\nu_{\varepsilon,2}(dv)/\nu_{\varepsilon,2}(\mathbb{R})$. Note that T_n, U_n and $V_{\varepsilon,n}$ are independent. Since $X_{\varepsilon,1}^f$ and $X_{\varepsilon,2}^f$ are independent, W^f is assumed to be independent of $(T_n, U_n, V_{\varepsilon,n})$. As a result, in the case where $\nu(\mathbb{R}) = +\infty$, under the assumptions of Proposition 5.1, an approximation of X^f is

$$Y_{\varepsilon,N}^f(x) = \sigma(\varepsilon)W^f(x) + \sum_{n=1}^{N} f\left(x, \left(\frac{T_n}{c_d \, \nu_{\varepsilon,2}(\mathbb{R})}\right)^{1/d} U_n\right) V_{\varepsilon,n}, \quad x \in \mathbb{R}^d.$$

6 Examples

This section illustrates with various examples the range of application of the preceding results. In all the following examples, $K \subset \mathbb{R}^d$ is a compact set and (13) is only fulfilled for $\|\xi\| \geq A$. Then, as noticed in Remark 4.5, we may split

$$Y_N^f = Y_N^g + Y_N^{f-g},$$

with $g(x,\xi) = f(x,\xi)\mathbf{1}_{\|\xi\|\geq A}$. Since, Y_N^g is in fact a finite sum (almost surely), the rate of convergence described below is actually the rate of convergence of Y_N^{f-g}.

6.1 Moving Average Fractional Lévy Motions

Let $H \in (0,1)$ such that $H \neq d/2$. Suppose that

$$f_{H,2}(x,\xi) = \|x-\xi\|^{H-d/2} - \|\xi\|^{H-d/2}$$

and that for every $p \geq 2$, $\int_{\mathbb{R}} |v|^p \, \nu(dv) < +\infty$. Then, $X_{H,2} = X^{f_{H,2}}$ is a MAFLM in the sense of [4].

6.1.1 Case of finite control measures

An approximation, in law, of the MAFLM X_H is given by

$$Y_N^{f_{H,2}}(x) = \sum_{n=1}^{N} \left(\left\| x - \left(\frac{T_n}{c_d \nu(\mathbb{R})}\right)^{1/d} U_n \right\|^{H-d/2} - \left(\frac{T_n}{c_d \nu(\mathbb{R})}\right)^{H/d-1/2} \right) V_n.$$

Let $A = \max_K \|y\| + 1$, $x \in K$ and $\|\xi\| \geq A$. The mean value inequality leads to

$$|f_{H,2}(x,\xi)| \leq \left| H - \frac{d}{2} \right| (A-1) \sup_{0<\theta<1} \|\xi - \theta x\|^{H-d/2-1}.$$

Remark that $\|\xi - \theta x\| \geq \|\xi\| - \|x\| \geq \|\xi\|/A$. Therefore, since $H - d/2 - 1 < 0$, for every $x \in K$, for $\|\xi\| \geq A$,

$$|f_{H,2}(x,\xi)| \leq \frac{C}{\|\xi\|^{1-H+d/2}} \tag{27}$$

with $C = |H - d/2|(A-1)A^{1-H+d/2}$.

Let $\beta_1 = 1 - H + d/2$ and $g_{H,2}(x,\xi) = f_{H,2}(x,\xi)\mathbf{1}_{\|\xi\|\geq \max_K \|y\|+1}$. Note that $\beta_1 > d/2$ since $1 > H$. Then, the assumptions of Theorem 4.3 are satisfied with $r = 2$ and

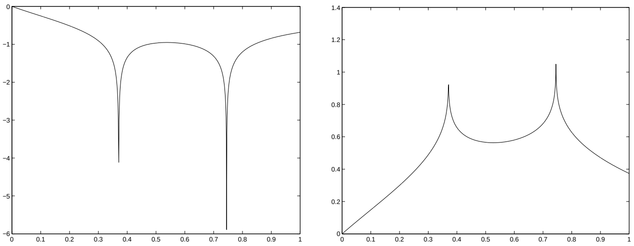
$$\mathbb{E}\Big(\big| Y_N^{g_{H,2}}(x) - Y^{g_{H,2}}(x) \big|^2 \Big) \leq \frac{C(2,\beta_1) D(N,2,\beta_1)}{N^{2(1-H)/d}}$$

where $C(2,\beta_1)$ and $D(N,2,\beta_1)$ are defined by (16) and (15). Therefore, the mean square error converges at the rate $N^{(1-H)/d}$.

We now focus on the uniform convergence of $Y^{g_{H,2}}$. For every integer $q \geq 1$, by a Taylor expansion, one can prove that for every $x \in K$ and for $\|\xi\| \geq A$,

$$\left| f_{H,2}(x,\xi) - \sum_{j=1}^{q-1} \|\xi\|^{H-j-d/2} d_j(x, \xi/\|\xi\|) \right| \leq B_{q,A,H} \|\xi\|^{H-d/2-q}, \tag{28}$$

for some positive constant $B_{q,A,H}$ and where the $d_j's$ are polynomial functions in x_i and u_i, $i = 1 \ldots d$, $j = 1 \ldots d$. Since the d_j's are polynomial functions, one can easily see that $g_{H,2}$ satisfies the assumption (19) taking $\beta_1 = 1 - H + d/2$ and $\beta_p = q - H + d/2$. Since (28) holds for every integer $q \geq 1$, by Theorem 4.6, $Y_N^{g_{H,2}}$ converges uniformly at the rate N^{ε} for every $\varepsilon \in (0, (1-H)/d)$.



$H = 0.3$ $\qquad$ $H = 0.7$

Figure 1: Examples of MAFLMs

Let us now present one example (see Figure 1) taking $\nu(dv) = (\delta_{-1} + \delta_1)/2$. In this example, we first simulate a realization of the random variables (T_n, U_n, V_n). Then, for these values of $(T_n, U_n, V_n)_{1\leq n\leq N}$, we evaluate $Y_N^{f_{H,2}}$ for $H = 0.3$ and $H = 0.7$. We observe that the trajectory regularity does not depend on the value of H. Actually, one can see that the derivatives of $Y_N^{g_{H,2}}$ at each order converge uniformly on each compact set. Therefore, $Y^{g_{H,2}}$ has $\mathcal{C}^\infty$ sample paths almost surely. As a consequence, the sample paths of $Y^{f_{H,2}}$ are $\mathcal{C}^\infty$ except at points $\xi_n = (T_n/c_d\nu(\mathbb{R}))^{1/d}U_n$. At these points, the behavior depends on H: while when $H < d/2$, $Y_N^{f_{H,2}}$ is not defined, when $H > d/2$ the pointwise Hölder exponent of $Y_N^{f_{H,2}}$ is given by $H - d/2$. In Figure 1, we observe that the sample paths are smooth on $[0, 1]$ except at two points.

6.1.2 Case of infinite control measures

In this example,

$$\nu(dv) = \frac{\mathbf{1}_{|v|\leq 1}\, dv}{|v|^{1+\alpha}} \quad \text{with} \quad 0 < \alpha < 2.$$

Let $(V_{\varepsilon,n})_{n\geq 1}$ be a sequence of i.i.d variables with common law

$$\frac{\alpha\, \mathbf{1}_{\varepsilon<|v|<1}\, dv}{2(\varepsilon^{-\alpha} - 1)|v|^{1+\alpha}}.$$

Moreover, let B_H be a standard fractional Brownian motion (in short FBM) with index H and assume that B_H, $(U_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(V_{\varepsilon,n})_{n\geq 1}$ are independent. An approximation of the MAFLM X_H is thus given by

$$Y_{\varepsilon,N}^{f_{H,2}}(x) = \sum_{n=1}^{N} \left(\left\| x - \left(\frac{T_n}{c_d \nu_{\varepsilon,2}(\mathbb{R})} \right)^{1/d} U_n \right\|^{H-d/2} - \left(\frac{T_n}{c_d \nu_{\varepsilon,2}(\mathbb{R})} \right)^{H/d-1/2} \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^{f_{H,2}}(x), \tag{29}$$

where

$$W^{f_{H,2}}(\cdot) = \int_{\mathbb{R}^d} f_{H,2}(\cdot, \xi)\, W(d\xi) \overset{(d)}{=} C_{H,d} B_H(\cdot)$$

with

$$C_{H,d} = \left(\int_{\mathbb{R}^d} |f_{H,2}(e_1, \xi)|^2\, d\xi \right)^{1/2}$$

and $e_1 = (1, 0, \ldots, 0)$. Actually, by a Fourier transform argument

$$C_{H,d} = \frac{2^{H-2}|d - 2H|\Gamma(H/2 + d/4)}{\Gamma(d/4 + 1 - H/2)} \left(\int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\lambda} - 1\right|^2}{\|\lambda\|^{2H+d}}\, d\lambda \right)^{1/2}.$$

As a result, due to [21] for $d = 1$ and to [11] for $d \geq 2$,

$$C_{H,d} = \frac{2^{H-2}|d-2H|\Gamma(H/2+d/4)}{\Gamma(d/4+1-H/2)} \left(\frac{\pi^{(d+1)/2}\Gamma(H+1/2)}{H\Gamma(2H)\sin(\pi H)\Gamma(H+d/2)} \right)^{1/2}. \tag{30}$$

Since $H > 0$, there exists $\delta \in (0,1]$ such that $H > d/2 - d/(2+\delta)$, which implies that $f_{H,2}(x,\cdot) \in L^{2+\delta}(\mathbb{R}^d)$. Then, by Theorem 5.2, in terms of Berry-Esseen bounds, the rate of convergence of the error due to the approximation of $X^f_{\varepsilon,1}(x)$ is of the order

$$\delta(\varepsilon) = \frac{(2-\alpha)^{1+\delta/2}\varepsilon^{\alpha\delta/2}}{(2+\delta-\alpha)2^{\delta/2}}.$$

Except at points $\xi_n = (T_n/c_d \nu_{\varepsilon,2}(\mathbb{R}))^{1/d} U_n$, the trajectory regularity of $Y^{f_{H,2}}_{\varepsilon,N}$ is given by the trajectory regularity of $W^{f_{H,2}}$. Between two points ξ_n, the pointwise Hölder exponent of $Y^{f_{H,2}}_{\varepsilon,N}$ is equal to H. When $H > d/2$, the trajectories of $Y^{f_{H,2}}_{\varepsilon,N}$ are thus H'-Hölder on each compact set for every $H' < H - d/2$. Following [4], this is exactly what we expect for the trajectory regularity of a MAFLM X_H associated to an infinite control measure. Figure 2 yields illustration of these facts in the case where $H = 0.8$, $\alpha = 1$ and for the preceding control measure.

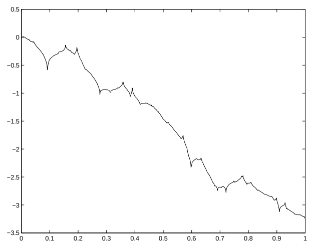


Figure 2: Example of MAFLM with index $H = 0.8$

6.2 Moving Average Fractional Stable Motions

In this example,

$$\nu(dv) = \frac{dv}{|v|^{1+\alpha}} \text{ with } 0 < \alpha < 2, \tag{31}$$

and

$$f_{H,\alpha}(x,\xi) = D(\alpha)^{-1/\alpha}\left(\|x-\xi\|^{H-d/\alpha} - \|\xi\|^{H-d/\alpha}\right),$$

with $0 < H < 1$ and $H \neq d/\alpha$ and where $D(\alpha)$ is defined by (6). Note that

$$D(\alpha) = \begin{cases} \dfrac{\Gamma(2-\alpha)|\cos(\pi\alpha/2)|}{\alpha|\alpha-1|} & \text{if } \alpha \neq 1 \\ \dfrac{\pi}{2} & \text{if } \alpha = 1. \end{cases} \tag{32}$$

Here $\sigma^2(\varepsilon) = 2\varepsilon^{2-\alpha}/(2-\alpha)$ and $\nu_{\varepsilon,2}(\mathbb{R}) = 2/(\alpha\varepsilon^\alpha)$. The approximation of the MAFSM is given by formula (29), replacing $d/2$ by d/α in the summation and with

$$W^{f_{H,\alpha}}(\cdot) = \int_{\mathbb{R}^d} f_{H,\alpha}(\cdot,\xi)\, W(d\xi) \overset{(d)}{=} D(\alpha)^{-1/\alpha} C_{H+d/2-d/\alpha,d} B_{H+d/2-d/\alpha}(\cdot).$$

More precisely, as previously, $B_{H+d/2-d/\alpha}$ is a standard FBM with index $H + d/2 - d/\alpha$ and $C_{H+d/2-d/\alpha,d}$ is defined by (30). Furthermore, $\nu_{\varepsilon,2}(dv) = \mathbf{1}_{|v|>\varepsilon}\, \nu(dv)$ and $(V_{\varepsilon,n})_{n\geq 1}$, is a sequence of i.i.d variables with common law $\nu_{\varepsilon,2}(dv)/\nu_{\varepsilon,2}(\mathbb{R})$. Let us recall that the sequences B_H, $(U_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(V_{\varepsilon,n})_{n\geq 1}$ are independent. Thus, the approximation of the MAFSM $X_{H,\alpha} = X^{f_{H,\alpha}}$ is given by

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) = D(\alpha)^{-1/\alpha}\sum_{n=1}^{N}\left(\left\|x - \left(\frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)^{1/d} U_n\right\|^{H-d/\alpha} - \left(\frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)^{H/d-1/\alpha}\right)V_{\varepsilon,n} + \sigma(\varepsilon)W^{f_{H,\alpha}}(x).$$

However, this approximation only holds if $f_{H,\alpha}(x,\cdot) \in L^2(\mathbb{R}^d)$, i.e. the FBM $B_{H+d/2-d/\alpha}$ is defined, that is if $1 > H > d/\alpha - d/2$.

Observe that the asymptotic expansion of $f_{H,\alpha}$ is given by (28), replacing $d/2$ by d/α. Then, let $g_{H,\alpha}(x,\xi) = f_{H,\alpha}(x,\xi)\mathbf{1}_{\|\xi\|\geq \max_K \|y\|+1}$ and note that $Y_{\varepsilon,N}^{f_{H,\alpha}} = Y_{\varepsilon,N,2}^{g_{H,\alpha}} + Y_{\varepsilon,N,2}^{f_{H,\alpha}-g_{H,\alpha}} + \sigma(\varepsilon)W^{f_{H,\alpha}}(x)$ with

$$Y_{\varepsilon,N,2}^{h}(x) = D(\alpha)^{-1/\alpha}\sum_{n=1}^{N} h\left(x, \frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)V_{\varepsilon,N}.$$

As noticed in Remark 4.5, $Y_{\varepsilon,N,2}^{f_{H,\alpha}-g_{H,\alpha}}$ is a finite sum. In addition, $g_{H,\alpha}$ satisfies the assumptions of Theorem 4.3 for every $r < \alpha$. In this case therefore,

$$\mathbb{E}\left(\left|Y_{\varepsilon,N,2}^{g_{H,\alpha}}(x) - Y_{\varepsilon,2}^{g_{H,\alpha}}(x)\right|^r\right) \leq \frac{C(r,\beta)D(N,r,\beta)}{N^{r(1/d+1/\alpha-H/d)-1}},$$

where $\beta = 1 + d/\alpha - H$. Furthermore, by Theorem 4.6, $Y_{\varepsilon,N,2}^{g_{H,\alpha}}$ converges uniformly at the rate N^{ε} for every $\varepsilon \in (0,(1-H)/d)$.

Finally, when $H > d/\alpha - d/2$, there exists $\delta \in (0,1]$ such that $H > d/\alpha - d/(2+\delta)$. Then, $\mathbb{E}\left(\left|X_{\varepsilon,1}^{f}(x)\right|^{2+\delta}\right) < +\infty$ and as in the case of MAFLMs, in terms of Berry-Esseen bounds, the rate of convergence of the error due to the approximation of $X_{\varepsilon,1}^{f}(x)$ is of the order

$$\delta(\varepsilon) = \frac{(2-\alpha)^{1+\delta/2}\varepsilon^{\alpha\delta/2}}{(2+\delta-\alpha)2^{\delta/2}}.$$

Except at points $\xi_n = (T_n/c_d\nu_{\varepsilon,2}(\mathbb{R}))^{1/d}U_n$, the pointwise Hölder exponent of $Y_{\varepsilon,N,2}^{f_{H,\alpha}}$ is given by the $W^{f_{H,\alpha}}$'s one and thus is equal to $H - d/\alpha + d/2$. When $H > d/\alpha$, on each compact set, $Y_{\varepsilon,N}^{f_{H,\alpha}}$ has H'-Hölder sample paths for every $H' < H - d/\alpha$. Figure 3 presents a realization of a MAFSM when $\alpha = 1.5$ and $H = 0.7$.

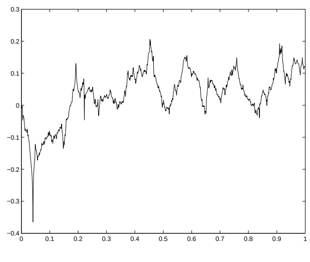


Figure 3: Example of MAFSM with index $H = 0.7$

6.3 Linear Fractional Stable Motions

Here $d = 1$ and we use the notation of Section 6.2. In particular, $\nu(dv)$ is given by (31). In this example, the kernel function is

$$f(x,\xi) = D(\alpha)^{-1/\alpha}\left((x-\xi)_+^{H-1/\alpha} - (-\xi)_+^{H-1/\alpha}\right),$$

where $(a)_+ = a \vee 0$, $D(\alpha)$ is given by (32), $H \in (0,1)$, $H \neq 1/\alpha$ (with the convention $0^{H-1/\alpha} = 0$). Hence, $L_{H,\alpha} = X^f$ is a linear fractional stable motion with index H (see [21] for more details on this process). Furthermore, we may approximate $L_{H,\alpha}$ in distribution by

$$Y^f_{\varepsilon,N}(x) = D(\alpha)^{-1/\alpha} \sum_{n=1}^{N} \left(\left(x - \frac{T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{H-1/\alpha} - \left(\frac{-T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{H-1/\alpha} \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^f(x),$$

where W^f is defined by (25). As previously, W^f is independent of $((U_n, T_n, V_{\varepsilon,n}))_{n\geq 1}$. Moreover,

$$W^f(\cdot) = \int_{\mathbb{R}^d} f(\cdot,\xi)\, W(d\xi) \overset{(d)}{=} D(\alpha)^{-1/\alpha} \widetilde{C_H} B_{H+1/2-1/\alpha}$$

where $B_{H+1/2-1/\alpha}$ is a FBM with index $H + 1/2 - 1/\alpha$ and

$$\begin{aligned} \widetilde{C_H} &= \left(\int_{\mathbb{R}} \left((x-\xi)_+^{H-1/\alpha} - (-\xi)_+^{H-1/\alpha} \right)^2 d\xi \right)^{1/2} \\ &= \Gamma(H+1/2) \sqrt{\frac{\sin\left((H-1/\alpha)\pi\right)\Gamma(1-2H+2/\alpha)}{2\pi(H+1/2-1/\alpha)(H-1/\alpha)}} \end{aligned} \tag{33}$$

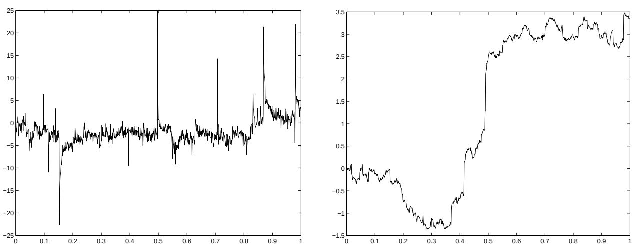
according to Lemma 4.1 in [25]. Obviously, this approximation only holds when $1 > H > 1/\alpha - 1/2$.

Furthermore, let us observe that

$$f(x,\xi) = \begin{cases} 0 & \text{if } \xi > \max_K |y| \\ f_{H,\alpha}(x,\xi) & \text{if } \xi < -\max_K |y|. \end{cases}$$

As a consequence, we obtain the same estimates for the almost sure, the L^r errors ($r < \alpha$) and the rate of convergence in terms of Berry-Esseen bounds as in the case of MAFSMs (see Section 6.2).

Figure 4 presents two realizations of LFSMs for $\alpha = 1.5$. As noticed in [23], when $H = 0.2$, we observe spikes which take place at points ξ_n. Actually, since $H = 0.2 < 1/\alpha$, when x tends to a point ξ_n, $Y^f_{\varepsilon,N}(x)$ tends to infinity, which explains that spikes appear. When $H = 0.7 > 1/\alpha$, as in the case of MAFSMs, the sample paths of the approximation are H'-Hölder on each compact set for every $H' < H - 1/\alpha$.



$H = 0.3$ $\quad\quad$ $H = 0.7$

Figure 4: Examples of LFSMs

6.4 Log-Fractional Stable Motion

Let $d = 1$ and $1 < \alpha < 2$ and assume that $\nu(dv)$ is given by (31). Furthermore, $(V_{\varepsilon,n})_{n\geq 1}$ and $\sigma(\varepsilon)$ are defined as in Section 6.2. Remark that here $(U_n)_{n\geq 1}$ is a sequence of i.i.d symmetric Bernoulli random variables. Then, let

$$f(x,\xi) = D(\alpha)^{-1/\alpha}(\ln|x-\xi| - \ln|\xi|).$$

Hence, X^f is a log-fractional stable motion and its approximation in law is given by

$$Y^f_{\varepsilon,N}(x) = D(\alpha)^{-1/\alpha} \sum_{n=1}^{N} \left(\ln \left| x - \frac{T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right| - \ln \left(\frac{T_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right) \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^f(x),$$

where

$$W^f(x) = D(\alpha)^{-1/\alpha} \int_{\mathbb{R}} (\ln|x-\xi| - \ln|\xi|)\, W(d\xi)$$

is independent of $((U_n, T_n, V_{\varepsilon,n}))_{n\geq 1}$. Note that $W^f \overset{(d)}{=} D(\alpha)^{-1/\alpha} C B_{1/2}$ where $B_{1/2}$ is a standard Brownian motion and

$$C = \int_{\mathbb{R}} (\ln|1-\xi| - \ln|\xi|)^2\, d\xi.$$

Furthermore, by a Fourier transform argument, one proves that [21]

$$C = \left(\frac{\pi}{2} \int_{\mathbb{R}} \frac{\left|\mathrm{e}^{-i\lambda} - 1\right|^2}{|\lambda|^2}\, d\lambda \right)^{1/2} = \pi.$$

As previously, the rate of almost sure convergence can be studied. In particular, if

$$g(x,\xi) = f(x,\xi) \mathbf{1}_{|\xi| \geq \max_K |y| + 1},$$

$Y^g_{\varepsilon,N,2}$ converges uniformly on K at least at the rate N^ε for every $\varepsilon \in (0, 1-1/\alpha)$. Furthermore, the L^r-error can be controlled and decreases in $N^{1-1/r}$ for every $r < \alpha$. Let us notice that X^f is a self-similar field with index $H = 1/\alpha$. Thus, we obtain the same rate of convergence for log-fractional stable motion and MAFSMs. Furthermore, since $f(x,\cdot) \in L^3(\mathbb{R})$, Theorem 5.2 gives the same rate of convergence in terms of Berry-Esseen bounds as in the cases of MAFSMs or MAFLMs (taking $\delta = 1$).

Figure 5 presents a trajectory of a log-fractional stable motion for $\alpha = 1.5$. Note that except at points $\xi_n = T_n U_n / (2\nu_{\varepsilon,2}(\mathbb{R}))$, the sample paths are locally H'-Hölder for every $H' < 1/2$: actually the regularity of the trajectories is given by the Brownian part. At points ξ_n, $Y^f_{\varepsilon,N}$ is not defined, which explains that spikes appear in Figure 5.

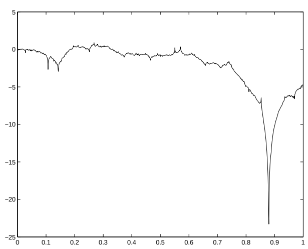


Figure 5: Example of log-fractional stable motion

6.5 Linear Multifractional Stable Motion

So far, the examples are fractional fields. However, our framework also contains multifractional fields. Let us now give one example. This example is defined replacing in the kernel of a LFSM the index H by $h(x)$.

Here $d = 1$ and $\nu(dv)$ is given by (31). Then, assume that the kernel function is defined by

$$f(x,\xi) = (x-\xi)_+^{h(x)-1/\alpha} - (-\xi)_+^{h(x)-1/\alpha}$$

where $h : \mathbb{R} \to (0,1)$. The process X^f is a linear multifractional stable motion in the sense of [26, 24]. The approximation of X^f is then given by

$$Y^f_{\varepsilon,N}(x) = D(\alpha)^{-1/\alpha} \sum_{n=1}^{N} \left(\left(x - \frac{T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{h(x)-1/\alpha} - \left(\frac{-T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{h(x)-1/\alpha} \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^f(x),$$

where W^f is defined by (25). As previously, W^f is independent of $((U_n, T_n, V_{\varepsilon,n}))_{n\geq 1}$. Moreover,

$$W^f(\cdot) = \int_{\mathbb{R}^d} f(\cdot, \xi)\, W(d\xi) \overset{(d)}{=} D(\alpha)^{-1/\alpha} \widetilde{C_{h(x)}} B_{h+1/2-1/\alpha}$$

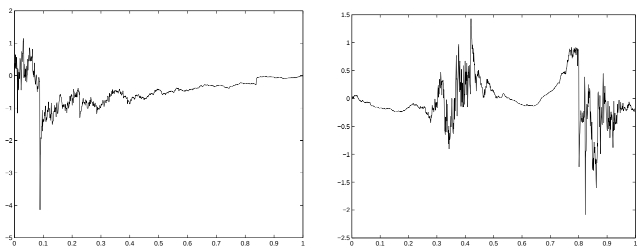
where $B_{h+1/2-1/\alpha}$ is a standard multifractional Brownian motion in the sense of [15] with multifractional function $h + 1/2 - 1/\alpha$ and $\widetilde{C_{h(x)}}$ is given by (33). This approximation only holds when $1 > h(x) > 1/\alpha - 1/2$.

As in the case of LFSM, we can observe that

$$f(x,\xi) = \begin{cases} 0 & \text{if } \|\xi\| > \max_K |y| \\ f_{h(x),\alpha}(x,\xi) & \text{if } \|\xi\| < -\max |y|. \end{cases}$$

Therefore, for a fixed x, we obtain the same estimates for the almost sure, the L^r errors ($r < \alpha$) and the rate of convergence in terms of Berry-Esseen bounds as in the case of LFSM (see Section 6.3) or MAFSMs (see Section 6.2), replacing H by $h(x)$. In particular, for a fixed x, the almost sure error converges at the rate N^{ε} for every $\varepsilon \in (0, (1-h(x))/d)$.

Figure 6 presents some trajectories of linear multifractional stable motions for $\alpha = 1.5$.



$h(t) = 0.3 + 0.6t$ $\qquad$ $h(t) = 0.6 + 0.3\sin(4\pi t)$

Figure 6: Examples of linear multifractional stable motion

7 Extension to complex random measure

Thanks to arguments used in Section 4 and 5, the results obtained in the case of RHMLMs in [11] can be extended to a larger class of infinitely divisible fields, in particular to the complex case. More precisely, let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with intensity $n(d\xi, dz) = d\xi\nu(dz)$. Assume that the σ-finite measure $\nu(dz)$ satisfies

$$\int_{\mathbb{C}} \left(|z|^2 \wedge 1 \right) \nu(dz) < +\infty.$$

Furthermore, the control measure $\nu(dz)$ is assumed to be rotationally invariant, i.e.

$$P(\nu(dz)) = d\theta\, \nu_\rho(d\rho), \tag{34}$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$ and $P(\rho e^{i\theta}) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}^+_*$.

Then, following the definition of complex Lévy random measure (see [3]), we can consider a complex random measure $\Lambda(d\xi)$ on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} g(\xi)\,\Lambda(d\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} (g(\xi)z + g(-\xi)\overline{z}) \left(N(d\xi,dz) - (|g(\xi)z + g(-\xi)\overline{z}| \vee 1)^{-1}\, n(d\xi,dz)\right) \tag{35}$$

for every $g : \mathbb{R}^d \to \mathbb{C}$ such that $\displaystyle\int_{\mathbb{R}^d\times\mathbb{C}} \left(|g(\xi)z|^2 \wedge 1\right) d\xi\nu(dz) < +\infty$.
Hence, following the arguments used in [11] in the case of RHFLMs, analogous results to those of Sections 4 and 5 can be obtained and a way to simulate

$$X^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda(d\xi).$$

can be proposed. However, in this part, we will just focus on the case where

$$\nu_\rho(d\rho) = \frac{\mathbf{1}_{\rho>0}\,d\rho}{\rho^{1+\alpha}}, \quad \alpha \in (0,2),$$

and the kernel function is

$$f_{H,\alpha}(x,\xi) = \frac{\left(2^{\alpha+1}\pi D(\alpha)\right)^{-1/\alpha}\left(\mathrm{e}^{-ix\cdot\xi} - 1\right)}{\|\xi\|^{H+d/\alpha}}$$

with $D(\alpha)$ given by (32). In this case,

$$X_{H,\alpha}(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,\Lambda(d\xi), \quad x \in \mathbb{R}^d,$$

is a real harmonizable fractional stable motion with index $H \in (0,1)$, i.e.

$$\{X_{H,\alpha}(x),\, x \in \mathbb{R}^d\} \overset{(d)}{=} \{S_{H,\alpha}(x),\, x \in \mathbb{R}^d\}$$

where

$$S_H(x) = \Re\left(\int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,M_\alpha(d\xi)\right)$$

with $M_\alpha(d\xi)$ a complex isotropic α-stable random measure with control measure the Lebesgue measure in the sense of [21].

Furthermore, in the case we are interested in, $\nu(\mathbb{C}) = +\infty$. As we know, we have to split in this case the random field $X_{H,\alpha} = X^{f_{H,\alpha}}_{\varepsilon,1} + X^{f_{H,\alpha}}_{\varepsilon,2}$ into two random fields where

$$X^{f_{H,\alpha}}_{\varepsilon,1}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(f_{H,\alpha}(x,\xi)z)\mathbf{1}_{|z|<\varepsilon} \left(N(d\xi,dz) - (|2\Re(f_{H,\alpha}(x,\xi)z)| \vee 1)^{-1}\, n(d\xi,dz)\right) \tag{36}$$

and

$$X^{f_{H,\alpha}}_{\varepsilon,2}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(f_{H,\alpha}(x,\xi)z)\mathbf{1}_{|z|\geq\varepsilon} \left(N(d\xi,dz) - (|2\Re(f_{H,\alpha}(x,\xi)z)| \vee 1)^{-1}\, n(d\xi,dz)\right). \tag{37}$$

As previously, $X^{f_{H,\alpha}}_{\varepsilon,1}$ and $X^{f_{H,\alpha}}_{\varepsilon,2}$ can be simulated independently. Furthermore,

$$X^f_{\varepsilon,2}(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,\Lambda_{\varepsilon,2}(d\xi), \quad x \in \mathbb{R}^d,$$

where the complex random measure $\Lambda_{\varepsilon,2}$ is associated by (35) to a Poisson random measure $N_{\varepsilon,2}$ whose control measure $\nu_{\varepsilon,2}(dz) = \mathbf{1}_{|z|\geq\varepsilon}\,\nu(dz)$ is finite. Therefore, $X^f_{\varepsilon,2}$ can be simulated as a generalized shot noise series. More precisely, let $(Z_{\varepsilon,n})_{n\geq 1}$ be a sequence of i.i.d. random variables with common law $\nu_{\varepsilon,2}(dz)/\nu_{\varepsilon,2}(\mathbb{C})$. Moreover, $(Z_{\varepsilon,n})_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are independent. Then, as in the case of RHMLMs, a series expansion of $X^f_{\varepsilon,2}$ can be given and this series converges in the space of continuous functions endowed with the topology of the uniform convergence on compact sets.

Proposition 7.1. *For every $x \in \mathbb{R}^d$,*

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) = 2\sum_{n=1}^{N} \Re\left(f_{H,\alpha}\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_{\varepsilon,n}\right) \tag{38}$$

converges almost surely to $Y_\varepsilon^{f_{H,\alpha}}(x)$ as $N \to +\infty$. Furthermore, $Y_{\varepsilon,N}^{f_{H,\alpha}}$ converges uniformly on each compact set almost surely and

$$\{X_{\varepsilon,2}^{f_{H,\alpha}}(x) : x \in \mathbb{R}^d\} \overset{(d)}{=} \{Y_\varepsilon^{f_{H,\alpha}}(x) : x \in \mathbb{R}^d\}.$$

Proof. The arguments of proof of Proposition 4.1 lead to the almost sure convergence of $Y_{\varepsilon,N}^{f_{H,\alpha}}(x)$ for each fixed x. They also give the equality of the finite dimensional marginals of $X_{\varepsilon,2}^{f_{H,\alpha}}$ and $Y_\varepsilon^{f_{H,\alpha}}$. In order to obtain the uniform convergence, one may follow the proof of Proposition 3.1 in [11]. □

Due to the rotational invariance of $Z_{\varepsilon,n}$ and to Theorem 4.3, a rate of almost sure convergence for $Y_{\varepsilon,N}^{f_{H,\alpha}}(x)$ can be given and the L^r-error can be controlled.

Proposition 7.2. *Let $x \in \mathbb{R}^d$.*

1. *Then, for every $\varepsilon \in (0, H/d)$, almost surely,*

$$\sup_{N\geq 1} N^\varepsilon \left| Y_\varepsilon^{f_{H,\alpha}}(x) - Y_{\varepsilon,N}^{f_{H,\alpha}}(x)\right| < +\infty.$$

2. *Moreover, for every $r < \alpha$ and every integer $N > r(1/\alpha + H/d)$,*

$$\mathbb{E}\left(\left|Y_{\varepsilon,N}^{f_{H,\alpha}}(x) - Y_\varepsilon^{f_{H,\alpha}}(x)\right|^r\right) \leq C(r)\,\frac{D(N, r, H + d/\alpha)}{N^{r/\alpha + rH/d - 1}}, \tag{39}$$

where $D(N, r, \beta)$ is defined by (15) *and*

$$C(r) = \frac{\left(2^{1-\alpha}\pi D(\alpha)\right)^{-r/\alpha} d(c_d\nu(\mathbb{R}))^{rH/d + r/\alpha}\,\mathbb{E}(|\Re(V_1)|^r)}{rH - d + rd/\alpha}.$$

Proof. Since $(Z_{\varepsilon,n})_{n\geq 1}$ is a sequence of i.i.d. with common law invariant by rotation,

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) \overset{(d)}{=} 2\sum_{n=1}^{N}\left| f_{H,\alpha}\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right)\right| \Re(Z_{\varepsilon,n}).$$

Hence, taking $V_n = \Re(Z_{\varepsilon,n})$, $C = 2^{1-1/\alpha}(\pi D(\alpha))^{-1/\alpha}$ and $\beta = H + d/\alpha$, the proof of Theorem 4.3 leads to the conclusion. □

Finally, the next proposition gives the expected approximation of $X_{\varepsilon,1}$. Let

$$\sigma(\varepsilon) = \left(\int_0^\varepsilon \rho^2\,\nu_\rho(d\rho)\right)^{1/2} = \sqrt{\frac{2\varepsilon^{2-\alpha}}{2-\alpha}}. \tag{40}$$

Proposition 7.3. *Assume that $0 < H + d/\alpha - d/2 < 1$ then*

$$\lim_{\varepsilon\to 0_+}\left(\frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)}\right)_{x\in\mathbb{R}^d} \overset{(d)}{=} \left(A_{H+d/\alpha-d/2} B_{H+d/\alpha-d/2}(x)\right)_{x\in\mathbb{R}^d},$$

where the convergence is in distribution on the space of continuous functions endowed with the topology of uniform convergence on compact sets, $B_{H+d/\alpha-d/2}$ is a standard FBM with index $H + d/\alpha - d/2$ and for $u \in (0,1)$

$$A_u = \left(2^{\alpha+1}\pi D(\alpha)\right)^{-1/\alpha}\left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(u+1/2)}{u\,\Gamma(2u)\sin(\pi u)\,\Gamma(u+d/2)}\right)^{1/2}. \tag{41}$$

Remark 7.4. *In Proposition 7.3,* $0 < H + d/\alpha - d/2 < 1$ *means that* $f_{H,\alpha}(x,\cdot) \in L^2(\mathbb{R}^d)$ *for every* $x \in \mathbb{R}^d$*.*

Proof. Actually

$$X^f_{\varepsilon,1}(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,\Lambda_{\varepsilon,1}(d\xi), \quad x \in \mathbb{R}^d,$$

where the complex random measure $\Lambda_{\varepsilon,1}$ is associated by (35) with a Poisson random measure $N_{\varepsilon,1}$ whose control measure $\nu_{\varepsilon,1}(dz) = \mathbf{1}_{|z|<\varepsilon}\,\nu(dz)$. Also, for every $p \geq 2$,

$$\int_{\mathbb{C}} |z|^p\,\nu_{\varepsilon,1}(dz) < +\infty$$

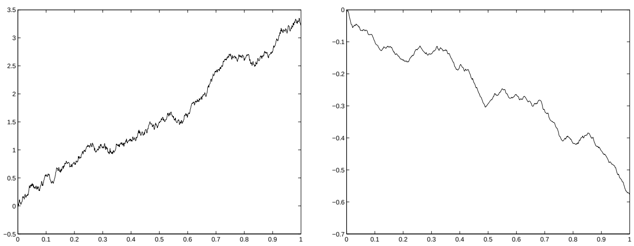
and then $\left(2^{\alpha+1}\pi D(\alpha)\right)^{1/\alpha} X^f_{\varepsilon,1}(x)$ is a RHFLM (real harmonizable fractional Lévy motion) since $f_{H,\alpha}(x,\cdot) \in L^2(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. Then, Proposition 4.1 in [11] yields the conclusion. □

As a consequence, as soon as the assumptions of Proposition 7.3 are fulfilled, we may approximate the RHFSM $X_{H,\alpha}$ by

$$Y_{\varepsilon,N}(x) = 2\sum_{n=1}^{N} \Re\left(f_{H;\alpha}\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_{\varepsilon,n}\right) + \sigma(\varepsilon) A_{H+d/\alpha-d/2} B_{H+d/\alpha-d/2}(x), \quad x \in \mathbb{R}^d,$$

where $B_{H+d/\alpha-d/2}$, T_n, U_n and $Z_{\varepsilon,n}$ are independent.

Figure 7 exhibits some examples of trajectories of RHFSMs for $\alpha = 1.5$.



$H = 0.3$ $\qquad$ $H = 0.7$

Figure 7: Examples of RHFSMs

Appendix

A Proof of Theorem 2.1

Since $0 \leq r \wedge 2 \leq r$ and $\mathbb{E}(|X_1|^r) < +\infty$, we also have that $\mathbb{E}\left(|X_1|^{r\wedge 2}\right) < +\infty$. Then, we can assume that $r \leq 2$.

Set $R_N = \sum_{n=N+1}^{+\infty} Y_n X_n$ and $r = r \wedge 2 \in (0,2)$. Then, let us fix $M > 0$ and set

$$\Omega_M = \left\{ \sup_{n\geq 1} n^{-1/r} |X_n| \leq M \right\}.$$

Hence for any $\varepsilon > 0$,

$$\mathbb{P}\left(\Omega_M \cap \{|R_N| \geq N^{-\varepsilon}\}\right) \leq \mathbb{P}\left(\left| \sum_{n=N+1}^{+\infty} Y_n W_n \right| \geq N^{-\varepsilon} \right),$$

where $W_n = X_n \mathbf{1}_{|X_n| \leq M n^{1/r}}$. Since X_n, $n \geq 1$, are i.i.d. and symmetric, $(W_n)_{n\geq 1}$ is a sequence of independent symmetric random variables. Then, since $(Y_n)_{n\geq 1}$ satisfies the assumption (2) and is independent of $(W_n)_{n\geq 1}$, by the contraction principle for symmetric random variables sequences, see [12] page 95,

$$\mathbb{P}\big(\Omega_M \cap \{|R_N| \geq N^{-\varepsilon}\}\big) \leq 2\mathbb{P}\left(C\left|\sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} W_n\right| \geq N^{-\varepsilon}\right).$$

Hence,

$$\mathbb{P}\big(\Omega_M \cap \{|R_N| \geq N^{-\varepsilon}\}\big) \leq 2\mathbb{P}\left(\sup_{n\geq N+1} \frac{n}{T_n} \geq 10\right) + 2A_N \tag{42}$$

where

$$A_N = \mathbb{P}\left(\left\{\sup_{n\geq N+1} \frac{n}{T_n} < 10\right\} \cap \left\{C\left|\sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} W_n\right| \geq N^{-\varepsilon}\right\}\right).$$

Step 1

$$\mathbb{P}\left(\sup_{n\geq N+1} \frac{n}{T_n} \geq 10\right) \leq \sum_{n=N+1}^{+\infty} \mathbb{P}(T_n \leq n/10) \leq \sum_{n=N+1}^{+\infty} \frac{n^n}{10^n n!}.$$

Hence, by the Stirling formula,

$$\mathbb{P}\left(\sup_{n\geq N+1} \frac{n}{T_n} \geq 10\right) \leq C_1 \exp\left(-C_2 N\right), \tag{43}$$

with $C_1 > 0$ and $C_2 > 0$.

Step 2 By the assumptions of independence, $(T_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ are independent. Therefore, by the contraction principle for symmetric random variables sequences,

$$A_N \leq 2\mathbb{P}\left(C\left|\sum_{n=N+1}^{+\infty} n^{-1/\gamma} W_n\right| \geq 10^{-1/\gamma} N^{-\varepsilon}\right).$$

Furthermore, by independence and symmetry,

$$\begin{aligned} A_N &\leq 4\mathbb{P}\left(C\sum_{n=N+1}^{+\infty} n^{-1/\gamma} W_n \geq 10^{-1/\gamma} N^{-\varepsilon}\right) \\ &\leq 4\exp\left(-\frac{10^{-1/\gamma}\lambda N^{-\varepsilon}}{C}\right) \prod_{n=N+1}^{+\infty} \mathbb{E}\Big(\exp\left(\lambda n^{-1/\gamma} W_n\right)\Big), \end{aligned}$$

where $\lambda > 0$. Moreover, since W_n is a symmetric random variable,

$$\mathbb{E}\Big(\exp\left(\lambda n^{-1/\gamma} W_n\right)\Big) = 1 + \sum_{j=1}^{+\infty} \frac{\lambda^{2j}}{(2j)!} n^{-2j/\gamma} \mathbb{E}\big(W_n^{2j}\big).$$

Then let $a = 1/\gamma - 1/r$ and $n \geq N+1$. Note that for $j \geq 1$, $2j \geq r$ and

$$\mathbb{E}\big(W_n^{2j}\big) \leq \mathbb{E}(|X_1|^r)\Big(M n^{1/r}\Big)^{2j-r}.$$

Therefore,

$$\begin{aligned} \mathbb{E}\big(\exp\left(\lambda n^{-1/\gamma} W_n\right)\big) &\leq 1 + \frac{\mathbb{E}(|X_1|^r)\lambda^2 M^{2-r} \exp\left(\lambda^2 M^2 n^{-2a}\right)}{2n^{1+2a}} \\ &\leq \exp\left(\frac{\mathbb{E}(|X_1|^r)\lambda^2 M^{2-r} \exp\left(\lambda^2 M^2 N^{-2a}\right)}{2n^{1+2a}}\right) \end{aligned}$$

As a consequence, taking $\lambda = 10^{1/\gamma} N^a$, there exist $C_3 > 0$ and $C_4 > 0$, which do not depend on N, such that

$$A_N \leq C_3 \exp\big(-C_4 N^{a-\varepsilon}\big). \tag{44}$$

Step 3 In view of (42), (43) and (44), for every $M > 0$ and every $\varepsilon \in (0, 1/\gamma - 1/r)$,

$$\sum_{N=1}^{+\infty} \mathbb{P}\big(\Omega_M \cap \big\{|R_N| \geq N^{-\varepsilon}\big\}\big) < +\infty.$$

Hence, by the Borel Cantelli lemma, for almost all $\omega \in \Omega_M$,

$$\sup_{N \geq 1} N^{\varepsilon} |R_N| < +\infty.$$

Furthermore, since $X_n \in L^r$,

$$\lim_{M \to +\infty} \mathbb{P}(\Omega_M) = \lim_{M \to +\infty} \mathbb{P}\left(\sup_{n \geq 1} |X_n| n^{-1/r} \leq M\right) = 1.$$

Then, for every $\varepsilon \in (0, 1/\gamma - 1/r)$, almost surely,

$$\sup_{N \geq 1} N^{\varepsilon} |R_N| < +\infty,$$

which concludes the proof.

B Proof of Theorem 2.2

It is a simple modification of the proof of Theorem 2.1.

Let $M > 0$, $\Omega_M = \left\{\sup_{n \geq 1} \left| n^{-1/r} X_n \right| \leq M\right\}$, $W_n = X_n \mathbf{1}_{|X_n| \leq n^{1/r} M}$ and

$$R_N = \sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |X_n|.$$

As in proof of Theorem 2.1,

$$\mathbb{P}\big(\Omega_M \cap \big\{|R_N| \geq N^{-\varepsilon}\big\}\big) \leq \mathbb{P}\left(\sup_{n \geq N+1} \frac{n}{T_n} \geq 10\right) + A_N \tag{45}$$

where

$$A_N = \mathbb{P}\left(\left\{\sup_{n \geq N+1} \frac{n}{T_n} < 10\right\} \cap \left\{\sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |W_n| \geq N^{-\varepsilon}\right\}\right).$$

Remark now that the contraction principle used in the proof of Theorem 2.1 can not be applied since $|W_n|$ is not a symmetric random variable. However, since $|W_n| \geq 0$,

$$\begin{aligned} A_N &\leq \mathbb{P}\left(\sum_{n=N+1}^{+\infty} n^{-1/\gamma} |W_n| \geq 10^{-1/\gamma} N^{-\varepsilon}\right) \\ &\leq \exp\left(-10^{-1/\gamma} \lambda N^{-\varepsilon}\right) \prod_{n=N+1}^{+\infty} \mathbb{E}\Big(\exp\left(\lambda n^{-1/\gamma} |W_n|\right)\Big), \end{aligned}$$

where $\lambda > 0$. Furthermore,

$$\begin{aligned}\mathbb{E}\Big(\exp\left(\lambda n^{-1/\gamma}|W_n|\right)\Big) &= 1+\sum_{j=1}^{+\infty}\frac{\lambda^j}{j!}n^{-j/\gamma}\mathbb{E}\Big(|W_n|^j\Big) \\ &\leq 1+\mathbb{E}(|X_1|^r)\sum_{j=1}^{+\infty}\frac{\lambda^j}{j!}n^{-j/\gamma}\Big(Mn^{1/r}\Big)^{j-r} \text{ since } r\leq 1 \\ &\leq \exp\left(\mathbb{E}(|X_1|^r)\lambda n^{-1-a}M^{1-r}\exp\left(\lambda M N^{-a}\right)\right),\end{aligned}$$

where $a = 1/\gamma - 1/r$ and $n \geq N+1$. Hence, choosing $\lambda = 10^{1/\gamma}N^a$, there exists C, which does not depend on N, such that

$$A_N \leq C\exp\left(-N^{a-\varepsilon}\right).$$

Consequently, the arguments used in step 3 of the proof of Theorem 2.1 lead to the conclusion.

Acknowledgments

The authors would like to thank Jan Rosiński for very interesting discussions concerning the simulation of moving average fractional Lévy motions. Part of this work was accomplished while Céline Lacaux was a member of the laboratory CEREMADE (Paris Dauphine University).

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