On locally self-similar fractional random fields indexed by a manifold

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November 25, 2007

Abstract

Local self-similarity for Euclidean random fields has been introduced since a while. In this paper, we define local self-similarity for manifold indexed random fields. We study the properties of the tangent field. In the Gaussian ($\alpha = 2$) and α-stable ($0 < \alpha < 2$) cases, we obtain the expected relations between the fractional index H and the stability index α. We then give examples.

Keywords: Local self-similarity, random field, manifold.
Subject Classification MSC-2000: 60F05, 60G10, 60G18, 60G60.

1 Introduction

Self-similar random fields are widely used to model natural phenomena. Let us recall that a random field $X = (X(x))_{x\in\mathbb{R}^n}$ is self-similar with index H iff

$$\forall c > 0,\ (X(cx))_{x\in\mathbb{R}^n} \overset{(d)}{=} c^H (X(x))_{x\in\mathbb{R}^n}, \tag{1}$$

where $\overset{(d)}{=}$ stands for equality of finite dimensional distributions. For instance, persistent phenomena in Internet traffic, hydrology, geophysics or financial markets, e.g. [1, 19, 25, 28], are known to be self-similar. The most famous and classical self-similar model is provided by the fractional Brownian motion $B_H = (B_H(x))_{x\in\mathbb{R}^n}$, in short FBM, with Hurst index $H \in (0, 1)$. The FBM B_H, introduced in [16] and developed in [21], is a Gaussian centered random field with stationary increments. Self-similar α-stable random fields, see [25] for an introduction, have been proposed to model some natural phenomena with heavy tails.

Self-similarity is a global property and is then too restrictive for some applications, see [27] and references therein. Therefore, it has been weakened since a while in [9, 22]. More precisely, [9, 22] introduce the so-called locally asymptotically self-similar property for random fields indexed by $\mathbb{R}^n$. Roughly speaking, a field $X = (X(x))_{x\in\mathbb{R}^n}$ is locally asymptotically self-similar (in short lass) at point x_0 if its increments around x_0, suitably normalized, converge to a non degenerate random field, called tangent field to X at point x_0. The properties of this tangent field have been studied in [11, 12]. The most famous lass random fields are multifractional Brownian motions, introduced in [9, 22]. However, many other examples have been studied, e.g. [2, 5, 6, 7, 8, 17, 18].

Self-similarity and lass property have been defined for random fields indexed by the Euclidean space $\mathbb{R}^n$. In order to avoid confusion, we will call random fields indexed by the Euclidean space $\mathbb{R}^n$ Euclidean random fields.

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Nevertheless, various spatial data (e.g. [10, 24]) are indexed by a manifold and not by the Euclidean space $\mathbb{R}^n$. For instance, geostatistical Earth data or cosmic microwave background are indexed by a sphere. Fractional random fields indexed by a sphere or an hyperbolic space have recently been introduced in [14, 15]. Let us recall that [14, 15] propose to define fractional manifold indexed fields as follows : a distance d, for instance the geodesic one, is defined on the manifold $\mathcal{M}$; then one says that the field $X = (X(M))_{M\in\mathcal{M}}$ is fractional with index $H > 0$ iff there exists a random variable Z such that for all $M, N \in \mathcal{M}$,

$$\frac{X(M) - X(N)}{d(M,N)^H} \overset{(d)}{=} Z,$$

i.e. iff its normalized increments with respect to the distance are constant in distribution. This fractional property is of course a global one, as is the global self-similarity property (1). In this paper, we consider random fields whose increments satisfy a localized fractional property. In other words, we extend the lass property to the framework of manifold indexed random fields.

Then, we investigate the properties of the tangent field at a given point $M_0 \in \mathcal{M}$. This field will be indexed by a subspace of the tangent space to the manifold $\mathcal{M}$ at point M_0. In general, it can not be defined on whole the tangent space. As expected, the tangent field is self-similar but not at any scale: (1) may not be fulfilled for any $c > 0$. Moreover, if the manifold indexed field has weak stationary increments, so has the tangent field. Some expected consequences are derived. In particular, if the tangent field is Gaussian, then it is a restriction of an Euclidean fractional Brownian motion. If the tangent field is α-stable with lass index H, then $0 < H \leq \max(1, 1/\alpha)$.

We then describe examples of manifold indexed fields, Gaussian and stable. Gaussian and stable random fields are of common use for modelling spatial data. We prove that the tangent field of a manifold indexed fractional Brownian motion, field defined in [14], is an Euclidean fractional Brownian motion. We then focus on spherical examples. Plenty of Gaussian or stable random fields indexed by spheres can be considered. Moving average fractional fields are on common use for modelling purpose, see [25]. Therefore, we focus on spherical moving average fractional fields, defined by an integration of a spherical moving average fractional kernel against a Gaussian or stable random measure. We prove that the tangent field of a spherical moving average fractional field is an Euclidean moving average fractional field. We then extend these constructions to multifractional fields.

In Section 2, notation in the framework of manifold indexed lass random fields are given. Section 3 is devoted to the definition of weak stationarity and lass property. Then, some properties of the tangent field of a lass field are studied in Section 4. Examples are developed in Section 5.

2 Preliminaries and Notation

In this paper, we consider $(X(M))_{M\in\mathcal{M}}$ a real valued random field indexed by a $\mathcal{C}^\infty$-complete Riemannian manifold $(\mathcal{M}, g)$ of dimension n. The distance $d(x, y)$ is defined as the length of the shortest curve between x and y.

Let $M_0 \in \mathcal{M}$. $T_{M_0}\mathcal{M}$ is the tangent space to $\mathcal{M}$ at M_0. Then, there exists a neighborhood $\mathcal{V}(M_0)$ of M_0 and $\delta > 0$, see for example [13], such that

1. for all $M \in \mathcal{V}(M_0)$, there exists an unique minimal geodesic between M and M_0,

2. the exponential map $\exp_{M_0}$ at point M_0 is a diffeomorphism between the open ball $\mathcal{B}(0, \delta) \subset T_{M_0}\mathcal{M}$ and $\mathcal{V}(M_0)$.

3 Definitions

Weak stationarity and local asymptotically self-similarity (in short lass) are well known properties for fields indexed by the Euclidean space $\mathbb{R}^n$ (e.g. [9, 22, 25]). We extend them in framework of manifold indexed fields.

3.1 Weak stationarity

Definition 3.1. *The increments of the field $X = (X(M))_{M\in\mathcal{M}}$ are weakly stationary if for all $(M,N) \in \mathcal{M}^2$, the distribution of $X(M) - X(N)$ only depends on the geodesic distance $d(M,N)$, i.e. if there exists a function ψ such that for all $(M,N) \in \mathcal{M}^2$ and all $\lambda \in \mathbb{R}$,*

$$\mathbb{E}\left[e^{i\lambda(X(M)-X(N))}\right] = \psi(\lambda, d(M,N)).$$

3.2 Local asymptotically self-similarity

Let us recall that a random field $(X(x))_{x\in\mathbb{R}^n}$ is locally asymptotically self-similar (in the sense of [9, 22]) at point x_0 with index H if

$$\lim_{\varepsilon\to 0_+} \left(\frac{X(x_0+\varepsilon x) - X(x_0)}{\varepsilon^H}\right)_{x\in\mathbb{R}^n} \overset{(d)}{=} (Z_{x_0}(x))_{x\in\mathbb{R}^n} \tag{2}$$

where $\overset{(d)}{=}$ stands for equality of finite dimensional distributions and Z_{x_0} is a non degenerate field, that is for almost ω, there exists x, such that

$$Z_{x_0}(x,\omega) \neq 0.$$

The random field Z_{x_0} is called tangent field at point x_0 of X.

We extend this notion to fields indexed by a manifold which is not in general a vector space. Hence, we first have to interpret $x_0 + \varepsilon x$ as a point of the manifold $\mathbb{R}^n$ without the help of the addition on $\mathbb{R}^n$. Note that the tangent space to $\mathbb{R}^n$ at any point is identified to $\mathbb{R}^n$. On one hand, x_0 is a point of $\mathbb{R}^n$, which correspond to the point M_0 for the manifold $\mathcal{M}$. On the other hand, $x_0 + \varepsilon x$ is the shift of M_0 by the vector $\varepsilon x \in T_{x_0}\mathbb{R}^n \approx \mathbb{R}^n$. Also, since the geodesics in $\mathbb{R}^n$ are the segments, we have

$$x_0 + \varepsilon x = \exp_{x_0}(\varepsilon x).$$

Then, we propose to replace in (2) the point x_0 by M_0 and its translate $x_0 + \varepsilon x$ by

$$M_0 + \varepsilon v \overset{def}{=} \exp_{M_0}(\varepsilon v). \tag{3}$$

Note that $M_0 + \varepsilon v$ is well defined as soon as $v \in \mathcal{B}(0,\delta)$ and $\varepsilon \in [0,1]$. Let us fix $M \in \mathcal{V}(M_0)$. Then, there exists an unique $v \in \mathcal{B}(0,\delta)$ such that $\exp_{M_0}(v) = M$ so that $M_0 + v = M$. Moreover, $M_0 + \varepsilon v$ is the only point of the geodesic between M_0 and M such that

$$d(M_0, M_0 + \varepsilon M) = \varepsilon d(M_0, M).$$

$M_0 + \varepsilon v$ describes the geodesic between M_0 and M as ε varies in $[0,1]$. In addition, as ε tends to zero, $M_0 + \varepsilon v$ tends to M_0 in the direction given by this geodesic.

Definition 3.2. $X = (X(M))_{M\in\mathcal{M}}$ *is locally asymptotically self-similar (lass in short) at point M_0 with index $H > 0$ if*

$$\lim_{\varepsilon\to 0_+} \left(\frac{X(M_0+\varepsilon v) - X(M_0)}{\varepsilon^H}\right)_{v\in B(0,\delta)} \overset{(d)}{=} (Z_{M_0}(v))_{v\in B(0,\delta)}$$

where Z_{M_0} is a non degenerate field and $\overset{(d)}{=}$ stands for equality of finite dimensional distributions. Z_{M_0} is called the tangent field at point M_0 of X.

As one could expect, the definition of a lass random field at point M_0 and the definition of its tangent field coincide with the definitions of [9, 22] in the framework of random fields indexed by $\mathbb{R}^n$. In particular, if $X = (X(x))_{x\in\mathbb{R}^n}$ is a self-similar random field with index $H > 0$ then $X(0) = 0$ almost surely and X is lass at point $M_0 = 0$ with index H and with itself as tangent field.

The most classical examples of such fields are the Euclidean fractional Brownian motions [16, 21]. Many examples of lass random fields indexed by $\mathbb{R}^n$ at any point have been introduced, e.g. [2, 5, 6, 7, 8, 9, 17, 18, 22].

4 Properties of the tangent field

Theorem 4.1. *If X is lass at point M_0 with index H and tangent field Z_{M_0}, then $Z_{M_0}(0) = 0$ almost surely and*

$$\forall\lambda \in (0, 1],\ (Z_{M_0}(\lambda v))_{v\in B(0,\delta)} \overset{(d)}{=} \lambda^H (Z_{M_0}(v))_{v\in B(0,\delta)}. \tag{4}$$

Moreover, if X has weakly stationary increments so has Z_{M_0}.

Corollary 4.2. *Let X be a lass random field at point M_0 with index H and tangent field Z_{M_0}. Assume that X has weakly stationary increments and that for some $u \in \mathcal{B}(0, \delta)\backslash\{0\}$ and some $\gamma > 0$,*

$$\mathbb{P}(Z_{M_0}(u) \neq 0) = 1 \quad \textit{and} \quad \mathbb{E}|Z_{M_0}(u)|^\gamma < +\infty.$$

1. *If $0 < \gamma < 1$, then $0 < H < \frac{1}{\gamma}$.*
2. *If $\gamma \geq 1$, then $0 < H \leq 1$.*

Corollary 4.3. *Assume that X has weakly stationary increments and is lass at point M_0 with index H and tangent field Z_{M_0}.*

1. *If Z_{M_0} is a centered Gaussian random field, then*
   (a) *$H \in (0, 1]$*
   (b) *there exists a constant $C > 0$ such that the covariance function of Z_{M_0} is given by*

$$\forall v, w \in \mathcal{B}(0, \delta),\ \mathbb{E}(Z_{M_0}(v)Z_{M_0}(w)) = \frac{C^2}{2}\Big(\|v\|^{2H} + \|w\|^{2H} - \|v - w\|^{2H}\Big).$$

   Moreover, Z_{M_0} is an Euclidean fractional Brownian motion restricted to $B(0, \delta)$.

2. *If Z_{M_0} is an α-stable random field, then $0 < H \leq \max\left(1, \frac{1}{\alpha}\right)$.*

Proof of Theorem 4.1. Since for $v = 0$, $M_0 + \varepsilon v = M_0$, it is straightforward that $Z_{M_0}(0) = 0$ almost surely. Let $\lambda \in (0, 1]$. Then, by definition

$$\begin{aligned}
(Z_{M_0}(\lambda v))_{v\in B(0,\delta)} &\overset{(d)}{=} \lim_{\varepsilon\to 0_+} \left(\frac{X(M_0 + \varepsilon\lambda v) - X(M_0)}{\varepsilon^H}\right)_{v\in B(0,\delta)} \\
&\overset{(d)}{=} \lambda^H \lim_{\varepsilon\to 0_+} \left(\frac{X(M_0 + \varepsilon\lambda v) - X(M_0)}{(\lambda\varepsilon)^H}\right)_{v\in B(0,\delta)} \\
&\overset{(d)}{=} \lambda^H (Z_{M_0}(\lambda v))_{v\in B(0,\delta)}.
\end{aligned}$$

Let us now assume that X has weakly stationary increments. Let $v, w \in \mathcal{B}(0, \delta)$. Then, by the lass property

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \lim_{\varepsilon\to 0_+} \frac{X(M_0 + \varepsilon v) - X(M_0 + \varepsilon w)}{\varepsilon^H}. \tag{5}$$

Let us fix $u \in \mathcal{B}(0,\delta)$ such that $u \neq 0$. We recall that $M_0 + \varepsilon v = \exp_{M_0}(v)$ and that $M_0 + \varepsilon w = \exp_{M_0}(w)$. Then, by continuity of the distance and of the exponential map $\exp_{M_0}$, there exists $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \leq \varepsilon_0,\ d(M_0 + \varepsilon v, M_0 + \varepsilon w) < \|u\|.$$

Therefore, the point

$$M_0 + \frac{d(M_0 + \varepsilon v, M_0 + \varepsilon w)}{\|u\|} u = \exp_{M_0}\left(\frac{d(M_0 + \varepsilon v, M_0 + \varepsilon w)}{\|u\|} u\right)$$

is well defined. By construction,

$$d\left(M_0, M_0 + \frac{d(M_0 + \varepsilon v, M_0 + \varepsilon w)}{\|u\|} u\right) = d(M_0 + \varepsilon v, M_0 + \varepsilon w).$$

Hence, by (5) and by the weakly stationarity of the increments of X,

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \lim_{\varepsilon \to 0_+} \frac{X\left(M_0 + \frac{d(M_0+\varepsilon v, M_0+\varepsilon v)}{\|u\|} u\right) - X(M_0)}{\varepsilon^H}.$$

Applying the lass property, we then have that

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \|u\|^{-H} Z_{M_0}(u) \lim_{\varepsilon \to 0_+} \frac{d(M_0 + \varepsilon v, M_0 + \varepsilon w)^H}{\varepsilon^H}.$$

The following Lemma, see [23, Chapter 5], establishes the behaviour of $d(M_0 + \varepsilon v, M_0 + \varepsilon w)$ as ε tends to 0.

Lemma 4.4. *For any $v, w \in \mathcal{B}(0,\delta)$,*

$$\lim_{\varepsilon \to 0_+} \frac{d(M_0 + \varepsilon v, M_0 + \varepsilon w)}{\varepsilon} = \|v - w\|.$$

This Lemma leads to

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \|u\|^{-H} \|v - w\|^H Z_{M_0}(u)$$

for any $v, w \in \mathcal{B}(0,\delta)$, which establishes that the increments of Z_{M_0} are weakly stationary. □

Proof of Corollary 4.2. By the proof of Theorem 4.1,

$$\forall v \in \mathcal{B}(0,\delta),\ Z_{M_0}(v) \overset{(d)}{=} \|v\|^H \|u\|^{-H} Z_{M_0}(u). \tag{6}$$

Since $Z_{M_0}(u) \neq 0$ almost surely,

$$\mathbb{E}|Z_{M_0}(u)|^\gamma \neq 0.$$

Moreover, by (6), for every $v \in \mathcal{B}(0,\delta) \backslash \{0\}$

$$Z_{M_0}(v) \neq 0 \text{ almost surely.}$$

Furthermore, let us remark that $u/2 \in \mathcal{B}(0,\delta)$ and then that $Z_{M_0}(u/2)$ is well defined.

1. Assume that $0 < \gamma < 1$. Since $0 < \gamma < 1$,

$$|Z_{M_0}(u)|^\gamma \leq \left|Z_{M_0}\left(\frac{u}{2}\right)\right|^\gamma + \left|Z_{M_0}(u) - Z_{M_0}\left(\frac{u}{2}\right)\right|^\gamma. \tag{7}$$

By Theorem 4.1, Z_{M_0} has weakly stationary increments and $Z_{M_0}(0) = 0$ almost surely. In particular,

$$Z_{M_0}(u) - Z_{M_0}\Big(\frac{u}{2}\Big) \overset{(d)}{=} Z_{M_0}\Big(\frac{u}{2}\Big). \tag{8}$$

Hence, since $\mathbb{P}(Z_{M_0}(u/2) \neq 0) = 1$,

$$\mathbb{P}\Big(Z_{M_0}\Big(\frac{u}{2}\Big) \neq 0,\, Z_{M_0}(u) \neq Z_{M_0}\Big(\frac{u}{2}\Big)\Big) = \mathbb{P}\Big(Z_{M_0}(u) \neq Z_{M_0}\Big(\frac{u}{2}\Big)\Big) = 1 > 0.$$

Therefore,

$$\mathbb{E}|Z_{M_0}(u)|^\gamma < \mathbb{E}\Big|Z_{M_0}\Big(\frac{u}{2}\Big)\Big|^\gamma + \mathbb{E}\Big|Z_{M_0}(u) - Z_{M_0}\Big(\frac{u}{2}\Big)\Big|^\gamma$$

since the inequality (7) is strict on $\{Z_{M_0}(u) \neq 0,\, Z_{M_0}(u) \neq Z_{M_0}(u/2)\}$. Furthermore, by (8),

$$\mathbb{E}\Big|Z_{M_0}(u) - Z_{M_0}\Big(\frac{u}{2}\Big)\Big|^\gamma = \mathbb{E}\Big|Z_{M_0}\Big(\frac{u}{2}\Big)\Big|^\gamma.$$

Therefore,

$$\mathbb{E}|Z_{M_0}(u)|^\gamma < 2\mathbb{E}\Big|Z_{M_0}\Big(\frac{u}{2}\Big)\Big|^\gamma.$$

Moreover, by Theorem 4.1, Z_{M_0} satisfies the self-similarity property (4). This property applied with $\lambda = 1/2$ leads to

$$\mathbb{E}|Z_{M_0}(u)|^\gamma < 2^{1-\gamma H}\mathbb{E}|Z_{M_0}(u)|^\gamma.$$

Then, since $\mathbb{E}|Z_{M_0}(u)|^\gamma \neq 0$, $1 < 2^{1-\gamma H}$ which means that $H < 1/\gamma$.

2. Assume now that $\gamma \geq 1$. Furthermore, for any $0 < \eta < 1$, $\mathbb{E}|Z_{M_0}(u)|^\eta < +\infty$. The first part of this proof implies that $H < 1/\eta$ for every $\eta \in (0,1)$. Also, taking the asymptotics as $\eta \to 1$, $H \leq 1$.

□

Proof of Corollary 4.3.

1. Assume that Z_{M_0} is a centered Gaussian random field. By (4), since the Gaussian field Z_{M_0} is non degenerate, for every $u \in \mathcal{B}(0,\delta)$,

$$\mathbb{P}(Z_{M_0}(u) \neq 0) = 1.$$

Moreover, for any $u \in \mathcal{B}(0,\delta)$,

$$\mathbb{E}Z_{M_0}(u)^2 < +\infty.$$

Then by Corollary 4.2, $H \leq 1$.

Let us consider $u \in \mathcal{B}(0,\delta)\backslash\{0\}$ and recall (see proof of Theorem 4.1) that

$$Z_{M_0}(v_1) - Z_{M_0}(v_2) \overset{(d)}{=} \|u\|^{-H}\|v_1 - v_2\|^H Z_{M_0}(u)$$

for every $v_1, v_2 \in \mathcal{B}(0,\delta)$. As a consequence, for every $v_1, v_2 \in \mathcal{B}(0,\delta)$,

$$\mathrm{Var}(Z_{M_0}(v_1) - Z_{M_0}(v_2)) = C^2\|v_1 - v_2\|^{2H}$$

with $C = \|u\|^{-H}\sqrt{\mathrm{Var(Z_{M_0}(u))}}$. Since $Z_{M_0}(0) = 0$ almost surely, we then have that

$$\begin{aligned}\mathrm{Cov}(Z_{M_0}(v), Z_{M_0}(w)) &= \frac{1}{2}[\mathrm{Var}\, Z_{M_0}(v) + \mathrm{Var}\, Z_{M_0}(w) - \mathrm{Var}(Z_{M_0}(v) - Z_{M_0}(w))] \\ &= \frac{C^2}{2}\Big(\|v\|^{2H} + \|w\|^{2H} - \|v-w\|^{2H}\Big)\end{aligned}$$

for every $v, w \in \mathcal{B}(0,\delta)$.

This covariance is the covariance of an Euclidean fractional Brownian motion restricted to $\mathcal{B}(0,\delta)$. So that, in this case, if Z_{M_0} is a Gaussian random field, it is an Euclidean fractional Brownian motion restricted to $\mathcal{B}(0,\delta)$.

2. Assume that Z_{M_0} is an α-stable random field. Hence, by (4) and since the α-stable random field Z_{M_0} is non degenerate, for every $u \in \mathcal{B}(0,\delta)$,

$$\mathbb{P}(Z_{M_0}(u) \neq 0) = 1.$$

Hence, for any $\gamma < \alpha$ and any $u \in \mathcal{B}(0,\delta)$,

$$\mathbb{E}|Z_{M_0}(u)|^{\gamma} < +\infty.$$

Assertion 2 is then a consequence of Corollary 4.2.

□

5 Examples

5.1 Fractional Brownian motion indexed by a manifold

The Euclidean fractional Brownian motions are Gaussian centered random fields. Then they can be characterized by their covariance function. Replacing in this covariance function the Euclidean distance by the distance d of the manifold $\mathcal{M}$, [14] extends the fractional Brownian motion in the indexed manifold fields realm. Indeed, a random field $X_H = (X_H(M))_{M\in\mathcal{M}}$ is a fractional Brownian motion with index H iff X is a Gaussian centered random field such that

$$\exists O \in \mathcal{M},\ \forall (M,N) \in \mathcal{M}^2,\ \mathbb{E}(X_H(M)X_H(N)) = \frac{1}{2}\big(d^{2H}(O,M) + d^{2H}(O,N) - d^{2H}(M,N)\big). \tag{9}$$

Note that (9) is equivalent to

$$\left\{\begin{array}{l} \exists O \in \mathcal{M},\ X_H(O) = 0 \text{ almost surely}, \\ \forall (M,N) \in \mathcal{M}^2,\ \mathbb{E}(X_H(M) - X_H(N))^2 = d^{2H}(M,N). \end{array}\right. \tag{10}$$

[14, 15] prove that the fractional Brownian motion exists for $H \in (0, \beta_{\mathcal{M}}]$, where $\beta_{\mathcal{M}}$ is a constant depending on the manifold. For instance, $\beta_{\mathcal{M}}$ is equal to 1 for Euclidean space, and is equal to 1/2 for spheres and hyperbolic spaces.

Since a fractional Brownian motion X_H is a Gaussian centered random field, by (10), X_H has weakly stationary increments. Let us now fix $M_0 \in \mathcal{M}$ and $u, v \in \mathcal{B}(0,\delta) \subset T_{M_0}\mathcal{M}$. Then, for every $\varepsilon \in (0,1]$, let

$$R(M_0+\varepsilon v, M_0+\varepsilon w) = \mathrm{Cov}\bigg(\frac{X_H(M_0+\varepsilon v) - X_H(M_0)}{\varepsilon^H}, \frac{X_H(M_0+\varepsilon w) - X_H(M_0)}{\varepsilon^H}\bigg).$$

By definition of X_H,

$$\begin{aligned} R(M_0+\varepsilon v, M_0+\varepsilon w) &= \frac{1}{2\varepsilon^{2H}}\big(d^{2H}(M_0+\varepsilon v, M_0) + d^{2H}(M_0+\varepsilon w, M_0) - d^{2H}(M_0+\varepsilon v, M_0+\varepsilon w)\big) \\ &= \frac{1}{2}\bigg(\|v\|^{2H} + \|w\|^{2H} - \frac{d^{2H}(M_0+\varepsilon v, M_0+\varepsilon w)}{\varepsilon^{2H}}\bigg) \end{aligned}$$

Hence, by Lemma 4.4,

$$\lim_{\varepsilon\to 0_+} R(M_0+\varepsilon v, M_0+\varepsilon w) = \frac{1}{2}\Big(\|v\|^{2H} + \|w\|^{2H} - \|v-w\|^{2H}\Big) \tag{11}$$

for every $v, w \in \mathcal{B}(0, \delta)$. Since X_H is a Gaussian centered random field, (11) implies that X_H is lass at point M_0 with index H. Note that (11) also characterizes the tangent field at X_H at point M_0. As expected, it is an Euclidean fractional Brownian motion B_H with index H restricted to $\mathcal{B}(0, \delta)$. Since the Euclidean fractional Brownian motion B_H exists iff $H \in (0, 1]$, we get a bound for $\beta_{\mathcal{M}}$.

Corollary 5.1. *There is no fractional Brownian motion indexed by a manifold $\mathcal{M}$ with index $H > 1$: $\beta_{\mathcal{M}} \leq 1$.*

Especially, fractional Brownian motions indexed by a sphere or an hyperbolic space are examples of Gaussian lass fields with index $H \in (0, 1/2]$. The following section gives examples of lass Gaussian or stable fields with index $H \in (0, 1]$.

5.2 Spherical moving average fractional fields

Let $n \in \mathbb{N} \backslash \{0\}$ and $\mathbb{S}_n = \{x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}, \sum_{i=1}^{n+1} x_i^2 = 1\}$ is the n-dimensional unit sphere. The distance d on $\mathbb{S}_n$ is its geodesic distance. By convention, $\mathbb{S}_0 = \{-1, 1\}$.

In this part, we introduce some fields indexed by $\mathbb{S}_n$ owing a moving average representation. Let us recall this representation for the Euclidean fractional α-stable motion (see [20, 25]) with index $H \in (0, 1)$ and $\alpha \in (0, 2]$:

$$B_{H,\alpha}(M) \quad = \quad \int_{\mathbb{R}^n} \left(||MM'||^{H-n/\alpha} - ||\widetilde{O}M'||^{H-n/\alpha} \right) dW_\alpha(M') \tag{12}$$

where $\widetilde{O}$ is the origin of $\mathbb{R}^n$ and where W_α is a symmetric α-stable random measure on $\mathbb{R}^n$ with Lebesgue measure as control measure (see [25, ch.3] for general results on random measures). In the case $\alpha = 2$, W_2 is a Brownian measure and up to a normalizing constant, $B_{H,2}$ is the standard Euclidean fractional Brownian motion.

In order to define spherical moving average α-stable fields, we will replace W_α by a random measure on $\mathbb{S}_n$ and the Euclidean norm by the distance d on $\mathbb{S}_n$. Let dx be the Lebesgue measure on $\mathbb{R}^n$. Then, in (12), the term $||\widetilde{O}M'||^{H-n/\alpha}$ implies that the kernel $M' \mapsto ||MM'||^{H-n/\alpha} - ||\widetilde{O}M'||^{H-n/\alpha}$ is in $L^\alpha(\mathbb{R}^n, dx)$ for any $H \in (0, 1)$ and then that $B_{H,\alpha}$ is well-defined. Without this compensative term, the kernel will not be in $L^\alpha(\mathbb{R}^n, dx)$ in view of its behaviour as $||MM'|| \to +\infty$. Since the sphere is compact, we don't need to reproduce this term in our framework.

Let σ_n be the uniform measure on $\mathbb{S}_n$, $\alpha \in (0, 2]$. Let W_α be a symmetric α-stable random measure on $\mathbb{S}_n$ with σ_n as control measure when $0 < \alpha < 2$ and let W_2 be the Brownian random measure on $\mathbb{S}_n$ with σ_n as control measure. Let us precise that $\int_{\mathbb{S}_n} f(M) dW_\alpha(M)$ exists iff $f \in L^\alpha(\mathbb{S}_n, \sigma_n)$. Furthermore, if $f \in L^\alpha(\mathbb{S}_n, \sigma_n)$, then $\int_{\mathbb{S}_n} f(M) dW_\alpha(M)$ is an α-stable symmetric random variable and

$$\forall u \in \mathbb{R}, \, \mathbb{E}\left[\exp\left(iu \int_{\mathbb{S}_n} f(M) dW_\alpha(M) \right) \right] = \exp\left[-|u|^\alpha \int_{\mathbb{S}_n} |f(M)|^\alpha d\sigma_n(M) \right].$$

Let $H \in \mathbb{R}$ such that $H \neq n/\alpha$. As soon as

$$X_{H,\alpha}(M) \quad = \quad \int_{\mathbb{S}_n} d(M, M')^{H-n/\alpha} dW_\alpha(M'), \, M \in \mathbb{S}_n \tag{13}$$

is well-defined, with convention $0^\beta = +\infty$ for $\beta < 0$, $X_{H,\alpha}$ is called spherical moving average fractional α-stable random field. Note that $X_{H,2}$ is a Gaussian field.

Proposition 5.2. *Let $H \in \mathbb{R}$ such that $H \neq n/\alpha$. Then, the spherical moving average fractional α-stable random field $X_{H,\alpha}$ is well-defined if and only if $H > 0$.*

Proof of Proposition 5.2. Let us recall that $X_{H,\alpha}(M)$ exits iff the function $M' \mapsto d(M,M')^{H-n/\alpha}$ belongs to $L^\alpha(\mathbb{S}_n, \sigma_n)$. Let

$$I(M) = \int_{\mathbb{S}_n} d(M,M')^{\alpha H - n} d\sigma_n(M').$$

Using the exponential map $\exp_M$ at point M, $I(M)$ can be rewritten as follows

$$I(M) = \iint_{[0,\pi]\times \mathbb{S}_{n-1}} d(M, \exp_M(ru))^{\alpha H - n} \sin^{n-1}(r) dr d\sigma_{n-1}(u),$$

where by convention if $n = 1$, $\sigma_0 = \delta_{-1} + \delta_1$. Hence,

$$I(M) = \sigma_{n-1}(\mathbb{S}_{n-1}) \int_{[0,\pi]} r^{\alpha H - n} \sin^{n-1}(r) dr.$$

Then,

$$I(M) < +\infty \iff \alpha H - 1 > -1 \iff H > 0$$

since $\alpha > 0$. □

Proposition 5.3. *Let $H > 0$ such that $H \neq n/\alpha$.*

1. *Then $X_{H,\alpha}$ has weakly stationary increments.*
2. (a) *Assume $H \in (0,1)$. Then $X_{H,\alpha}$ is lass at each point with index H. Furthermore, the tangent field at point M_0 is an Euclidean moving average α-stable random field with index H. More precisely, for every $M_0 \in \mathbb{S}_n$,*

   $$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\alpha}(M_0 + \varepsilon v) - X(M_0)}{\varepsilon^H} \right)_{v \in B(0,\pi)} \stackrel{(d)}{=} (B_{H,\alpha}(v))_{v\in B(0,\pi)},$$

   where $B_{H,\alpha}$ is defined by (12).

   (b) *Assume $H > 1$. Then $X_{H,\alpha}$ is lass at each point with index 1. More precisely, for every $M_0 \in \mathbb{S}_n$,*

   $$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\alpha}(M_0 + \varepsilon v) - X(M_0)}{\varepsilon} \right)_{v \in B(0,\pi)} \stackrel{(d)}{=} (Z_{M_0,\alpha}(v))_{v\in B(0,\pi)},$$

   where for every $v \in \mathcal{B}(0,\pi)$,

   $$Z_{M_0,\alpha}(v) = \left(\frac{n}{\alpha} - H\right) \int_{\mathbb{S}_n} \frac{< v, \Pi_{M_0}(M') >}{\|\Pi_{M_0}(M')\|} d\big(M_0, M'\big)^{H-1-n/\alpha} dW_\alpha\big(M'\big),$$

   with Π_{M_0} the inverse of the exponential map $\exp_{M_0}$ at point M_0.

Proof of Proposition 5.3.

1. Let $H > 0$ and $(M_1, M_2) \in \mathbb{S}_n^2$. Let us first notice that

$$\mathbb{E}[\exp(iu(X_{H,\alpha}(M_1) - X_{H,\alpha}(M_2)))] = \exp\left[-|u|^\alpha \int_{\mathbb{S}_n} \left| d\big(M_1, M'\big)^{H-n/\alpha} - d\big(M_2, M'\big)^{H-n/\alpha} \right|^\alpha d\sigma_n\big(M'\big) \right]$$

for every $u \in \mathbb{R}$. Then, we have to prove that

$$I(M_1, M_2) = \int_{\mathbb{S}_n} \left| d(M_1, M')^{H-n/\alpha} - d(M_2, M')^{H-n/\alpha} \right|^\alpha d\sigma_n(M')$$

only depends on $d(M_1, M_2)$. Let $(N_1, N_2) \in \mathbb{S}_n^2$ such that

$$d(M_1, M_2) = d(N_1, N_2).$$

Then, there exists a rotation r such that $r(N_1) = M_1$ and $r(N_2) = M_2$. By invariance by rotation of σ_n, the change of variable $N' = r^{-1}(M')$ leads to

$$I(M_1, M_2) = \int_{\mathbb{S}_n} \left| d(M_1, r(N'))^{H-n/\alpha} - d(M_2, r(N'))^{H-n/\alpha} \right|^\alpha d\sigma_n(N').$$

Since $d(M_i, r(N')) = d(r(N_i), r(N')) = d(N_i, N')$, $i = 1, 2$,

$$\begin{aligned} I(M_1, M_2) &= \int_{\mathbb{S}_n} \left| d(N_1, M')^{H-n/\alpha} - d(N_2, M')^{H-n/\alpha} \right|^\alpha d\sigma_n(M') \\ &= I(N_1, N_2), \end{aligned}$$

which establishes the weakly stationarity of the increments of $X_{H,\alpha}$.

2. Let $M_0 \in \mathcal{M}$ and consider the open ball $\mathcal{B}(0, \pi) \subset T_{M_0}\mathbb{S}_n$. Then, let $\varepsilon \in (0, 1]$, $k \in \mathbb{N}\backslash\{0\}$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ and $v = (v_1, \ldots, v_k) \in \mathcal{B}(0, \pi)^k$. Let us denote

$$I(\lambda, v, \varepsilon) = \int_{\mathbb{S}_n} f_{\varepsilon,\lambda,v}(M') d\sigma_n(M')$$

where

$$f_{\varepsilon,\lambda,v}(M') = \left| \sum_{j=1}^k \lambda_j \left(d(M_0 + \varepsilon v_j, M')^{H-n/\alpha} - d(M_0, M')^{H-n/\alpha} \right) \right|^\alpha$$

(a) Assume now that $H \in (0, 1)$. By definition of $X_{H,\alpha}$,

$$\mathbb{E}\left(\exp\left(i \sum_{j=1}^k \frac{\lambda_j (X_{H,\alpha}(M_0 + \varepsilon v_j) - X_{H,\alpha}(M_0))}{\varepsilon^H} \right) \right) = \exp\left(-\varepsilon^{-\alpha H} I(\lambda, v, \varepsilon)\right).$$

Using the exponential map $\exp_{M_0}$ at point M_0, $I(\lambda, v, \varepsilon)$ can be rewritten as follows

$$I(\lambda, v, \varepsilon) = \int_{(0,\pi]\times\mathbb{S}_{n-1}} f_{\varepsilon,\lambda,v}(\exp_{M_0}(ru)) \sin^{n-1}(r) dr d\sigma_{n-1}(u)$$

i.e.

$$I(\lambda, v, \varepsilon) = \int_{(0,\pi]\times\mathbb{S}_{n-1}} \left| \sum_{j=1}^k \lambda_j \left(d(M_0 + \varepsilon v_j, M_0 + ru)^{H-n/\alpha} - r^{H-n/\alpha} \right) \right|^\alpha \sin^{n-1}(r) dr d\sigma_{n-1}(u).$$

Then the change of variable $\rho = r/\varepsilon$ leads to

$$I(\lambda, v, \varepsilon) = \int_{(0,\pi/\varepsilon]\times\mathbb{S}_{n-1}} \widetilde{f}_{\varepsilon,\lambda,v}(\rho, u) d\rho d\sigma_{n-1}(u)$$

with

$$\widetilde{f}_{\varepsilon,\lambda,v}(\rho,u)=\varepsilon\left|\sum_{j=1}^{k}\lambda_j\Big(d(M_0+\varepsilon v_j,M_0+\varepsilon\rho u)^{H-n/\alpha}-(\varepsilon\rho)^{H-n/\alpha}\Big)\right|^{\alpha}\sin^{n-1}(\varepsilon\rho). \quad (14)$$

By Lemma 4.4,

$$\widetilde{f}_{\lambda,v}(\rho,u)=\lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha H}\widetilde{f}_{\varepsilon,\lambda,v}(\rho,u)=\left|\sum_{j=1}^{k}\lambda_j\Big(\|v_j-\rho u\|^{H-n/\alpha}-\rho^{H-n/\alpha}\Big)\right|^{\alpha}\rho^{n-1}.$$

Note that $\widetilde{f}_{\lambda,v}$ is integrable iff $H\in(0,1)$. In order to conclude, we will use the dominated convergence theorem. Let us remark that for $\rho\in(0,\pi/\varepsilon)$ and $u\in\mathbb{S}_{n-1}$,

$$|d(M_0+\varepsilon v_j,M_0)-d(M_0,M_0+\varepsilon\rho u)|\le d(M_0+\varepsilon v_j,M_0+\varepsilon\rho u)$$

and that

$$d(M_0+\varepsilon v_j,M_0+\varepsilon\rho u)\le d(M_0+\varepsilon v_j,M_0)+d(M_0,M_0+\varepsilon\rho u)$$

i.e.

$$\varepsilon\big|\|v_j\|-\rho\big|\le d(M_0+\varepsilon v_j,M_0+\varepsilon\rho u)\le\varepsilon(\|v_j\|+\rho)$$

since $d(M_0,M_0+\varepsilon w)=d\big(M_0,\exp_{M_0}(\varepsilon w)\big)=\varepsilon\|w\|$ as soon as $w\in\mathcal{B}(0,\pi)$. Then, let

$$\begin{aligned}\widetilde{g}_{\lambda,v}(\rho,u) &= \rho^{n-1}\left[\sum_{j=1}^{k}|\lambda_j|\Big|\big|\|v_j\|-\rho\big|^{H-n/\alpha}-\rho^{H-n/\alpha}\Big|\right]^{\alpha}\\ &\quad+\rho^{n-1}\left[\sum_{j=1}^{k}|\lambda_j|\Big|\big|\|v_j\|+\rho\big|^{H-n/\alpha}-\rho^{H-n/\alpha}\Big|\right]^{\alpha} \quad (15)\end{aligned}$$

One easily proves that $\widetilde{g}_{\lambda,v}\in L^1((0,+\infty[\times\mathbb{S}_{n-1},d\rho d\sigma_{n-1}(u))$ and that

$$\sup_{\varepsilon\in(0,1]}\left|\varepsilon^{-\alpha H}\widetilde{f}_{\varepsilon,\lambda,v}\right|\le\widetilde{g}_{\lambda,v}\text{ almost surely.}$$

Furthermore, using polar coordinates, one easily sees that

$$\int_{(0,+\infty[\times\mathbb{S}_{n-1}}\widetilde{f}_{\lambda,v}(\rho,v)d\rho d\sigma_{n-1}(u)=\int_{\mathbb{R}^n}\left|\sum_{j=1}^{k}\lambda_j\Big(\|v_j-\xi\|^{H-n/\alpha}-\|\xi\|^{H-n/\alpha}\Big)\right|^{\alpha}d\xi.$$

Hence,

$$\lim_{\varepsilon\to 0_+}\mathbb{E}\left(\exp\left(i\sum_{j=1}^{k}\frac{\lambda_j(X_{H,\alpha}(M_0+\varepsilon v_j)-X_{H,\alpha}(M_0))}{\varepsilon^H}\right)\right)=\mathbb{E}\left(\exp\left(i\sum_{j=1}^{k}\lambda_j B_{H,\alpha}(v_j)\right)\right),$$

which concludes the proof of the lass property when $H\in(0,1)$.

(b) Assume that $H>1$. Then, by definition of $X_{H,\alpha}$,

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^{k}\frac{\lambda_j(X_{H,\alpha}(M_0+\varepsilon v_j)-X_{H,\alpha}(M_0))}{\varepsilon^H}\right)\right)=\exp\big(-\varepsilon^{-\alpha}I(\lambda,v,\varepsilon)\big).$$

Let us split $I(\lambda, v, \varepsilon)$ into $I(\lambda, v, \varepsilon) = I_1(\lambda, v, \varepsilon) + I_2(\lambda, v, \varepsilon)$ with

$$I_1(\lambda, v, \varepsilon) = \int_{\mathbb{S}_n} f_{\varepsilon,\lambda,v}\big(M'\big)\mathbf{1}_{d(M_0,M')\leq 2\pi\varepsilon}\, d\sigma_n\big(M'\big)$$

and

$$I_2(\lambda, v, \varepsilon) = \int_{\mathbb{S}_n} f_{\varepsilon,\lambda,v}\big(M'\big)\mathbf{1}_{d(M_0,M')>2\pi\varepsilon}\, d\sigma_n\big(M'\big).$$

Then, proceeding as in the proof of the lass property when $H < 1$, one establishes that for every $\varepsilon \in (0, 1/2]$,

$$I_1(\lambda, v, \varepsilon) = \int_{[0,2\pi]\times\mathbb{S}_{n-1}} \widetilde{f}_{\varepsilon,\lambda,v}(\rho, u) d\rho d\sigma_{n-1}(u)$$

where $\widetilde{f}_{\varepsilon,\lambda,v}$ is defined by (14). Hence, for every $\varepsilon \in (0, 1/2]$,

$$I_1(\lambda, v, \varepsilon) \leq \varepsilon^{\alpha H} \int_{(0,2\pi]\times\mathbb{S}_{n-1}} \widetilde{g}_{\lambda,v}(\rho, u) d\rho d\sigma_{n-1}(u)$$

where $g_{\lambda,v}$ is defined by (15). Remark that $\widetilde{g}_{\lambda,v} \in L^1((0, 2\pi] \times \mathbb{S}_{n-1}, d\rho d\sigma_{n-1}(u))$. Hence, since $H > 1$,

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-\alpha} I_1(\lambda, v, \varepsilon) = 0. \tag{16}$$

Let us now study the asymptotics of $I_2(\lambda, v, \varepsilon)$ as ε tends to 0_+. Let $SP(M_0) \in \mathbb{S}_n$ be the antipodal point of M_0. If $M' \notin \{M_0, SP(M_0)\}$,

$$\lim_{\varepsilon\to 0_+} \frac{d(M_0 + \varepsilon v_j, M') - d(M_0, M')}{\varepsilon} = -\frac{< \Pi_{M_0}(M'), v_j >}{\|\Pi_{M_0}(M')\|}$$

where Π_{M_0} is the inverse of the exponential map $\exp_{M_0}$ at point M_0.
Then, a Taylor expansion leads to

$$f_{\lambda,v}(M) = \lim_{\varepsilon\to 0_+} \varepsilon^{-\alpha} f_{\varepsilon,\lambda,v}\big(M'\big) = \frac{|n - H\alpha|^\alpha d(M_0, M')^{H\alpha-\alpha-n} \Big|< \Pi_{M_0}(M'), \sum_{j=1}^k \lambda_j v_j >\Big|^\alpha}{\alpha^\alpha \|\Pi_{M_0}(M')\|^\alpha}.$$

Since for every $x \geq 0$ and $y \geq 0$,

$$\Big|x^{H-n/\alpha} - y^{H-n/\alpha}\Big| \leq \Big|H - \frac{n}{\alpha}\Big| \Big(x^{H-1-n/\alpha} + y^{H-1-n/\alpha}\Big)|x - y|,$$

and since

$$\big|d\big(M_0 + \varepsilon v_j, M'\big) - d\big(M_0, M'\big)\big| \leq d(M_0 + \varepsilon v_j, M_0) = \varepsilon\|v_j\|,$$

we have that

$$\big|f_{\varepsilon,\alpha,v}\big(M'\big)\big| \leq \Big|H - \frac{n}{\alpha}\Big|^\alpha \varepsilon^\alpha \left|\sum_{j=1}^k |\lambda_j|\|v_j\| \Big(d\big(M_0 + \varepsilon v_j, M'\big)^{H-1-n/\alpha} + d\big(M_0, M'\big)^{H-1-n/\alpha}\Big)\right|^\alpha$$

If $d(M_0, M') \geq 2\pi\varepsilon \geq 2\varepsilon\|v_j\|$,

$$d\big(M_0 + \varepsilon v_j, M'\big) \geq d\big(M_0, M'\big) - \varepsilon\|v_j\| \geq \frac{d(M_0, M')}{2}$$

since $\varepsilon\|v_j\| = d(M_0 + \varepsilon v_j, M_0)$. Then, if $d(M_0, M') \geq 2\pi\varepsilon$,

$$\big|\varepsilon^{-\alpha} f_{\varepsilon,\alpha,v}\big(M'\big)\big| \leq g_{\lambda,v}\big(M'\big)$$

with

$$g_{\lambda,v}\big(M'\big) = \begin{cases} 2^{\alpha}\pi^{H\alpha-\alpha-n}\Big|H-\dfrac{n}{\alpha}\Big|^{\alpha}\Bigg|\displaystyle\sum_{j=1}^{k}|\lambda_j|\|v_j\|\Bigg|^{\alpha} & \text{if } H\geq 1+n/\alpha,\\ 2^{n+2\alpha-H\alpha}\Big|H-\dfrac{n}{\alpha}\Big|^{\alpha}\Bigg|\displaystyle\sum_{j=1}^{k}|\lambda_j|\|v_j\|\Bigg|^{\alpha} d\big(M_0,M'\big)^{\alpha H-\alpha-n} & \text{if } H<1+n/\alpha. \end{cases}$$

Note that $g_{\lambda,v}\in L^1(\mathbb{S}_n,d\sigma_n(u))$ since $H>1$. Therefore, the dominated convergence theorem and (16) yield that

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha}I(\varepsilon,\lambda,v) &= \lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha}I(\varepsilon,\lambda,v)\\ &= \Big|\frac{n}{\alpha}-H\Big|^{\alpha}\int_{\mathbb{S}_n} d\big(M_0,M'\big)^{H\alpha-\alpha-n}\Bigg|<\frac{\Pi_{M_0}(M')}{\|\Pi_{M_0}(M')\|},\sum_{j=1}^{k}\lambda_j v_j>\Bigg|^{\alpha} d\sigma_n\big(M'\big),\end{aligned}$$

which establishes the lass property when $H>1$ and concludes the proof.

□

5.3 Spherical moving average multifractional stable fields

All the previous examples, the index of the lass property at point M_0 does not depend on M_0. However, for modelization purpose, it is sometime a constraining condition. We therefore introduce some multifractional lass random fields: the index of the lass property will then vary. In the case of Euclidean random fields, multifractional random fields have been defined by replacing the index H by a function $h(\cdot)$ in some integral representations of fractional Euclidean fields, e.g. [2, 3, 4, 8, 9, 17, 22, 26]. The most famous examples are multifractional Brownian fields, introduced either by replacing the Hurst index H by a function in the moving average representation of a fractional Brownian motion [22] and in its harmonizable representation [9]. Following this approach, we define spherical moving average multifractional α-stable and Gaussian fields.

Let us recall that $\alpha\in(0,2]$ and consider $h:\mathbb{S}_n\longrightarrow(0,+\infty)$ such that $h(M)\neq n/\alpha$ for every $M\in\mathbb{S}_n$. Then,

$$X_{h,\alpha}(M) = \int_{\mathbb{S}_n} d(M,M')^{h(M)-n/\alpha}dW_\alpha(M'),\ M\in\mathbb{S}_n \tag{17}$$

is well-defined and $X_{h,\alpha}$ is called spherical moving average multifractional α-stable random field with multifractional function h. If $\alpha=2$, $X_{h,2}$ is a centered Gaussian random field.

Before we study the lass property for $X_{h,\alpha}$, let us introduce

$$Y_\alpha(M,H)=\int_{\mathbb{S}_n} d(M,M')^{H-n/\alpha}dW_\alpha(M'),\ M\in\mathbb{S}_n,\ H>0$$

and notice that

$$X_{h,\alpha}(M)=Y_\alpha(M,h(M)).$$

Then, let $M_0\in\mathbb{S}_n$. The random field $X_{h,\alpha}$ will be split into $X_{h,\alpha}=X_{h(M_0),\alpha}+R_{M_0}$ with

$$X_{h(M_0),\alpha}(M)=Y_\alpha(M,h(M_0))$$

and

$$R_{M_0}(M)=Y_\alpha(M,h(M))-Y_\alpha(M,h(M_0)). \tag{18}$$

Remark that $X_{h(M_0),\alpha}$ is a spherical moving average fractional α-stable random field. Then, Section 5.2 gives the behaviour of its increments around M_0.

Lemma 5.4. *Assume that the function h is $\mathcal{C}^1$. Then, for every $M_0 \in \mathbb{S}_n$ and every $\gamma \in (0,1)$,*

$$\lim_{M\to M_0} \frac{R_{M_0}(M)}{d(M,M_0)^\gamma} \overset{(d)}{=} 0,$$

where R_{M_0} is defined by (18).

Proposition 5.5. *Let $M_0 \in \mathbb{S}_n$. Assume that the function h is $\mathcal{C}^1$ and that $h(M_0) < 1$. Then, $X_{h,\alpha}$ is lass at point M_0 with index $h(M_0)$ and its tangent field at point M_0 is an Euclidean moving average α-stable random field with index $h(M_0)$. More precisely,*

$$\lim_{\varepsilon\to 0_+} \left(\frac{X_{h,\alpha}(M_0+\varepsilon v) - X(M_0)}{\varepsilon^{h(M_0)}}\right)_{v\in B(0,\pi)} \overset{(d)}{=} \left(B_{h(M_0),\alpha}(v)\right)_{v\in B(0,\pi)},$$

where $B_{h(M_0),\alpha}$ is defined by (12).

Proof of Lemma 5.4.
Let $M_0 \in \mathbb{S}_n$ and $\gamma \in (0,1)$. For every $u \in \mathbb{R}$ and $M \in \mathbb{S}_n$,

$$\mathbb{E}(\exp(iuR_{M_0}(M))) = \exp(-|u|^\alpha I_{M_0}(M))$$

with

$$I_{M_0}(M) = \int_{\mathbb{S}_n} \left| d^{h(M)-n/\alpha}(M,M') - d^{h(M_0)-n/\alpha}(M,M')\right|^\alpha d\sigma_n(M') \geq 0.$$

Using the exponential map $\exp_M$ at point M, one easily sees that

$$I_{M_0}(M) = \sigma_{n-1}(S_{n-1}) \int_0^\pi \left| r^{h(M)-n/\alpha} - r^{h(M_0)-n/\alpha}\right|^\alpha dr.$$

Let $\delta > 0$. Then, by continuity of h there exists $a,b \in (0,+\infty)$ such that for $d(M,M_0) \leq \delta$, $a \leq h(M) \leq b$. Hence, applying the Taylor-Lagrange inequality, one proves that

$$I_{M_0}(M) \leq |h(M)-h(M_0)|^\alpha \sigma_{n-1}(S_{n-1}) \int_0^\pi |\ln r| \left(r^{a-n/\alpha} + r^{b-n/\alpha}\right)^\alpha dr$$

as soon as $d(M,M_0) \leq \delta$. Since h is a $\mathcal{C}^1$ function and $\gamma \in (0,1)$,

$$\lim_{M\to M_0} \frac{I_{M_0}(M)}{d(M,M_0)^{\alpha\gamma}} = 0,$$

which implies

$$\lim_{M\to M_0} \mathbb{E}\left(\exp\left(\frac{iuR_{M_0}(M)}{d(M,M_0)^\gamma}\right)\right) = 0$$

for every $u \in \mathbb{R}$. The proof of Lemma 5.4 is then complete. □

Proof of Proposition 5.5. For every $v \in \mathcal{B}(0,\pi)$ and every $\varepsilon \in (0,1)$

$$\frac{X_{h,\alpha}(M_0+\varepsilon v) - X_{h,\alpha}(M_0)}{\varepsilon^{h(M_0)}} = \frac{X_{h(M_0),\alpha}(M_0+\varepsilon v) - X_{h(M_0),\alpha}(M_0)}{\varepsilon^{h(M_0)}} + \frac{R_{M_0}(M_0+\varepsilon v)}{\varepsilon^{h(M_0)}}.$$

Since $h(M_0) < 1$, by Lemma 5.4,

$$\lim_{\varepsilon\to 0_+} \left(\frac{R_{M_0}(M_0+\varepsilon v)}{\varepsilon^{h(M_0)}}\right)_{v\in\mathcal{B}(0,\pi)} \overset{(d)}{=} 0.$$

Let us recall that $X_{h(M_0),\alpha}$ is a spherical moving average α-stable random fields with index $h(M_0) < 1$. Hence, Proposition 5.3 leads to the conclusion. □

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