Space-Time Regularity of the Solution to Maxwell's Equations in Non-Convex Domains

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Abstract. We present various space-time regularity results for Maxwell's equations. The general results, valid for all Lipschitz domains, are optimal in the absence of singularities. For singular domains that are invariant by translation or rotation, we prove precise results by extending Grisvard's singularity theory to Maxwell's equations.

1 'Basic' Results

Let Ω be a three-dimensional domain with a Lipschitz boundary Γ, $\boldsymbol{n}$ the unit outgoing normal to Γ, $\mathbf{X}$ and $\mathbf{Y}$ the 'natural' spaces of electromagnetic fields

$$\mathbf{X} = \mathbf{H}_0(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega), \quad \mathbf{Y} = \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}_0(\mathrm{div}\,;\Omega).$$

1.1 The First-Order Formulation

Let c and ε_0 be the speed of light and the dielectric permittivity, ϱ and $\boldsymbol{J}$ the charge and current densities. The Maxwell system consists of *evolution* equations

$$\partial_t \boldsymbol{E} - c^2\,\mathbf{curl}\,\boldsymbol{B} = -(1/\varepsilon_0)\,\boldsymbol{J}, \qquad \partial_t \boldsymbol{B} + \mathbf{curl}\,\boldsymbol{E} = 0 \quad \text{in } \Omega \times \,]0,T[\,, \tag{1}$$

and *constraint* equations (divergence and boundary conditions)

$$\mathrm{div}\,\boldsymbol{E} = \varrho/\varepsilon_0,\ \mathrm{div}\,\boldsymbol{B} = 0 \text{ in } \Omega\times]0,T[\,; \quad \boldsymbol{E}\times\boldsymbol{n} = 0,\ \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\times]0,T[\,. \tag{2}$$

They are supplemented with *initial conditions*

$$\boldsymbol{E}(0) = \boldsymbol{E}_0,\ \boldsymbol{B}(0) = \boldsymbol{B}_0 \text{ in } \Omega, \tag{3}$$

which, to be consistent, should satisfy the constraints, i.e.

$$\mathrm{div}\,\boldsymbol{E}_0 = \varrho(0)/\varepsilon_0,\ \mathrm{div}\,\boldsymbol{B}_0 = 0 \text{ in } \Omega\,; \quad \boldsymbol{E}_0 \times \boldsymbol{n} = 0,\ \boldsymbol{B}_0 \cdot \boldsymbol{n} = 0 \text{ on } \Gamma. \tag{4}$$

To have a well-posed problem, the *charge conservation equation* must hold:

$$\mathrm{div}\,\boldsymbol{J} + \partial_t \varrho = 0 \text{ in } \Omega \times \,]0,T[\,. \tag{5}$$

Theorem 1. *Assume $\boldsymbol{J} \in C^1(0,T;\mathbf{L}^2(\Omega))$ and $(\boldsymbol{E}_0,\boldsymbol{B}_0) \in \mathbf{H}_0(\mathbf{curl}\,;\Omega) \times \mathbf{H}(\mathbf{curl}\,;\Omega)$. The problem (1,3) has a unique solution $(\boldsymbol{E},\boldsymbol{B}) \in C^0(0,T;\ \mathbf{H}_0(\mathbf{curl}\,;\Omega)\times \mathbf{H}(\mathbf{curl}\,;\Omega)) \cap C^1(0,T;\ \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega))$. If, moreover, $\varrho \in C^0(0,T;\ L^2(\Omega))$ satisfies (5) in the sense of $\mathcal{D}'(\Omega \times \,]0,T[)$, and $(\boldsymbol{E}_0,\boldsymbol{B}_0)$ satisfy (4), then the constraints (2) are satisfied at any time and $(\boldsymbol{E},\boldsymbol{B}) \in C^0(0,T;\mathbf{X}\times\mathbf{Y})$.*

The proof is based on the semi-group theory [1] applied to the Maxwell evolution operator associated to (1), whose domain is $\mathbf{H}_0(\mathbf{curl}\,;\Omega) \times \mathbf{H}(\mathbf{curl}\,;\Omega)$ within the pivot space $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$.

1.2 The Second-Order Formulation

The two equations in (1) can be reformulated as:

$$\Box \boldsymbol{E} = -(1/\varepsilon_0)\,\partial_t \boldsymbol{J}, \quad \Box \boldsymbol{B} = (1/\varepsilon_0)\,\mathbf{curl}\,\boldsymbol{J} \quad \text{in } \Omega \times]0,T[\,, \tag{6}$$

where $\Box = \partial_t^2 + c^2\mathbf{curl}\,\mathbf{curl}$. This model still needs the equations (2–5) plus the extra initial condition:

$$\partial_t \boldsymbol{E}|_{t=0} = c^2\mathbf{curl}\,\boldsymbol{B}_0 - (1/\varepsilon_0)\,\boldsymbol{J}(0), \quad \partial_t \boldsymbol{B}|_{t=0} = -\mathbf{curl}\,\boldsymbol{E}_0 \text{ in } \Omega. \tag{7}$$

Theorem 2. *Assume $\boldsymbol{J} \in H^1(0,T;\mathbf{L}^2(\Omega))$ and (4–5) hold, with $\varrho \in C^0(0,T;\ L^2(\Omega))$. Then the problem (6,3,7) has a unique solution $(\boldsymbol{E},\boldsymbol{B}) \in C^0(0,T;\ \mathbf{X}\times\mathbf{Y}) \cap C^1(0,T;\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega))$. Moreover, it is equivalent to the first-order system.*

The key argument is the Lions–Magenes theory [2], which is applied to a variational setting of (6,3,7) in the same spaces as in Theorem 1.

2 Decomposition into Regular and Singular Parts

It is well known [3] that, when the domain Ω is smooth or convex, the regularity of the spaces $\mathbf{X}$ and $\mathbf{Y}$ is exactly H^1. Thus the results of Theorems 1 and 2 are optimal in terms of $C^\alpha(0,T;\mathbf{H}^s(\Omega))$ regularity, for $\alpha, s \geq 0$.

Now, when Ω is *singular*, i.e. neither smooth nor convex, $\mathbf{X}$ and $\mathbf{Y}$ are generally not included in $\mathbf{H}^1(\Omega)$. The interesting point is that the *regularised* subspaces $\mathbf{X}_R = \mathbf{X} \cap \mathbf{H}^1(\Omega)$, $\mathbf{Y}_R = \mathbf{Y} \cap \mathbf{H}^1(\Omega)$ are *closed* within the natural spaces [4–6]. Thus we introduce direct-sum decompositions as follows

$$\mathbf{X} = \mathbf{X}_R \oplus \mathbf{X}_S, \quad \mathbf{Y} = \mathbf{Y}_R \oplus \mathbf{Y}_S,$$

and we split the fields $(\boldsymbol{E}(t),\boldsymbol{B}(t))$ onto them as

$$\boldsymbol{E}(t) = \boldsymbol{E}_R(t) + \boldsymbol{E}_S(t), \quad \boldsymbol{B}(t) = \boldsymbol{B}_R(t) + \boldsymbol{B}_S(t). \tag{8}$$

It stems from the smoothness of projections onto closed subspaces that:

Theorem 3. *Under the hypotheses of Theorems 1 or 2, there holds:*
$(\boldsymbol{E}_R,\boldsymbol{B}_R) \in C^0(0,T;\mathbf{X}_R\times\mathbf{Y}_R)$ *and* $(\boldsymbol{E}_S,\boldsymbol{B}_S) \in C^0(0,T;\mathbf{X}_S\times\mathbf{Y}_S)$.

This result is of little practical use for a general 3D domain, because the *singular* spaces $\mathbf{X}_S$, $\mathbf{Y}_S$ lack so far a practical description, even though some interesting characterisations of them are known [5,7].

On the other hand, a much clearer picture appears when the Maxwell equations can be reduced to two-dimensional problems, i.e. in presence of an invariance by translation or by rotation. Namely, under this condition:

- The singular spaces are *finite-dimensional* and *explicit* bases are known.
- There exists a simplified Hodge decomposition of the fields in $\mathbf{X}$ and $\mathbf{Y}$.
- Precise regularity results are available for solutions to scalar wave equations.

3 Case of a Cylindrical Domain: Cartesian Geometry

3.1 Dimension Reduction and Basic Results

We assume that: (i) $\Omega = \omega \times \mathbb{R}$, with ω a (curvilinear) polygon in the (x, y) plane; (ii) $(\varrho, \boldsymbol{J})$ and $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ are invariant by translation in the direction z. This induces the invariance in z of the solution $(\boldsymbol{E}, \boldsymbol{B})$, which in turn allows the decoupling of the system (1–2) into two first-order systems. The **TE** mode links $\boldsymbol{E}_\perp = (E_x, E_y)$ and B_z, and the **TM** mode links $\boldsymbol{B}_\perp$ and E_z. If we denote $\Gamma = \gamma \times \mathbb{R}$, and $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ the normal and tangent vectors to γ, we have:

$$\textbf{TE:}\quad \partial_t \boldsymbol{E}_\perp - c^2\, \mathbf{curl}\, B_z = -\boldsymbol{J}_\perp/\varepsilon_0, \quad \partial_t B_z + \mathrm{rot}\, \boldsymbol{E}_\perp = 0 \text{ in } \omega \times]0, T[, \tag{9}$$

$$\mathrm{div}\, \boldsymbol{E}_\perp = \varrho/\varepsilon_0 \text{ in } \omega, \quad \boldsymbol{E}_\perp \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma. \tag{10}$$

$$\textbf{TM:}\quad \partial_t \boldsymbol{B}_\perp + \mathbf{curl}\, E_z = 0, \quad \partial_t E_z - c^2 \mathrm{rot}\, \boldsymbol{B}_\perp = -J_z/\varepsilon_0 \text{ in } \omega \times]0, T[, \tag{11}$$

$$\mathrm{div}\, \boldsymbol{B}_\perp = 0 \text{ in } \omega, \quad \boldsymbol{B}_\perp \cdot \boldsymbol{\nu} = 0 \text{ and } E_z = 0 \text{ on } \gamma. \tag{12}$$

These **TE** and **TM** modes can be rewritten as fully decoupled second-order systems similar to (6). $\mathbf{X}$ and $\mathbf{Y}$ are redefined as:

$$\mathbf{X} = \mathbf{H}_0(\mathrm{rot}\,; \omega) \cap \mathbf{H}(\mathrm{div}\,; \omega), \quad \mathbf{Y} = \mathbf{H}(\mathrm{rot}\,; \omega) \cap \mathbf{H}_0(\mathrm{div}\,; \omega).$$

With these modifications in the hypotheses of Theorems 1 and 2 there holds:

$$(\boldsymbol{E}_\perp, B_z) \in C^0(0, T; \mathbf{X} \times H^1(\omega)/\mathbb{R}) \cap C^1(0, T; \mathbf{L}^2(\omega) \times L^2(\omega)/\mathbb{R})\,; \tag{13}$$

$$(\boldsymbol{B}_\perp, E_z) \in C^0(0, T; \mathbf{Y} \times H^1_0(\omega)) \cap C^1(0, T; \mathbf{L}^2(\omega) \times L^2(\omega)). \tag{14}$$

We notice that the z-components are *regular* in space. Now we focus on the transversal components, for which the splitting (8) yields:

$$\boldsymbol{E}_\perp(t) = \boldsymbol{E}_R(t) + \sum_{k=1}^{\dim \mathbf{X}_S} \kappa_k^E(t)\, \boldsymbol{x}_{S*}^k, \quad \boldsymbol{B}_\perp(t) = \boldsymbol{B}_R(t) + \sum_{k=1}^{\dim \mathbf{Y}_S} \kappa_k^B(t)\, \boldsymbol{y}_{S*}^k. \tag{15}$$

Here, the dimension of $\mathbf{X}_S$ and $\mathbf{Y}_S$ is n_C, the number of reentrant corners in ω; $\boldsymbol{x}_{S*}^k$ (resp. $\boldsymbol{y}_{S*}^k$) is the basis function of $\mathbf{X}_S$ (resp. $\mathbf{Y}_S$) associated to the k-th reentrant corner. (See [4,7] for explicit expressions.) The space regularity of the fields depends on that of $\boldsymbol{x}_{S*}^k$ and $\boldsymbol{y}_{S*}^k$. Indeed, if π/α_k denotes the angle at the k-th reentrant corner, and $\alpha_{\min}$ and $\alpha_{\max}$ the least and the greatest of the α_k, $1 \le k \le n_C$, then $\mathbf{X} \subset \mathbf{H}^s(\omega)$ and $\mathbf{Y} \subset \mathbf{H}^s(\omega)$ iff $s < \alpha_{\min}$. On the other hand, the time regularity of $\boldsymbol{E}$, $\boldsymbol{B}$ depends on those of the regular parts $\boldsymbol{E}_R$, $\boldsymbol{B}_R$ and of the *singularity coefficients* κ_k.

3.2 Hodge Decomposition of the Electromagnetic Field

Independently of the splitting (8), we introduce the Hodge decomposition for $\boldsymbol{E}_\perp$:

$$\boldsymbol{E}_\perp(t) = -\mathbf{grad}\, V(t) + \mathbf{curl}\, W(t). \tag{16}$$

It can be easily proven that, if $\boldsymbol{E}_\perp(t) \in \mathbf{X}$, $V(t)$ and $W(t)$ are unique provided they are chosen within the 'natural' spaces of potentials defined as

$$\boldsymbol{\Phi} = \left\{\phi \in H_0^1(\omega) : \Delta\phi \in L^2(\omega)\right\}, \ \boldsymbol{\Psi} = \left\{\psi \in H^1(\omega)/\mathbb{R} : \Delta\psi \in L^2(\omega), \ \partial_\nu\psi|_\gamma = 0\right\},$$

and that they satisfy, under the hypotheses of Theorems 1 and 2:

$$(V, W) \in C^0(0,T;\boldsymbol{\Phi}\times\boldsymbol{\Psi}) \cap C^1(0,T;H_0^1(\omega)\times H^1(\omega)/\mathbb{R}). \tag{17}$$

The *regular* subspaces are defined as $\boldsymbol{\Phi}_R = \boldsymbol{\Phi}\cap H^2(\omega)$, $\boldsymbol{\Psi}_R = \boldsymbol{\Psi}\cap H^2(\omega)$; they are *closed*, their codimension is equal to n_C, and explicit bases ϕ_{S*}^k, ψ_{S*}^k of the singular complements are known [8]. Thus we have the splittings:

$$V(t) = V_R(t) + \sum \kappa_k^V(t)\,\phi_{S*}^k, \quad W(t) = W_R(t) + \sum \kappa_k^W(t)\,\psi_{S*}^k. \tag{18}$$

Moreover, it is possible [7] to choose the various singular bases such as to have:

$$\kappa_k^E(t) = \kappa_k^V(t) + \kappa_k^W(t), \quad k = 1,\ldots,n_C. \tag{19}$$

Similar (and simpler) results hold for $\boldsymbol{B}_\perp$, which has no gradient part.

3.3 Reduction to Scalar Problems

Simple computations show that $V(t)$ is solution to the elliptic problem

$$-\Delta V(t) = \varrho(t)/\varepsilon_0 \text{ in } \omega, \quad V(t) = 0 \text{ on } \gamma, \tag{20}$$

at any time t, while $W(t)$ is a solution to the hyperbolic problem

$$\partial_t^2 W - c^2\Delta W = \partial_t f \text{ in } \omega\times]0,T[, \quad \partial_\nu W = 0 \text{ on } \gamma\times]0,T[, \tag{21a}$$

$$W(0) = W_0, \quad \partial_t W|_{t=0} = W_1 \text{ in } \omega. \tag{21b}$$

where the right-hand side $\partial_t f$ and the initial conditions W_0, W_1 satisfy

$$\mathbf{curl}\, f = -\boldsymbol{J}_\perp + \partial_t \mathbf{grad}\, V, \quad \mathbf{curl}\, W_0 = \boldsymbol{E}_{0\perp} + \mathbf{grad}\, V(0), \quad W_1 = c^2 B_{0z} + f(0).$$

Thanks to (17), W can be identified to the strong solution of (21a-21b) if $\partial_t f \in L^1(0,T;H^1(\omega))$. This is equivalent to $\boldsymbol{J}_\perp \in W^{1,1}(0,T;\mathbf{L}^2(\omega))$.

Theorem 4. *Under the hypotheses* $\boldsymbol{J}_\perp \in W^{1,1}(0,T;\mathbf{L}^2(\omega))$ *and* $\varrho \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;L^2(\omega))$*, there holds for any* ϵ, $\epsilon' > 0$*:*

$$\boldsymbol{E} \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;H^{\alpha_{\min}-\epsilon'}(\omega)^3), \quad \kappa_k^E \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbb{R}). \tag{22}$$

Proof: The solution W to the scalar wave equation satisfies [8, Thm 5.3.1]

$$W_R \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;H^{1+\alpha_{\max}+\delta}(\omega)), \quad \kappa_k^W \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbb{R}), \quad \forall\epsilon > \delta > 0.$$

Moreover (20) implies $V \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\boldsymbol{\Phi})$, hence: $V_R \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\boldsymbol{\Phi}_R)$, $\kappa_k^V \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbb{R})$. So, by (19), (18) and (15) we have

$$\kappa_k^E \in C^{0,1-\alpha_{\max}-\epsilon}(\mathbb{R}), \quad \boldsymbol{E}_\perp \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbf{H}^{\alpha_{\min}-\epsilon'}(\omega))$$

We recall that the space regularity $\mathbf{H}^{\alpha_{\min}-\epsilon'}(\omega)$ is optimal. For the z-component, there holds by interpolation: $E_z \in C^{0,1-\alpha_{\min}+\epsilon'}(0,T;H^{\alpha_{\min}-\epsilon'}(\omega))$; hence (22).

Similarly one can prove:

Theorem 5. *If* $J_z \in L^1(0,T;H_0^1(\Omega))$*, then* $\boldsymbol{B} \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;H^{\alpha_{\min}-\epsilon'}(\omega)^3)$ *and* $\kappa_k^B \in C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbb{R})$.

4 Case of an Axisymmetric Domain

Now we assume that (i) Ω is an axisymmetric domain generated by the rotation of a polygon [6]; (ii) $(\varrho, \boldsymbol{J})$ and $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ are invariant by rotation in the direction θ. This induces the invariance in θ of the solution $(\boldsymbol{E}, \boldsymbol{B})$. The results of Section 3 remain qualitatively valid [6,9]: the decoupling of components concerns $\boldsymbol{E}_m = (E_r, E_z)$ and B_θ on one hand, $\boldsymbol{B}_m$ and E_θ on the other hand.

The dimension of $\mathbf{Y}_S$ is n_A, the number of reentrant circular edges in Ω; that of $\mathbf{X}_S$ is $n_A + n_P$, where n_P is the number of *sharp* conical vertices, i.e. those whose angle is greater than the value $\pi/\beta_\star$ defined by $P_{1/2}(\cos\pi/\beta_\star) = 0$, where P_ν denotes the Legendre function. For any sharp vertex one defines ν_k as the unique $\nu \in]0, 1/2[$ such that $P_\nu(\cos\pi/\beta_k) = 0$; then $\sigma_{\min}$ and $\sigma_{\max}$ are the minimum and maximum of the set $\{\alpha_k,\ 1 \le k \le n_A\,;\, \nu_k + 1/2,\ 1 \le k \le n_P\}$.

Theorem 6. *If* $\boldsymbol{J}_m \in W^{1,1}(0,T;\mathbf{L}^2(\Omega))$*, then for a reentrant edge, resp. a sharp vertex:* $\kappa_k^E \in C^{0,1-\alpha_k-\epsilon}(0,T;\ \mathbb{R})$*, resp.* $\kappa_k^E \in C^{0,1/2-\nu_k-\epsilon}(0,T;\ \mathbb{R})$*,* $\forall \epsilon > 0$*. Moreover, if* $\boldsymbol{J}_\theta \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ *then* $\kappa_k^B \in C^{0,1-\alpha_k-\epsilon}(0,T;\ \mathbb{R})$*. And the electromagnetic field satisfies for any* $\epsilon, \epsilon' > 0$*:*

$$(\boldsymbol{E}, \boldsymbol{B}) \in C^{0,1-\sigma_{\max}-\epsilon}(0,T;\mathbf{H}^{\sigma_{\min}-\epsilon'}(\Omega)) \times C^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathbf{H}^{\alpha_{\min}-\epsilon'}(\Omega)).$$

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