Comments on our article *Global solutions for the one-dimensional Vlasov–Maxwell system for laser-plasma interaction*

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The various notations are the same as in the article; particularly, $\|\cdot\|_t$ is the L^∞ norm on $(0,t)\times\mathbb{R}$. The equation and page numbers refer to the final version of the paper, published in *Mathematical Models and Methods in the Applied Sciences* [1].

There is an error in Lemma 2.2, where we estimate the divergence of characteristics associated to different forces. We claim that Eq. (2.7), namely:

$$\left|P^1(0;t,x,p)-P^2(0;t,x,p)\right| \le \int_0^t \|F_1-F_2\|_s \,\mathrm{d}s.$$

(as well as (2.6), which is a straightforward consequence of it) directly follows from the integration of the characteristic system (2.2). This is dubious, since

$$P^1(\tau;t,x,p)-P^2(\tau;t,x,p)=\int_t^\tau \left[F_1(s,X^1(s;t,x,p))-F_2(s,X^2(s;t,x,p))\right]\,\mathrm{d}s.$$

The forces are taken at two different points, so one cannot bound the integrand by $\|F_1(s)-F_2(s)\|_{L^\infty}$ or $\|F_1-F_2\|_s$. Cooper and Klimas [2, p. 310] make the same mistake; we copied it without thinking it over.

Indeed, if we calculate like on p. 26, we find:

.../...

$$
\begin{aligned}
&\left|X^1(\tau) - X^1(\tau)\right| + \left|P^1(\tau) - P^2(\tau)\right| \\
&\le \int_\tau^t \left\{\left|\widehat{P}^1(s) - \widehat{P}^2(s)\right| + \left|F_1(s, X^1(s)) - F_2(s, X^2(s))\right|\right\} \,\mathrm{d}s \\
&\le \int_\tau^t \left\{\left|P^1(s) - P^2(s)\right| + \left|F_1(s, X^1(s)) - F_1(s, X^2(s))\right| + \left|F_1(s, X^2(s)) - F_2(s, X^2(s))\right|\right\} \,\mathrm{d}s \\
&\le \int_\tau^t \left\{\left|P^1(s) - P^2(s)\right| + \Lambda F_1(s)\, \left|X^1(s) - X^2(s)\right| + \|F_1(s) - F_2(s)\|_{L^\infty}\right\} \,\mathrm{d}s \\
&\le \int_\tau^t \|F_1 - F_2\|_s \,\mathrm{d}s + (1 + \|\partial_x F_1\|_t) \int_\tau^t \left\{\left|X^1(s) - X^2(s)\right| + \left|P^1(s) - P^2(s)\right|\right\} \,\mathrm{d}s.
\end{aligned}
$$

We have denoted by $\Lambda F_1(s)$ the Lipschitz constant of $F_1(s)$. Hence by Gronwall:

$$
\left|X^1(0) - X^2(0)\right| + \left|P^1(0) - P^2(0)\right| \le \mathrm{e}^{t\,(1+\|\partial_x F_1\|_t)} \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s\,. \qquad (1)
$$

A similar bound is found in [3, Eq (1.7) p. 353]; strangely, the function sh must be a hyperbolic *cosine*.

If we now compare a generic force with the free streaming case, we take $F_1 = 0$, and drop the subscript 2 in $X^2,\ P^2,\ F_2$. Then (1) becomes:

$$
|X(0) - (x - \widehat{p}\,t)| + |P(0) - p| \le \mathrm{e}^T \int_0^t \|F\|_s \,\mathrm{d}s\,. \qquad (2)
$$

The latter inequality allows to bound *a priori* the solution of the forced Vlasov equation, in a much better way than (1). For Vlasov–Poisson, Iordanski [3] thus obtains a solution by a fixed-point process. The proof goes as follows: the force is only made of the electric field E, with $\partial_x E = n_{\mathrm{ext}} - n$, and $n = \int f \,\mathrm{d}p$. Thanks to (2) one bounds

$$
\|n_{k+1}\|_t \le C_0 + C_1 \int_0^t \|E_k\|_s \,\mathrm{d}s \quad \text{and hence} \quad \|E_{k+1}\|_t \le C_2 + C_3 \int_0^t \|E_k\|_s \,\mathrm{d}s.
$$

So, the sequence (E_k) is bounded, first in L^∞ norm, then in $W^{1,\infty}$ norm. Then, Eq. (1) yields:

$$
\|E_{k+1} - E_k\|_t \le C_4 \int_0^t \|E_k - E_{k-1}\|_s \,\mathrm{d}s,
$$

and the sequence converges as a telescopic series. Finally, using (1) once more, the sequence (f_k) also converges. The article [2] can be "repaired" similarly, by carefully distinguishing between *a priori* and *a posteriori* bounds. (This is hardly surprising, since Cooper–Klimas are dealing with the same problem as Iordanski, even though they are unwilling to admit it.)

In the case of the reduced Vlasov–Maxwell system analysed in our article, one can use (2) and bound uniformly the sequences n_k, E_k, $\partial_x E_k$, A_k, $\partial_x A_k$, $\partial_t A_k$ and F_k: Eqs. (2.15) and (3.3) to (3.7) are proven, without any difficulty. Moreover, we have the bound (2.20) for $\partial_x n_k$. But the inequalities (3.8) to (3.13), upon which the convergence proof is based, are false. They lack an exponential term which stems from (1). Therefore, it appears necessary to bound uniformly $\partial_x F_k$ — or equivalently $\partial_x n_k$ or $\partial_x^2 A_k$ — *before* beginning the convergence proof.

To this end, the ideas of §§4 and 5 can be applied, with little or no change. Thus the proof is achieved; we *have* proven the existence of strong solutions, locally in time in the NR case, and globally in the QR case. On the other hand, global existence of weak solutions of the NR model is *not* guaranteed. The argument of Lemma 3.6 breaks down, since the operator $\mathcal{L}$ is actually *not* continuous in the norm of the space X, but merely in that of Y.

References

[1] J.A. Carrillo, S. Labrunie. Global solutions for the one-dimensional Vlasov–Maxwell system for laser-plasma interaction. *Math. Models Methods Appl. Sci.* **16** (2006), 19–57.

[2] J. Cooper, A. Klimas. Boundary-value problem for the Vlasov–Maxwell equation in one dimension. *J. Math. Anal. Appl.* **75** (1980), 306–329.

[3] S.V. Iordanskiĭ. The Cauchy problem for the kinetic equation of plasma. *Amer. Math. Soc. Transl.* **35** (1964), 351–363.