

Resolution of axisymmetric Maxwell equations

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Abstract

In this article, we study the static and time-dependent Maxwell equations in axisymmetric geometry. Using the mathematical tools introduced in [3], we investigate the decoupled problems induced in a meridian half-plane, and the splitting of the solution in a regular part and a singular part, the former being in the Sobolev space H^1 component-wise. It is proven that the singular parts are related to singularities of Laplace-like or wave-like operators. We infer from these characterisations: (i) the finite dimension of the space of singular fields; (ii) global space and space-time regularity results for the electromagnetic field. This paper is the continuation of [5, 3].

Introduction.

In the recent years, an ever-growing number of engineering problems requiring to model and to simulate numerically devices working with or within electromagnetic fields have come out. The mathematical models describing the physics of these devices are based on Maxwell's equations, in the stationary or time-dependent form, often coupled with other equations.

Moreover, many structures that are to be modelled have a complex three-dimensional geometry and often present a surface with edges or corners. These geometrical singularities can generate very strong fields that have to be taken into account, be they desired as an active part of the device (e.g. to extract electrons from a velvet cathode), or the consequence of *a priori* constraints on the design, whose potentially destructive effects have to be controlled.

However, three-dimensional computations are very expensive. In a number of cases, one reduces the problem to two-dimensional equations by assuming that both the geometry and the data and initial conditions are independent of one of the coordinates. As a first step, we considered in a previous article [5] problems independent of the transversal variable z to reduce them to two-dimensional Cartesian ones.

In this paper, we consider the case of an axisymmetric situation, that can be viewed as an intermediate between a full three-dimensional problem and a two-dimensional one. Indeed, while the geometry of real devices is very rarely Cartesian, it is much more common to have an axial symmetry, at least approximately or locally. In other words, the axisymmetric geometry can be considered as a zero-order approximation of a real three-dimensional case [6].

Nevertheless, very few mathematical analyses have been carried out in the framework of axisymmetric problems [15, 6]. In this paper, we propose to study the static and time-dependent Maxwell equations in axisymmetric geometry. It is the continuation of [3], where the mathematical tools were introduced, and will be followed by a forthcoming paper where numerical developments and applications will be shown.

The article is organised as follows. Section 1 recalls the basic facts about the static and time-dependent Maxwell equations. Section 2 gives the fundamental results and definitions associated to the axisymmetric geometry. In Section 3, we set the functional-analytic framework adapted to the study of boundary-value problems in this geometry, i.e. we describe various properties of axisymmetric and weighted Sobolev spaces as well as the variational elliptic problems defined on them. Then, in Section 4, we propose several closedness results, which are related to the lack of density of regular (i.e. H^1 component-wise) fields in the natural spaces of electromagnetic fields. This further leads to the splitting of the electromagnetic field in regular and singular parts. The latter are related to

the singular solutions of some Laplace-like problems through integration by parts formulae, which will be useful throughout the paper. Most of the material of the latter three Sections is dealt with in [3], to which we refer the reader interested in the details.

Section 5 is devoted to the study of the singular solutions of the Laplace-like problems, which allows to characterise the singular parts and especially their dimension. In Section 6, an in-depth examination of the static Maxwell equations is performed. We focus on two topics: the decoupling, induced by the axial symmetry, of the equations into two systems posed in the meridian half-plane; and its relationship with the splitting in regular and singular parts. As a byproduct, we determine the global space regularity of the electromagnetic field. This study is one of the two ingredients needed for the understanding of the time-dependent equations. The other one is the analysis of singularities of a wave-like problem, which is carried out in Section 7. Finally, Section 8 investigates the time-dependent equations, in the spirit of Section 6; we conclude with a space-time regularity result for the electromagnetic field.

1 The Maxwell problems.

Let Ω be a bounded and simply connected domain of \mathbb{R}^3 , Γ its Lipschitz boundary, and \mathbf{n} the unit outward normal to Γ . Note that the case of a domain, which is not simply connected, is treated very carefully in [2, 11]. We assume that Γ is made of a perfectly conducting material.

1.1 The static Maxwell equations.

There are two div-curl problems, depending on the boundary condition. The *electrostatic* (or Dirichlet) problem is, for $\mathbf{F}^n \in \mathcal{J}^n = \mathbf{H}_0(\operatorname{div} 0; \Omega)$, and $G^d \in \mathcal{L}^d = L^2(\Omega)$:
Find $\mathcal{U}^d \in \mathbf{L}^2(\Omega)$ such that

$$\operatorname{curl} \mathcal{U}^d = \mathbf{F}^n \text{ in } \Omega, \quad (1)$$

$$\operatorname{div} \mathcal{U}^d = G^d \text{ in } \Omega, \quad (2)$$

$$\mathcal{U}^d \times \mathbf{n} = 0 \text{ on } \Gamma. \quad (3)$$

The boundary condition on \mathbf{F}^n is imposed by the condition (3) (cf. [13]): the **curl** operator “swaps” the Dirichlet and Neumann boundary conditions, hence the change of superscripts n/d .

The *magnetostatic* or Neumann problem is, given $\mathbf{F}^d \in \mathcal{J}^d = \mathbf{H}(\operatorname{div} 0, \Omega)$, and $G^n \in \mathcal{L}^n = \{u \in L^2(\Omega) : \iiint_{\Omega} u \, d\Omega = 0\}$: *Find $\mathcal{U}^n \in \mathbf{L}^2(\Omega)$ such that*

$$\operatorname{curl} \mathcal{U}^n = \mathbf{F}^d \text{ in } \Omega, \quad (4)$$

$$\operatorname{div} \mathcal{U}^n = G^n \text{ in } \Omega, \quad (5)$$

$$\mathcal{U}^n \cdot \mathbf{n} = 0 \text{ on } \Gamma. \quad (6)$$

The fact that G^n has a mean zero value stems from (6). This condition is satisfied by the actual magnetic field, which is divergence-free (cf. (21) below).

In order to prove the existence and uniqueness of the solution $\mathcal{U}^{d/n}$ to these problems, a possible way is to reformulate the equations as a saddle-point formulation where (2) or (5) are seen as constraints, and to check that the Lagrange multiplier associated to them is equal to 0 (see [8] for details). The basic tool in both cases is the *Weber inequality*, which stems from the compactness result of [16]:

Proposition 1.1 In $\mathcal{X}^d = \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}(\mathbf{div}; \Omega)$ and $\mathcal{X}^n = \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}_0(\mathbf{div}; \Omega)$, the semi-norm $\mathbf{u} \mapsto (\|\mathbf{curl} \mathbf{u}\|_0^2 + \|\mathbf{div} \mathbf{u}\|_0^2)^{1/2}$ is a norm, which is equivalent to the canonical norm. In other words, there exists a constant $C > 0$ such that $\|\mathbf{u}\|_0^2 \leq C (\|\mathbf{curl} \mathbf{u}\|_0^2 + \|\mathbf{div} \mathbf{u}\|_0^2)$, for all \mathbf{u} in \mathcal{X}^d or \mathcal{X}^n .

We shall also need the *scalar* and *vector* potentials, which are associated to the Hodge decomposition

$$\mathcal{U}^{d/n} = -\mathbf{grad} V^{d/n} + \mathbf{curl} \mathcal{A}^{n/d}. \quad (7)$$

Notice, once more, the swap of superscripts n/d between the electromagnetic field and its vector potential, due to the **curl** operator. Indeed, both the gradient and the curl parts in (7) satisfy the electric or magnetic boundary conditions.

The scalar potential V is defined as the variational solution in $\mathcal{V}^d = H_0^1(\Omega)$, resp. $\mathcal{V}^n = \{u \in H^1(\Omega) : \iiint_{\Omega} u \, d\Omega = 0\}$, of the Dirichlet, resp. Neumann problem

$$-\Delta V^{d/n} = G^{d/n} \text{ in } \Omega, \quad (8)$$

$$V^d = 0 \text{ on } \Gamma, \quad (9)$$

$$\text{resp. } \frac{\partial V^n}{\partial n} = 0 \text{ on } \Gamma. \quad (10)$$

The vector potential \mathcal{A} is the solution in $\mathbf{L}^2(\Omega)$ of the Neumann resp. Dirichlet problem

$$-\Delta \mathcal{A}^{n/d} = \mathbf{F}^{n/d} \text{ in } \Omega, \quad (11)$$

$$\mathbf{div} \mathcal{A}^{n/d} = 0 \text{ in } \Omega, \quad (12)$$

$$\mathcal{A}^n \cdot \mathbf{n} = 0 \text{ on } \Gamma, \quad (13)$$

$$(\mathbf{curl} \mathcal{A}^n) \times \mathbf{n} = 0 \text{ on } \Gamma, \quad (14)$$

$$\text{resp. } \mathcal{A}^d \times \mathbf{n} = 0 \text{ on } \Gamma. \quad (15)$$

In both cases, the existence and uniqueness of the vector potential stem once more from a saddle-point approach [8].

Defining the spaces of potentials by

$$\Phi^{d/n} = \left\{ \varphi \in \mathcal{V}^{d/n} : \Delta \varphi \in \mathcal{L}^{d/n} \text{ and, in the Neumann case, } \partial_n \varphi = 0 \right\}$$

$$\mathcal{M}^{d/n} = \left\{ \mathbf{M} \in \mathcal{X}^{d/n} : \mathbf{curl} \mathbf{M} \in \mathcal{X}^{n/d} \text{ and } \mathbf{div} \mathbf{M} = 0 \right\}$$

the existence and uniqueness results are summarised in the

Theorem 1.2 *The following mappings are isomorphisms of vector spaces:*

$$\begin{array}{ccc} \Phi^{d/n} & & \mathcal{L}^{d/n} \\ \mathcal{M}^{n/d} & \xrightarrow[\mathbf{+ curl}]{\mathbf{- grad}} \mathcal{X}^{d/n} & \xrightarrow[\mathbf{curl}]{\mathbf{div}} \mathcal{J}^{n/d} \end{array} \quad (16)$$

The operators placed above and beneath a horizontal arrow have their source or target space at the same height. The two operators that come with the left arrow are to be added; with the right arrow, their are in tensor product.

As a consequence of these isomorphisms, the scalar, resp. vector Laplacian, is an isomorphism between $\Phi^{d/n}$ and $\mathcal{L}^{d/n}$, resp. $\mathcal{M}^{n/d}$ and $\mathcal{J}^{n/d}$.

We shall also be interested in the *divergence-free* problem, i.e. the case $G = 0$, for which there holds the simplified chain of isomorphisms:

$$\mathcal{M}^{n/d} \xrightarrow{\mathbf{curl}} \mathcal{X}^{0d/n} \xrightarrow{\mathbf{curl}} \mathcal{J}^{n/d} \quad (17)$$

with $\mathcal{X}^{0d/n} = \{\mathbf{u} \in \mathcal{X}^{d/n} : \mathbf{div} \mathbf{u} = 0\}$.

1.2 The time-dependent Maxwell equations.

Given $T > 0$, $Q = \Omega \times]0, T[$ and $\Sigma = \Gamma \times]0, T[$, let us recall Maxwell's equations in time. Let c and ε_0 be respectively the speed of light and the dielectric permittivity; ϱ and \mathcal{J} the *sources* (charge and current densities). First, there are the evolution equations:

$$\frac{\partial \mathcal{E}}{\partial t} - c^2 \operatorname{curl} \mathcal{B} = -\frac{1}{\varepsilon_0} \mathcal{J} \text{ in } Q, \quad (18)$$

$$\frac{\partial \mathcal{B}}{\partial t} + \operatorname{curl} \mathcal{E} = 0 \text{ in } Q. \quad (19)$$

Then, the constraint equations, viz. divergence and boundary conditions:

$$\operatorname{div} \mathcal{E} = \frac{\varrho}{\varepsilon_0} \text{ in } Q, \quad (20)$$

$$\operatorname{div} \mathcal{B} = 0 \text{ in } Q, \quad (21)$$

$$\mathcal{E} \times \mathbf{n} = 0 \text{ on } \Sigma, \quad (22)$$

$$\mathcal{B} \cdot \mathbf{n} = 0 \text{ on } \Sigma. \quad (23)$$

The charge conservation equation,

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \mathcal{J} = 0. \quad (24)$$

appears as a compatibility condition for (18), given (20).

Last, initial conditions are provided to close the system of equations,

$$\mathcal{E}(0) = \mathcal{E}_0, \quad \mathcal{B}(0) = \mathcal{B}_0, \quad (25)$$

where $f(t)$, $\mathbf{u}(t)$ denote the fields $\mathbf{x} \mapsto f(\mathbf{x}, t)$, $\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, t)$ —and similarly $f'(t) : \mathbf{x} \mapsto \partial_t f(\mathbf{x}, t)$, $\mathbf{u}'(t) : \mathbf{x} \mapsto \partial_t \mathbf{u}(\mathbf{x}, t)$. In the same spirit, we shall measure the space-time regularity of solutions of evolution problems with the following spaces: if X is some Banach space of functions over a given domain, we set:

- $\mathcal{C}^m(0, T; X)$, $m \in \mathbb{N}$, resp. $\mathcal{C}^{0,\alpha}(0, T; X)$, $\alpha \in [0, 1]$: the space of m times continuously differentiable, resp. Hölder continuous of exponent α , functions of $t \in [0, T]$, with values in X ;
- $L^p(0, T; X)$, $p \in [1, +\infty]$: the space of p -th power integrable functions of $t \in]0, T[$ with values in X ;
- $W^{s,p}(0, T; X)$, $s \in \mathbb{R}$, $p \in [1, +\infty]$: the Sobolev space of exponent s built upon $L^p(0, T; X)$.

In order to prove the existence and uniqueness of the electromagnetic field under suitable assumptions on the data and the initial conditions, one can use for instance the semi-group theory [10] to get the

Theorem 1.3 *Assume that $(\mathcal{E}_0, \mathcal{B}_0)$ belongs to $\mathbf{H}_0(\operatorname{curl}; \Omega) \times \mathbf{H}(\operatorname{curl}; \Omega)$, and that $\mathcal{J} \in \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega))$. Then, there exists one and only one solution to the time dependent problem (18)-(19), (22), (25), such that*

$$\mathcal{E} \in \mathcal{C}^0(0, T; \mathbf{H}_0(\operatorname{curl}; \Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega)), \quad (26)$$

$$\mathcal{B} \in \mathcal{C}^0(0, T; \mathbf{H}(\operatorname{curl}; \Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega)).$$

Assume moreover that ϱ belongs to $C^0(0, T; L^2(\Omega))$, that the charge conservation (24) holds and that the initial data satisfy

$$\operatorname{div} \mathcal{E}_0 = \frac{\varrho(0)}{\varepsilon_0}, \quad \operatorname{div} \mathcal{B}_0 = 0, \quad \mathcal{B}_0 \cdot \mathbf{n}|_{\Gamma} = 0.$$

Then, (20) and (21) are fulfilled, and in addition to (26),

$$\mathcal{E} \in C^0(0, T; \mathcal{X}^d), \quad \mathcal{B} \in C^0(0, T; \mathcal{X}^n) \cap C^1(0, T; \mathbf{H}(\operatorname{div}; \Omega)). \quad (27)$$

(The proof of the first part of the Theorem is a standard application of the semi-group theory whereas the second part can be obtained through some simple verifications.)

2 The axisymmetric geometry and operators.

2.1 Notations.

In the remainder of this article, we shall treat the special case of an axisymmetric domain Ω generated by the rotation of a polygon ω around one of its sides, denoted γ_a . The other sides are denoted γ_i , $1 \leq i \leq n + 1$, and generate the faces Γ_i , $1 \leq i \leq n + 1$, of Γ . The other notations are the same as in [3] and will be merely recalled on Figure 1. Of course, we shall mostly use the cylindrical coordinates (r, θ, z) .

Moreover, we assume that the data (\mathbf{F}, G) or (ϱ, \mathcal{J}) of the static or time-dependent

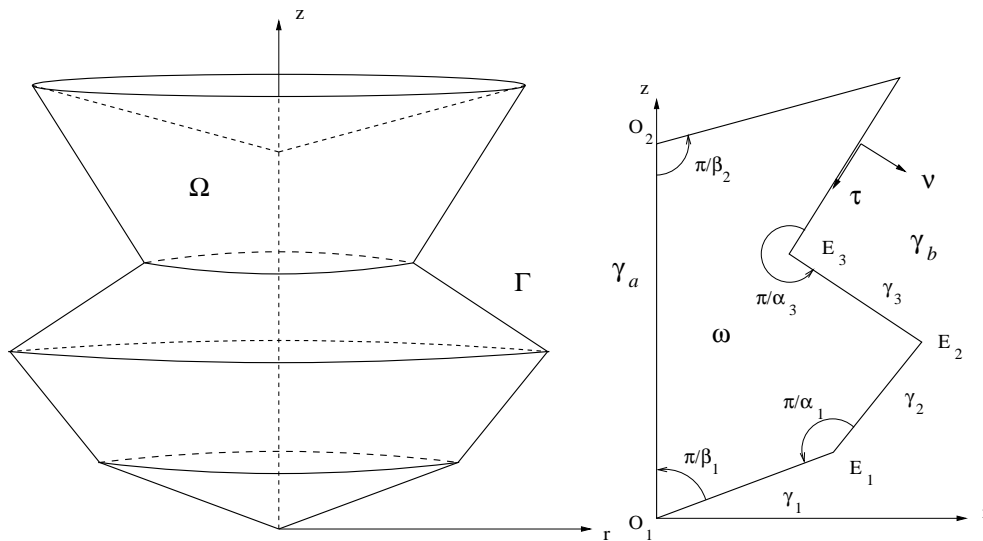


Figure 1: The domains Ω and ω .

problems, and the initial conditions $(\mathcal{E}_0, \mathcal{B}_0)$, possess an *axial symmetry*. As a consequence of the Curie principle (cf. [6]), the same will hold for their solutions \mathcal{U} or $(\mathcal{E}, \mathcal{B})$.

The definition of axial symmetry for general scalar- or vector-valued distributions is found in [6, 3]. In practice it means that they are entirely characterised by the data of their traces (or the traces of their cylindrical components) in a meridian half-plane, or equivalently that their derivative (or the derivatives of their cylindrical components) with respect to θ vanishes: $T(x, y, z) = T(r, z)$ or $\partial_{\theta} T = 0$.

We denote by $\check{\mathcal{D}}'(\Omega)$ and $\check{\mathbf{D}}'(\Omega)$ the spaces of axisymmetric scalar and vector distributions; $\check{\mathcal{D}}(\Omega) = \mathcal{D}(\Omega) \cap \check{\mathcal{D}}'(\Omega)$ and $\check{\mathbf{D}}(\Omega) = \mathcal{D}(\Omega)^3 \cap \check{\mathbf{D}}'(\Omega)$ the spaces of axisymmetric test functions and fields. The traces of those spaces in a meridian half-plane are characterised in [3]. Generally, we constantly denote the subspaces of axisymmetric fields by the sign $\check{}$:

- $\check{L}^2(\Omega) = L^2(\Omega) \cap \check{\mathcal{D}}'(\Omega)$, $\check{H}^s(\Omega) = H^s(\Omega) \cap \check{\mathcal{D}}'(\Omega)$ (for $s \in \mathbb{R}$);
- $\check{\mathbf{L}}^2(\Omega) = \mathbf{L}^2(\Omega) \cap \check{\mathbf{D}}'(\Omega)$, $\check{\mathbf{H}}(\mathbf{curl}; \Omega) = \mathbf{H}(\mathbf{curl}; \Omega) \cap \check{\mathbf{D}}'(\Omega)$;
- $\check{\mathcal{X}}^{d/n} = \mathcal{X}^{d/n} \cap \check{\mathbf{D}}'(\Omega)$, $\check{\Phi}^{d/n} = \Phi^{d/n} \cap \check{\mathcal{D}}'(\Omega)$, and so on.

For any vector field \mathbf{w} , we define its *meridian* and *azimuthal* components as $\mathbf{w}_m = w_r \mathbf{e}_r + w_z \mathbf{e}_z$ and $\mathbf{w}_\theta = w_\theta \mathbf{e}_\theta$. The expression of the differential operators in cylindrical coordinates, as well as the decoupling of meridian and azimuthal components induced on them by the axial symmetry, are recalled in [3]. In the remainder of this paper, we shall need the traces of these operators in the meridian half-plane (r, z) . To this end, we define the following operators:

$$\begin{aligned} \operatorname{div} \mathbf{u} &= \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}, & \operatorname{div}_+ \mathbf{u} &= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, \\ \operatorname{curl} \mathbf{u} &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, & \operatorname{curl}_- \mathbf{u} &= \frac{\partial u_r}{\partial z} + \frac{u_z}{r} - \frac{\partial u_z}{\partial r}, \\ \operatorname{curl} f &= -\frac{\partial f}{\partial z} \mathbf{e}_r + \frac{\partial f}{\partial r} \mathbf{e}_z, & \Delta^\pm f &= \frac{\partial^2 f}{\partial r^2} \pm \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

The Δ^\pm are called the *modified* Laplacians. Δ^+ is the trace of the 3D scalar Laplacian for an axisymmetric function. And for an axisymmetric, azimuthal vector field \mathbf{u} , there holds:

$$\mathbf{u} = \frac{\varphi}{r} \mathbf{e}_\theta \implies \Delta \mathbf{u} = \frac{1}{r} \Delta^- \varphi \mathbf{e}_\theta. \quad (28)$$

These operators satisfy the identities $\Delta^+ = \operatorname{div}_+ \mathbf{grad}$ and $\Delta^- = -\operatorname{curl}_- \operatorname{curl}$.

2.2 Basic results for the axisymmetric Maxwell problems.

As pointed out earlier, the axial symmetry of the domain, sources, and initial conditions induces that of the electromagnetic fields, hence the following results:

Proposition 2.1 *The following mappings are isomorphisms of vector spaces:*

$$\check{\Phi}^{d/n} \quad \begin{array}{c} \xrightarrow{-\mathbf{grad}} \\ \check{\mathcal{M}}^{n/d} \quad \xrightarrow{+\mathbf{curl}} \end{array} \check{\mathcal{X}}^{d/n} \quad \begin{array}{c} \xrightarrow{\operatorname{div}} \\ \check{\mathcal{J}}^{n/d} \end{array} \check{\mathcal{L}}^{d/n} \quad (29)$$

$$\check{\mathcal{M}}^{n/d} \quad \xrightarrow{\operatorname{curl}} \quad \check{\mathcal{X}}^{0d/n} \quad \xrightarrow{\operatorname{curl}} \quad \check{\mathcal{J}}^{n/d} \quad (30)$$

Proposition 2.2 *If ϱ , \mathcal{J} and $(\mathcal{E}_0, \mathcal{B}_0)$ are axisymmetric, so is the solution to (18)-(25). And, provided that $\mathcal{J} \in \mathcal{C}^1(0, T; \check{\mathbf{L}}^2(\Omega))$ and that $\varrho \in \mathcal{C}^0(0, T; \check{L}^2(\Omega))$, there holds*

$$\mathcal{E} \in \mathcal{C}^0(0, T; \check{\mathcal{X}}^d), \quad \mathcal{B} \in \mathcal{C}^0(0, T; \check{\mathcal{X}}^{0n}). \quad (31)$$

3 Some results of functional analysis.

3.1 Sobolev spaces and the axial symmetry.

We briefly recall some results about axisymmetric Sobolev spaces; the proofs can be found in [6]. To study the traces of those spaces in a meridian half-plane, we introduce the weighted Lebesgue spaces on ω

$$L_\alpha^2(\omega) = \left\{ f : f \text{ is measurable on } \omega, \int_\omega |f|^2 r^\alpha dr dz < +\infty \right\}, \quad \alpha \in \mathbb{R},$$

with its canonical norm $\|\cdot\|_{0,\alpha,\omega}$, and the related scales of Sobolev spaces $H_\alpha^s(\omega)$, with the canonical norms $\|\cdot\|_{s,\alpha,\omega}$. In the remainder of the paper, we shall only use the scale up to $s = 2$, so we give only those results. The more general ones can be found in [6].

Proposition 3.1 *The mapping $R : L_\alpha^2(\omega) \rightarrow L_{\alpha-2}^2(\omega)$, $f \mapsto r f$, is an isometry.*

Definition 3.2 *Let $s \in [0, 2]$ and set*

- if $s \neq 2$, $H_+^s(\omega) = H_1^s(\omega)$;
- if $s = 2$, $H_+^2(\omega) = \left\{ w \in H_1^2(\omega) : \frac{\partial^2 w}{\partial r^2} \in L_{-1}^2(\omega) \right\}$, which is a Hilbert space endowed with the norm $\|w\|_{2,+,\omega} = \left(\|w\|_{2,1,\omega}^2 + \left\| \frac{\partial^2 w}{\partial r^2} \right\|_{0,-1,\omega}^2 \right)^{1/2}$.

Then, for s in $[0, 2]$, one has the

Proposition 3.3 *The trace operator is an isomorphism from $\check{H}^s(\Omega)$ to $H_+^s(\omega)$. Moreover, when s is an integer, it is an isometry up to a factor $\sqrt{2\pi}$.*

Definition 3.4 *Let $s \in [0, 2]$ and set*

- if $s \neq 1$, $H_-^s(\omega) = H_1^s(\omega)$;
- if $s = 1$, $H_-^1(\omega) = H_1^1(\omega) \cap L_{-1}^2(\omega)$, which is a Hilbert space endowed with the norm $\|w\|_{1,-,\omega} = \left(\|w\|_{1,1,\omega}^2 + \|w\|_{0,-1,\omega}^2 \right)^{1/2}$.

Proposition 3.5 *The trace operator is an isomorphism from $\check{H}^s(\Omega)$ to $H_-^s(\omega) \times H_+^s(\omega) \times H_-^s(\omega)$, for s in $[0, 2]$; when s is an integer, it is an isometry up to a factor $\sqrt{2\pi}$.*

Proposition 3.6 *Let v belong to $H_-^1(\omega)$: $v|_{\gamma_a} \in L^2(\gamma_a)$ and $v|_{\gamma_a} = 0$. Hence, if u belongs to $H_+^2(\omega)$, then $\partial_r u|_{\gamma_a} \in L^2(\gamma_a)$ and $\partial_r u|_{\gamma_a} = 0$.*

Proposition 3.7 *The range of the trace operator from $\check{H}(\mathbf{curl}; \Omega)$ is:*

$$R(\omega) = \left\{ \mathbf{w} = (w_r, w_\theta, w_z) : \mathbf{w}_m \in L_1^2(\omega)^2, \text{curl } \mathbf{w}_m \in L_1^2(\omega), r w_\theta \in H_{-1}^1(\omega) \right\}.$$

3.2 Some properties of weighted Sobolev spaces.

In this Subsection, we state some criteria for characterising the $H_\alpha^s(\omega)$ and $H_\pm^s(\omega)$ spaces, especially their transformations by the isometry \mathbf{R} introduced in Proposition 3.1, as well as their consequences on axisymmetric Sobolev spaces $\check{\mathbf{H}}^s(\Omega)$.

Lemma 3.8 *The space $H_{-1}^1(\omega)$ is continuously imbedded into $L_{-3}^2(\omega)$, i.e. there exists a constant K_1 such that*

$$\forall u \in H_{-1}^1(\omega), \quad \|u\|_{0,-3,\omega}^2 \leq K_1 \|\mathbf{grad} u\|_{0,-1,\omega}^2. \quad (32)$$

Proof: Cf. [3], Lemma 4.9. ■

Proposition 3.9 *The range of the operator \mathbf{R} from $H_-^1(\omega)$ is $H_{-1}^1(\omega)$, and the norms $\|u\|_{1,-,\omega}$ and $\|ru\|_{1,-1,\omega}$ are equivalent. Hence, for any axisymmetric and azimuthal vector field \mathbf{u} , the following properties are equivalent*

$$\mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}; \Omega) \iff \mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}, \text{div}; \Omega) \iff \mathbf{u} \in \check{\mathbf{H}}^1(\Omega),$$

and the canonical norms of these spaces are equivalent.

Proof: The first assertion stems from the previous Lemma, plus a few straightforward calculations. Now let $\mathbf{u} = u \mathbf{e}_\theta$ be an axisymmetric azimuthal field; $\text{div} \mathbf{u} = r^{-1} \partial_\theta u = 0$, hence $\mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}, \text{div}; \Omega)$ iff $\mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}; \Omega)$ and the canonical norms are equal.

Setting $v = ru$, Proposition 3.7 states that $\mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}; \Omega) \iff v \in H_{-1}^1(\omega)$; and Proposition 3.5 that $\mathbf{u} \in \check{\mathbf{H}}^1(\Omega) \iff u \in H_-^1(\omega)$, with equivalent norms in both cases. Hence $\mathbf{u} \in \check{\mathbf{H}}(\mathbf{curl}; \Omega) \iff \mathbf{u} \in \check{\mathbf{H}}^1(\Omega)$, with equivalent norms. ■

Proposition 3.10 *One has:*

$$\mathbf{R} [H_-^s(\omega)] = H_{-1}^s(\omega) \text{ for } 0 \leq s \leq 1, \quad \mathbf{R} [H_-^s(\omega)] \subset H_{-1}^s(\omega) \text{ for } 1 < s < 2.$$

The canonical norm in $\mathbf{R} [H_-^s(\omega)]$ is $\|w\|_{\mathbf{R} [H_-^s(\omega)]} = \|w/r\|_{s,-,\omega}$.

Proof: We already know $\mathbf{R} [H_-^0(\omega)] = \mathbf{R} [L_1^2(\omega)] = L_{-1}^2(\omega)$ and $\mathbf{R} [H_-^1(\omega)] = H_{-1}^1(\omega)$. The trace of $[\check{\mathbf{H}}^0(\Omega), \check{\mathbf{H}}^1(\Omega)]_s = \check{\mathbf{H}}^s(\Omega)$ is: $[H_-^0(\omega), H_-^1(\omega)]_s = H_-^s(\omega)$. On the other hand, the spaces $H_{-1}^s(\omega)$ are also defined by interpolation, and given our norm for $\mathbf{R} [H_-^s(\omega)]$, we have $\mathbf{R} [H_-^s(\omega)] = H_{-1}^s(\omega)$ for $0 \leq s \leq 1$. Moreover, $\mathbf{R} [H_+^s(\omega)] = H_{-1}^s(\omega)$ for $0 \leq s < 1$ since, for such an s , $H_-^s(\omega)$ and $H_+^s(\omega)$ are both equal to $H_1^s(\omega)$.

Now let $1 < s < 2$ and \mathbf{W} the field $\mathbf{W} = (w/r) \mathbf{e}_\theta$. We easily check that $\mathbf{W} \in \check{\mathbf{H}}^s(\Omega) \iff w \in \mathbf{R} [H_-^s(\omega)]$ and $\mathbf{curl} \mathbf{W} \in \check{\mathbf{H}}^{s-1}(\Omega) \iff (\partial_r w, \partial_z w) \in \mathbf{R} [H_1^{s-1}(\omega)]^2 = H_{-1}^{s-1}(\omega)^2$. Hence, $w \in \mathbf{R} [H_-^s(\omega)]$ implies $w \in H_{-1}^s(\omega)$. ■

Finally, the following criterion will be useful to decide whether some concrete function belongs to a given Sobolev space or not.

Proposition 3.11 *Let A be a point in the meridian half-plane; (ρ, ϕ) local polar coordinates centred at A , and ω_A the bounded angular sector $\{(\rho, \phi) : 0 < \rho < \rho_0, 0 \leq \phi < \phi_0\}$. Let f be a function whose expression in ω_A is $f(\rho, \phi) = \rho^\alpha g(\phi)$, with $g(\phi) \in C^\infty([0, \phi_0])$.*

1. *If A is in the open half-plane ($r(A) > 0$), and ω_A is small enough to ensure $\overline{\omega_A} \cap \gamma_a = \emptyset$, then: $\forall s \in \mathbb{R}^+$, $H^s(\omega_A) = H_{\pm 1}^s(\omega_A)$ and $\forall s \in \mathbb{R}$, $f \in H^s(\omega_A) \iff s < \alpha + 1$.*
2. *If A stands on the axis ($r(A) = 0$), and the axis is taken as the origin of ϕ , then:*

$$\forall s \in \mathbb{R}^+, \quad \begin{cases} f \in H_1^s(\omega_A) & \iff s < \alpha + 3/2, \\ f \in H_{-1}^s(\omega_A) & \iff s < \alpha + 1/2 \text{ and } g^{(j)}(0) = 0 \text{ for all } j \in \mathbb{N}, j \leq s. \end{cases}$$

3.3 Variational spaces for the modified Laplacians.

The variational spaces for Δ^+ with Dirichlet, resp. Neumann boundary conditions are:

$$\mathbb{V}^{d+} = \overset{\diamond}{H}_1^1(\omega) = \{v \in H_1^1(\omega) : v = 0 \text{ on } \gamma_b\} \text{ resp. } \mathbb{V}^{n+} = H_1^1(\omega).$$

We shall denote $\mathbb{V}^{d/n+}$ or simply \mathbb{V}^+ any of these spaces. It stems from the Curie principle and the results of Section 3.1 that:

Proposition 3.12 *Let $f \in [\mathbb{V}^+]'$ and $u \in \mathbb{V}^+$ as well as $\lambda \geq 0$. The equality $-\Delta^+u + \lambda u = f$ in the sense of distributions, supplemented in the Neumann case with the boundary condition $\partial_\nu u|_{\gamma_b} = 0 \in H_1^{1/2}(\gamma_b)$, is equivalent to the variational formulation:*

$$\forall v \in \mathbb{V}^+, \quad \iint_{\omega} [\mathbf{grad} u \cdot \mathbf{grad} v + \lambda u v] r d\omega = \langle f, v \rangle_{[\mathbb{V}^+]', \mathbb{V}^+} \quad (33)$$

Remark 3.1 The space $[\mathbb{V}^+]'$ is not very easy to describe in 2D. The only types of $f \in [\mathbb{V}^+]'$ that we shall need in the following are: (i) the elements of $L_1^2(\omega)$, for which there holds: $\langle f, v \rangle_{[\mathbb{V}^+]', \mathbb{V}^+} = \iint_{\omega} f v r d\omega$; (ii) the elements of the dual of $H_0^1(\omega)$ or $H^1(\omega)$ whose support is away from the axis. In both cases, u seen as a function in Ω is of H^2 regularity near any segment included in γ_a , hence it satisfies $\partial_r u|_{\gamma_a} = 0$ (Proposition 3.6). \blacksquare

As a straightforward consequence, we have the following Green's formulae for Δ^+ :

Proposition 3.13 *Let $\Phi^{n+} = \{u \in \mathbb{V}^{n+} : \Delta^+u \in L_1^2(\omega) \text{ and } \partial_\nu u|_{\gamma_b} = 0\}$ and $\Phi^{d+} = \{u \in \mathbb{V}^{d+} : \Delta^+u \in L_1^2(\omega)\}$. One has:*

$$\forall u \in \mathbb{V}^{d/n+}, \forall v \in \Phi^{d/n+}, \quad \iint_{\omega} \{\Delta^+u v + \mathbf{grad} u \cdot \mathbf{grad} v\} r d\omega = 0, \quad (34)$$

$$\forall u, v \in \Phi^{d/n+}, \quad \iint_{\omega} \{\Delta^+u v - u \Delta^+v\} r d\omega = 0. \quad (35)$$

Similarly, the variational spaces associated to Δ^- are

$$\mathbb{V}^{d-} = \overset{\circ}{H}_{-1}^1(\omega) = \{v \in H_{-1}^1(\omega) : v = 0 \text{ on } \gamma\} \text{ and } \mathbb{V}^{n-} = H_{-1}^1(\omega).$$

We shall denote either space by $\mathbb{V}^{d/n-}$ or simply \mathbb{V}^- . Besides the difficulty of the description of the space $[\mathbb{V}^-]'$ in 2D, another technical point is that the product by r or $1/r$ of a distribution is not defined in general. Yet these operations are underlying in the use of Δ^- , cf. (28). Fortunately, all we need in the sequel are the three particular cases dealt with in the following Proposition.

Proposition 3.14 *Let $f \in [\mathbb{V}^-]'$ and $u \in \mathbb{V}^-$ as well as $\lambda \geq 0$. The equality $-\Delta^-u + \lambda u = f$ in the sense of distributions, supplemented in the Neumann case with the boundary condition $\partial_\nu u|_{\gamma_b} = 0 \in H_{-1}^{1/2}(\gamma_b)$, is equivalent to the variational formulation:*

$$\forall v \in \mathbb{V}^-, \quad \iint_{\omega} [\mathbf{grad} u \cdot \mathbf{grad} v + \lambda u v] \frac{d\omega}{r} = \langle f, v \rangle_{[\mathbb{V}^-]', \mathbb{V}^-} \quad (36)$$

in the three following cases:

1. f is an element of $L^2_{-1}(\omega)$, i.e. $\langle f, v \rangle_{[V_-]', V_-} = \iint_{\omega} f v \frac{d\omega}{r}$;
2. more generally, $f \in L^p_{1-p}(\omega) = \mathbf{R}[L^p_1(\omega)] \simeq \mathbf{R}[\check{L}^p(\Omega)]$, for $p \geq 6/5$;
3. f is an element of the dual of $H^1(\omega)$ or $H^1_0(\omega)$ whose support is away from the axis.

Remark 3.2 Because $H^1_{-1}(\omega) \subset H^1_-(\omega)$, u always satisfies $u|_{\gamma_a} = 0$ (Proposition 3.6). ■

Remark 3.3 The limiting value $6/5$ for the Lebesgue exponent p stems from the Sobolev imbedding in the 3D domain Ω : $H^1(\Omega) \subset L^6(\Omega)$, and by duality $L^{6/5}(\Omega) \subset H^1(\Omega)'$. ■

Green's formulae for Δ^- are a straightforward consequence:

Proposition 3.15 Let $\Phi^{n-} = \left\{ u \in V^{n-} : \Delta^- u \in L^2_{-1}(\omega) \text{ and } \partial_\nu u|_{\gamma_b} = 0 \right\}$ and $\Phi^{d-} = \left\{ u \in V^{d-} : \Delta^- u \in L^2_{-1}(\omega) \right\}$. One has:

$$\forall u \in V^{d/n-}, \forall v \in \Phi^{d/n-}, \quad \iint_{\omega} \left\{ \Delta^- u v + \mathbf{grad} u \cdot \mathbf{grad} v \right\} \frac{d\omega}{r} = 0, \quad (37)$$

$$\forall u, v \in \Phi^{d/n-}, \quad \iint_{\omega} \left\{ \Delta^- u v - u \Delta^- v \right\} \frac{d\omega}{r} = 0. \quad (38)$$

4 Principle of space decompositions.

We refer to [3] for the details and omitted proofs.

Definition 4.1 We denote by \mathcal{X}_R^d , \mathcal{X}_R^n the regular subspaces of \mathcal{X}^d and \mathcal{X}^n , i.e. their intersection with $\mathbf{H}^1(\Omega)$; Φ_R^d the regular subspace of Φ^d , i.e. $H^2(\Omega) \cap H^1_0(\Omega)$.

The axisymmetric subspaces of those spaces are denoted, as usually, by $\check{\mathcal{X}}_R^d, \dots, \check{\Phi}_R^d$.

Theorem 4.2 The spaces Φ_R^d and \mathcal{X}_R^d are closed within Φ^d and \mathcal{X} respectively, unless the aperture angle at one conical vertex has the exceptional value π/β_* , characterised by the presence of the eigenvalue $3/4$ in the spectrum of the local Laplace operator.

The same holds for the axisymmetric subspaces.

The spaces $\check{\mathcal{X}}_R^n$ and $\check{\mathcal{X}}_R^{0n}$ are closed within $\check{\mathcal{X}}^n$ and $\check{\mathcal{X}}^{0n}$ respectively, for all configurations.

The main steps for obtaining these closedness results are indicated in [7]. In the following, we shall assume that none of the conical vertices has an aperture angle equal to π/β_* .

Definition 4.3 The singular subspaces, denoted by $\check{\mathcal{X}}_S^{0n}$, Φ_S^d , etc. are the orthogonal complements of the regular ones. (Later on, we shall also use non-orthogonal complements.)

The consequence of these results is that the electromagnetic field can be split into regular and singular parts: $\mathcal{U} = \mathcal{U}_R + \mathcal{U}_S$, with $\mathcal{U}_R \in \check{\mathbf{H}}^1(\Omega)$ and \mathcal{U}_S in the suitable complement; in the time-dependent case, this decomposition is continuous with respect to time, i.e.

Proposition 4.4 Assume that $(\mathcal{E}, \mathcal{B})$ belongs to $\mathcal{C}^0(0, T; \check{\mathcal{X}}^d \times \check{\mathcal{X}}^{0n})$: one can write

$$\mathcal{E}(t) = \mathcal{E}_R(t) + \mathcal{E}_S(t), \quad (\mathcal{E}_R, \mathcal{E}_S) \in \mathcal{C}^0(0, T; \check{\mathcal{X}}_R^d \times \check{\mathcal{X}}_S^d), \quad (39)$$

$$\mathcal{B}(t) = \mathcal{B}_R(t) + \mathcal{B}_S(t), \quad (\mathcal{B}_R, \mathcal{B}_S) \in \mathcal{C}^0(0, T; \check{\mathcal{X}}_R^{0n} \times \check{\mathcal{X}}_S^{0n}). \quad (40)$$

The tools for characterising the singular spaces are “very weak” integration by parts formulae. These results parallel those of [4], but cannot be considered as a mere application of them, since the domain Ω is *not* a polyhedron (nor even a “curved polyhedron”). The strategy of proof has thus to be adapted, and specific treatments have to be designed to handle the conical vertices. We shall indicate the main steps of this derivation.

Definition 4.5 *On any face Γ_i , $1 \leq i \leq n+1$, let ρ_i be the distance to its boundary, and define*

$$H_{00}^{1/2}(\Gamma_i) = \left\{ f \in H^{1/2}(\Gamma_i) : \frac{f}{\sqrt{\rho_i}} \in L^2(\Gamma_i) \right\}, \text{ and } \check{H}(\Gamma_i) = H_{00}^{1/2}(\Gamma_i) \cap \check{\mathcal{D}}'(\Gamma).$$

Let γ_1^i be the trace mapping of the normal derivative on Γ_i .

Lemma 4.6 *The application γ_1^i is continuous from $\check{\Phi}_R^d$ to $\check{H}(\Gamma_i)$.*

Moreover, it is surjective from $G_i = \{u \in \check{\Phi}_R^d : \gamma_1^j u = 0, \forall j \neq i\}$ onto $\check{H}(\Gamma_i)$, and there exists a continuous lifting operator from $\check{H}(\Gamma_i)$ into G_i .

As a consequence, γ_1^i is surjective from $\check{\Phi}_R^d$ onto $\check{H}(\Gamma_i)$. This result permits to prove an integration by parts formula, between elements of $\check{\Phi}_R^d$ and elements of $D(\Delta, \Omega) = \{g \in L^2(\Omega) : \Delta g \in L^2(\Omega)\}$. One has the

Lemma 4.7 *Let $p \in D(\Delta, \Omega)$ and $u \in \check{\Phi}_R^d$. There holds:*

$$\iiint_{\Omega} (\Delta u p - u \Delta p) d\Omega = \sum_{i=1}^n \left\langle p, \frac{\partial u}{\partial \nu} \right\rangle_{\check{H}(\Gamma_i)', \check{H}(\Gamma_i)}. \quad (41)$$

Theorem 4.8 *There holds: $\mathcal{X} = \mathcal{X}_R^d \oplus \mathbf{grad} \Phi_S^d$, $\check{\mathcal{X}} = \check{\mathcal{X}}_R^d \oplus \mathbf{grad} \check{\Phi}_S^d$.*

Moreover, $\varphi \in \check{\Phi}_S^d$ iff $p = \Delta \varphi$ is a solution in $L_1^2(\omega)$ to the “very weak” Dirichlet problem

$$\Delta^+ p = 0 \text{ in } \omega, \quad (42)$$

$$p|_{\gamma_i} = 0, \quad 1 \leq i \leq n+1, \quad (43)$$

$$p \in C^\infty(\bar{\omega} \setminus \mathcal{V}_b), \text{ for any neighbourhood } \mathcal{V}_b \text{ of } \gamma_b. \quad (44)$$

(The trace on γ_i is understood in the suitable trace space of $\check{H}(\Gamma_i)$ '.)

The first assertion was proven in [7, 3]. The second easily follows from (41).

Remark 4.1 Since p is smooth up to any segment included in γ_a , one infers that $\partial_r p|_{\gamma_a} = 0$. This additional boundary condition is used in the actual computation of p . ■

In the magnetic case, let us introduce the space of regular vector potentials

$$\check{\mathcal{M}}_R^{n/d} = \{\mathcal{A} \in \check{\mathcal{X}}^{n/d} : \operatorname{div} \mathcal{A} = 0 \text{ and } \mathbf{curl} \mathcal{A} \in \check{\mathcal{X}}^{d/n}\}$$

of elements of $\check{\mathcal{X}}_R^{0d/n}$, and $\check{\mathcal{M}}_{\theta R}^{n/d} = \{\mathcal{A} \in \check{\mathcal{M}}_R^{n/d} : \mathcal{A} \parallel \mathbf{e}_\theta\}$; for all $\mathcal{A} \in \check{\mathcal{M}}_{\theta R}^{n/d}$, both \mathbf{A}_m and \mathbf{A}_θ belong to $\check{\mathcal{M}}_R^{n/d}$. Let Γ_i be a given face, and $\gamma_{1\theta}^i$ be defined as $\gamma_{1\theta}^i \mathbf{u} = \gamma_1^i u_\theta$.

Lemma 4.9 *$\gamma_{1\theta}^i$ is continuous from $\check{\mathcal{M}}_{\theta R}^d$ to $\check{H}(\Gamma_i)$. Moreover, it is surjective from $\mathbf{G}_i = \{\mathbf{u} \in \check{\mathbf{H}}^2(\Omega) \cap \check{\mathbf{H}}_0^1(\Omega) : \mathbf{u} \parallel \mathbf{e}_\theta, \gamma_{1\theta}^j \mathbf{u} = 0, \forall j \neq i\}$ onto $\check{H}(\Gamma_i)$, and there exists a continuous lifting operator from $\check{H}(\Gamma_i)$ into \mathbf{G}_i .*

As a consequence, $\gamma_{1\theta}^i$ is surjective from $\check{\mathcal{M}}_{\theta R}^d$ onto $\check{H}(\Gamma_i)$. There follows the

Lemma 4.10 Let $\mathbf{P} \in D(\Delta, \Omega)^3$ and $\mathbf{A} \in \check{\mathcal{M}}_{\theta R}^d$. There holds

$$\iint\int_{\Omega} (\mathbf{P} \cdot \Delta \mathbf{A} - \mathbf{A} \cdot \Delta \mathbf{P}) d\Omega = \sum_{i=1}^{n+1} \langle P_{\theta}, \gamma_{1\theta}^i \mathbf{A} \rangle_{\check{H}(\Gamma_i)', \check{H}(\Gamma_i)}. \quad (45)$$

This formula has two interesting consequences:

Theorem 4.11 $\mathcal{B} \in \check{\mathcal{X}}_S^{0n}$ iff $\mathbf{curl} \mathcal{B} = \mathbf{P}$, where $\mathbf{P} \parallel \mathbf{e}_{\theta}$ is a solution in $\check{\mathbf{L}}^2(\Omega)$ to the “very weak” Dirichlet problem

$$\Delta \mathbf{P} = 0 \text{ in } \Omega, \quad P_{\theta} = 0 \text{ in the sense of } \check{H}(\Gamma_i)', \quad 1 \leq i \leq n+1. \quad (46)$$

Setting $\mathbf{P} = (p/r) \mathbf{e}_{\theta}$, p is a solution in $L_{-1}^2(\omega)$ to the “very weak” Dirichlet problem

$$\Delta^- p = 0 \text{ in } \omega, \quad (47)$$

$$p|_{\gamma_i} = 0, \quad 1 \leq i \leq n+1, \quad (48)$$

$$\frac{p}{r} \in C^{\infty}(\bar{\omega} \setminus \mathcal{V}_b), \text{ for any neighbourhood } \mathcal{V}_b \text{ of } \gamma_b. \quad (49)$$

(The trace on γ_i is understood in the suitable trace space of $\check{H}(\Gamma_i)'$.)

Remark 4.2 The smoothness of p/r up to any segment included in γ_a yields $p|_{\gamma_a} = 0$. ■

Proposition 4.12 $\check{\mathcal{M}}_{\theta R}^d = \mathbf{V}_{\theta}^{2,d} \stackrel{def}{=} \left\{ \mathbf{A} \in \check{\mathbf{H}}^2(\Omega) \cap \check{\mathbf{H}}_0^1(\Omega) : \mathbf{A} \parallel \mathbf{e}_{\theta} \right\}$, algebraically and topologically.

Proof: Obviously, $\mathbf{V}_{\theta}^{2,d} \subset \check{\mathcal{M}}_{\theta R}^d$. Moreover: by the isomorphisms of Proposition 2.1, $\Delta \check{\mathcal{M}}_{\theta R}^d$ is a closed subspace of $\check{\mathbf{L}}^2(\Omega)$; and so is $\Delta \mathbf{V}_{\theta}^{2,d}$ by the well-known properties of the scalar Laplacian. Thanks to Lemmas 4.7 and 4.10, it is not difficult to show that $[\Delta \mathbf{V}_{\theta}^{2,d}]^{\perp} \subset [\Delta \check{\mathcal{M}}_{\theta R}^d]^{\perp}$. The conclusion follows. ■

The above characterisations of the singular spaces allow to precise their dimensions.

Theorem 4.13 The spaces $\check{\mathcal{X}}_S^d$ and $\check{\mathcal{X}}_S^{0n}$ are of finite dimension. The dimension of $\check{\mathcal{X}}_S^{0n}$ is equal to the number of reentrant edges. The dimension of $\check{\mathcal{X}}_S^d$ is equal to the number of reentrant edges, plus the number of conical vertices with aperture greater than π/β_{\star} .

This will be proven in Section 5 thanks to the analysis of the above very weak problems with modified Laplacians.

In the analysis of time-dependent problems, we shall also need statements similar to Lemma 4.10 and Proposition 4.12, with a Neumann boundary condition.

Definition 4.14 Let Γ be a circular cone, Ω its interior, and Γ_i a regular open subset of Γ . Define $H^{3/2}(\Gamma)$ as the trace space on Γ of functions in $H^2(\Omega)$; $H_{00}^{3/2}(\Gamma_i)$ as the space of functions such that their extension by 0 to the whole of Γ belongs to $H^{3/2}(\Gamma)$. $\check{H}(\Gamma_i)$ is the axisymmetric subspace of $H_{00}^{3/2}(\Gamma_i)$.

Lemma 4.15 For any face Γ_i , the trace mapping $\gamma_{0\theta}^i$ on Γ_i is surjective from

$$F_i = \left\{ u \in \check{H}^2(\Omega) : \partial_{\nu}(ru)|_{\Gamma} = 0 \text{ and } u|_{\Gamma_j} = 0, \forall j \neq i \right\},$$

onto $\check{H}(\Gamma_i)$; the mapping $\gamma_{0\theta}^i \mathbf{u} = \gamma_{0\theta}^i u_{\theta}$ is surjective from

$$\mathbf{F}_i = \left\{ \mathbf{u} \in \check{\mathbf{H}}^2(\Omega) : \mathbf{u} \parallel \mathbf{e}_{\theta} \text{ and } \partial_{\nu}(ru_{\theta})|_{\Gamma} = 0 \text{ and } u_{\theta}|_{\Gamma_j} = 0, \forall j \neq i \right\}$$

onto $\check{H}(\Gamma_i)$; and there exist continuous liftings from $\check{H}(\Gamma_i)$ to F_i and \mathbf{F}_i .

Lemma 4.16 *The operator $\text{cst} : \begin{cases} \check{\mathbf{H}}^2(\Omega) & \longrightarrow & \check{H}^{1/2}(\Gamma_i) \\ \mathbf{u} & \longmapsto & (\text{curl } \mathbf{u} \times \boldsymbol{\nu}) \cdot \mathbf{e}_\theta \end{cases}$ admits an extension as a continuous linear operator from $D(\Delta, \Omega)^3$ to $\check{H}(\Gamma_i)'$. For $\mathbf{P} \in D(\Delta, \Omega)^3$ and $\mathbf{A} \in \check{\mathcal{M}}_{\theta R}^n$ such that $A_\theta|_{\Gamma_i} \in \check{H}(\Gamma_i)$ for any face Γ_i , there holds the integration by parts formula*

$$\iiint_{\Omega} (\Delta \mathbf{A} \cdot \mathbf{P} - \mathbf{A} \cdot \Delta \mathbf{P}) \, d\Omega = - \sum_{i=1}^n \langle \text{cst} \mathbf{P}, A_\theta \rangle_{\check{H}(\Gamma_i)', \check{H}(\Gamma_i)}. \quad (50)$$

Proposition 4.17 $\check{\mathcal{M}}_{\theta R}^n = \mathbf{V}_\theta^{2,n} \stackrel{\text{def}}{=} \left\{ \mathbf{A} \in \check{\mathbf{H}}^2(\Omega) : \mathbf{A} \parallel \mathbf{e}_\theta \text{ and } \partial_\nu(r A_\theta)|_\Gamma = 0 \right\}$, algebraically and topologically.

5 Analysis of singularities of the modified Laplacians.

Let N^{d+} be the space of solutions to (42–44) in $L_1^2(\omega)$ and N^{d-} the space of solutions to (47–49) in $L_{-1}^2(\omega)$. Obviously, one has: $N^{d+} \cap H_1^1(\omega) = \{0\}$ and $N^{d-} \cap H_{-1}^1(\omega) = \{0\}$.

Lemma 5.1 *Any $p \in N^{d+}$, resp. $p \in N^{d-}$, belongs to $C^\infty(\bar{\omega} \setminus V)$, where V is any neighbourhood of the corners $E_1, \dots, E_n, O_1, O_2$.*

Proof: By (44), resp. (49), we only have to prove that p is C^∞ up to a neighbourhood of any open interval $]A_0 B_0[$ such that $[A_0 B_0] \subset \gamma_j$.

Let us consider a domain $D \subset \omega$ whose C^∞ boundary ∂D contains the segment $[A_0 B_0]$, and stays away from all other corners and sides of ω , including the axis γ_a . χ is a cut-off function which takes its values as shown on Figure 2. Of course, one has $\chi p \in L^2(D)$;

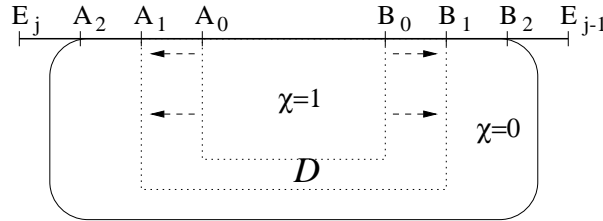


Figure 2: Notations of Lemma 5.1.

hence, since $\Delta^\pm p = 0$, there holds

$$f_p = \Delta(\chi p) = \mp \frac{\chi}{r} \frac{\partial p}{\partial r} + 2 \mathbf{grad} \chi \cdot \mathbf{grad} p + p \Delta \chi \in H^{-1}(D). \quad (51)$$

Moreover, the trace of χp vanishes smoothly on the whole of ∂D .

Now, introduce the solution $v \in H^1(D)$ to the variational problem

$$\Delta v = f_p \in H^{-1}(D), \quad v|_{\partial D} = 0 \in H^{1/2}(\partial D).$$

$v - \chi p$ belongs to $L^2(D)$, has a vanishing Laplacian in D and a vanishing trace on ∂D . Again, as D is smooth, one has $\Delta \{H^2(D) \cap H_0^1(D)\} = L^2(D)$: there are no singularities for the Laplacian in D . Hence, $v - \chi p = 0$, i.e. $\chi p \in H^1(D)$, and by (51) $f_p \in L^2(D)$. By applying a bootstrap argument, one shows that $\chi p \in H^m(D)$ for any $m \in \mathbb{N}$, i.e. $\chi p \in C^\infty(\bar{D})$ and p and C^∞ up to $[A_0 B_0]$. \blacksquare

5.1 Local study of singularities near the edges.

We look for a local analytical expression of the solution p to (42–44), resp. (47–49) in a neighbourhood of an edge, i.e. an off-axis corner $E = E_j$ with opening π/α . [We drop the corner subscript j .] In a meridian half-plane, we use local polar coordinates (ρ, ϕ) centred at E , the origin of ϕ being on the half-line $[E_{j+1}E_j]$; $a = r(E)$ is the distance between E to the (Oz) axis and ϕ_0 the angle between $[E_{j+1}E_j]$ and (Or) (see Fig. 3). The expression

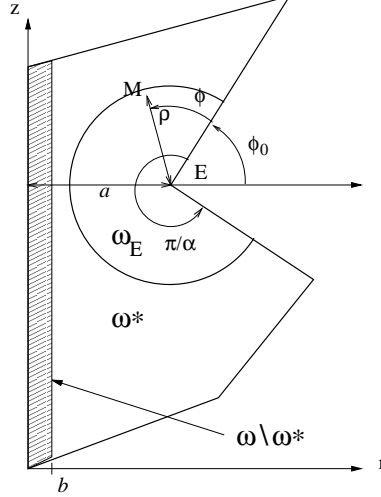


Figure 3: Edge singularity.

of the modified Laplacians in these coordinates reads, with $\phi' = \phi + \phi_0$:

$$\Delta^\pm p = \frac{\partial^2 p}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial p}{\partial \rho} \pm \frac{1}{a + \rho \cos \phi'} \left(\cos \phi' \frac{\partial p}{\partial \rho} - \frac{\sin \phi'}{\rho} \frac{\partial p}{\partial \phi} \right) + \frac{1}{\rho^2} \frac{\partial^2 p}{\partial \phi^2}. \quad (52)$$

We settle in a neighbourhood ω_E of E such that $\bar{\omega}_E$ is away from all corners except E and all sides except the ones which meet at E . $\eta \in C^\infty(\bar{\omega})$ is a cut-off function with the following properties: (i) $\eta \equiv 1$ in ω_E ; (ii) $\eta \equiv 0$ outside some neighbourhood $\omega'_E \supset \bar{\omega}_E$ which satisfies the same conditions as ω_E ; (iii) η depends on ρ only.

One has: $L_1^2(\omega_E) = L_{-1}^2(\omega_E) = L^2(\omega_E)$: there is locally no difference between functions in $L_1^2(\omega)$, $L_{-1}^2(\omega)$ or $L^2(\omega)$. Moreover, the modified Laplacians are locally equal to perturbations of the standard Laplacian by less singular terms. Thus, the singularities of Δ^\pm in $L_{\pm 1}^2(\omega)$ are “close” to the singularities of Δ in $L^2(\omega)$, which are well-known [14].

Lemma 5.2 *Let $p \in N^{d+}$, resp. $p \in N^{d-}$. There exist $c \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, $\ell \alpha > -1$, such that $\eta \left(p - c \rho^{\ell \alpha} \sin(\ell \alpha \phi) \right) \in H^1(\omega_E)$.*

Proof: Let ω^* be the polygonal domain $\{\mathbf{x} \in \omega : r(\mathbf{x}) > b\}$, and γ^* its boundary (see Figure 3). One has:

$$\Delta(\eta p) = \mp \frac{\eta}{r} \frac{\partial p}{\partial r} + 2 \mathbf{grad} \eta \cdot \mathbf{grad} p + p \Delta \eta \in H^{-1}(\omega^*).$$

Now let $v \in H^1(\omega^*)$ be the variational solution of the Dirichlet problem

$$\Delta v = \Delta(\eta p) \text{ in } \omega^*, \quad v = 0 \text{ on } \gamma^*,$$

and $p^* \stackrel{\text{def}}{=} \eta p - v$. p^* belongs to $L^2(\omega^*)$, has a vanishing Laplacian in ω^* and a vanishing trace on γ^* . Grisvard [14, pp. 45–56] has established that such a singularity satisfies

$$\eta \left(p^* - c \rho^{\ell \alpha} \sin(\ell \alpha \phi) \right) \in H^1(\omega^*).$$

for some c, ℓ such that $\ell \alpha > -1$. Hence $\eta \left(p - c \rho^{\ell \alpha} \sin(\ell \alpha \phi) \right) \in H^1(\omega_E)$. \blacksquare

By Proposition 3.11, the condition $\ell \alpha > -1$ is needed for the term $\rho^{\ell \alpha} \sin(\ell \alpha \phi)$ to be in $L^2(\omega_E)$. If the corner is outgoing ($\alpha > 1$), this implies $\ell \geq 0$ and the latter expression is indeed in $H^1(\omega_E)$. Hence any element of $N^{d\pm}$ is locally H^1 . For a reentrant corner ($1/2 < \alpha < 1$), however, the term $\rho^{\ell \alpha} \sin(\ell \alpha \phi)$ is locally L^2 but not H^1 for $\ell = -1$, and locally H^1 for $\ell \geq 0$. As a consequence, there exists a unique (up to a multiplication by a constant) *local* singular function, as shown by the

Lemma 5.3 *If the corner E is reentrant, there exists $\sigma^\pm \in N^{d\pm}$ such that*

$$\sigma^\pm(\rho, \phi) - \eta \rho^{-\alpha} \sin \alpha \phi \in H_{\pm 1}^1(\omega).$$

Proof: Let $u(\rho, \phi) = \eta \rho^{-\alpha} \sin \alpha \phi$; this function vanishes on γ_b and γ_a , and so does its normal derivative on γ_a . In ω_E , $\eta \equiv 1$ and by (52),

$$f_\pm = \Delta^\pm u = \mp \frac{\alpha \rho^{-\alpha-1}}{a + \rho \cos \phi'} \left(\cos \phi' \sin \alpha \phi + \sin \phi' \cos \alpha \phi \right).$$

As $-\alpha - 1 > -2$, $f_\pm \in H^{-1}(\omega_E)$; elsewhere it is C^∞ and vanishes near the axis. Hence, by Propositions 3.12 and 3.14, one can solve variationally the Dirichlet problems

$$\Delta^\pm w^\pm = f_\pm \text{ in } \omega, \quad w^\pm = 0 \text{ on } \gamma_b,$$

in $\mathring{H}_{-1}^1(\omega)$ or $\hat{H}_1^1(\omega)$. The difference $\sigma^\pm = u - w^\pm \in L_{\pm 1}^2(\omega)$ has a vanishing modified Laplacian Δ^\pm and a vanishing trace on γ_b ; on γ_a , σ^\pm satisfies the same boundary condition as w^\pm . So, $\sigma^\pm \in N^{d\pm}$. \blacksquare

5.2 Local study of singularities near the conical vertices.

Similarly to the previous Subsection, we look for an analytical expression of p near a conical vertex O . [Here, too, we drop the vertex subscript.] The coordinates used in a meridian half-plane are the polar coordinates (ρ, ϕ) centred at O , with the origin of ϕ on the (Oz) axis (see Figure 4). In 3D, (ρ, θ, ϕ) are a non-standard system of spherical coordinates. The expression of the modified Laplacians in these variables is:

$$\Delta^+ p = \frac{\partial^2 p}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial p}{\partial \rho} + \frac{\cot \phi}{\rho^2} \frac{\partial p}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 p}{\partial \phi^2}, \quad (53)$$

$$\Delta^- p = \frac{\partial^2 p}{\partial \rho^2} - \frac{\cot \phi}{\rho^2} \frac{\partial p}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 p}{\partial \phi^2}. \quad (54)$$

Unlike the previous situation, the complementary terms (with respect to the standard Laplacian) are not less singular. However, the whole of Δ^\pm enjoys another nice feature: variable separation. To take advantage of it, let us introduce the angular parts:

$$(\Lambda^\pm u)(\phi) = -u''(\phi) \mp \cot \phi u'(\phi),$$

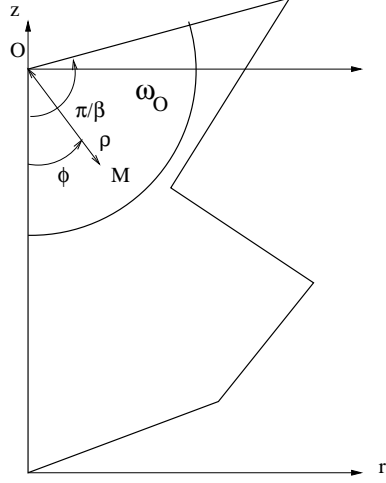


Figure 4: Conical singularity.

and the Hilbert spaces:

$$\mathcal{H}_+ = L^2 \left(\left[0, \frac{\pi}{\beta} \right], \sin \phi \, d\phi \right), \quad \mathcal{H}_- = L^2 \left(\left[0, \frac{\pi}{\beta} \right], \frac{d\phi}{\sin \phi} \right).$$

The boundary conditions in the definition of $N^{d\pm}$ suggest to consider the domains:

$$D(\Lambda^+) = \{u \in \mathcal{H}_+ : \Lambda^+ u \in \mathcal{H}_+ \text{ and } u'(0) = u(\pi/\beta) = 0\}, \quad (55)$$

$$D(\Lambda^-) = \{u \in \mathcal{H}_- : \Lambda^- u \in \mathcal{H}_- \text{ and } u(0) = u(\pi/\beta) = 0\}. \quad (56)$$

We notice that Λ^\pm defines an unbounded operator in \mathcal{H}_\pm , which is self-adjoint in the above domain, strictly positive and has a compact inverse; hence

Theorem 5.4 *There exists a Hilbert basis $(u_\ell^\pm)_{\ell \in \mathbb{N}^*}$ of \mathcal{H}_\pm made of eigenfunctions of Λ^\pm . The corresponding eigenvalues $(\lambda_\ell^\pm)_{\ell \in \mathbb{N}^*}$ are strictly positive and go to infinity. Moreover, $\left((u_\ell^\pm)' / \sqrt{\lambda_\ell^\pm} \right)_{\ell \in \mathbb{N}^*}$ is an orthonormal family in \mathcal{H}_\pm .*

Let us determine these eigenfunctions and -values. The eigenpairs (λ, u) of Λ^+ satisfy:

$$\begin{cases} u''(\phi) + \cot \phi \, u'(\phi) + \nu(\nu+1)u(\phi) = 0 \\ u'(0) = u(\pi/\beta) = 0 \end{cases} \quad (57)$$

where the eigenvalue $\lambda = \nu(\nu+1)$ with $\nu > 0$. Setting $x = \cos \phi$ and $u(\phi) = w(\cos \phi) = w(x)$ yields:

$$\begin{cases} (1-x^2)w''(x) - 2xw'(x) + \nu(\nu+1)w(x) = 0 \\ w(\cos(\pi/\beta)) = w'(1) = 0 \end{cases}$$

which is the Legendre equation with indices 0 and ν , so the general solution of (57) is $u(\phi) = aP_\nu^0(\cos \phi) + bQ_\nu^0(\cos \phi)$. The function $Q_\nu^0(x)$ is infinite for $x = 1$, i.e. $\phi = 0$; on the other hand, p is C^∞ on the axis, so $u(\phi)$ is C^∞ at 0. Hence, the boundary conditions are satisfied iff $b = 0$ and $P_\nu^0(\cos \frac{\pi}{\beta}) = 0$.

Let us number $(\nu_\ell^+)_{\ell \in \mathbb{N}^*}$ the increasing sequence of $\nu > 0$ satisfying this condition; $\lambda_\ell^+ = \nu_\ell^+(\nu_\ell^+ + 1)$ the corresponding eigenvalues of Λ^+ , and $u_\ell^+ = C_\ell^+ P_{\nu_\ell^+}^0(\cos \phi)$ the corresponding eigenfunctions of Λ^+ , where C_ℓ^+ is a normalisation factor.

Similarly, let $\lambda = \nu(\nu - 1)$, with $\nu > 1$, be some eigenvalue of Λ^- ; the corresponding eigenfunction u is solution of

$$\begin{cases} u''(\phi) - \cot \phi u'(\phi) + \nu(\nu - 1)u(\phi) = 0 \\ u(0) = u(\pi/\beta) = 0 \end{cases} \quad (58)$$

Setting $x = \cos \phi$ and $u(\phi) = \sin \phi w(\cos \phi) = (1 - x^2)^{1/2} w(x)$, we get:

$$\begin{cases} (1 - x^2) w''(x) - 2xw'(x) + (\nu(\nu - 1) - (1 - x^2)^{-1}) w(x) = 0 \\ w(\cos(\pi/\beta)) = w(1) = 0 \end{cases}$$

which is the Legendre equation with indices $\mu = 1$ and $\nu - 1$, so the general solution of (58) is $u(\phi) = \sin \phi (a P_{\nu-1}^1(\cos \phi) + b Q_{\nu-1}^1(\cos \phi))$. Using the expansion [1, Eq. 8.7.1 and 8.7.2] of $P(\cos \phi)$ and $Q(\cos \phi)$ for $\phi \rightarrow 0$, we get

$$\lim_{\phi \rightarrow 0} u(\phi) = \frac{b\nu(\nu - 1)\pi}{\Gamma(2 - \nu)\Gamma(\nu + 1)\sin \nu \pi} \text{ if } \nu \notin \mathbb{N}, \text{ and } -b \text{ if } \nu \in \mathbb{N}, \nu \geq 2.$$

Hence, the boundary conditions are satisfied iff $b = 0$ and $P_{\nu-1}^1(\cos \frac{\pi}{\beta}) = 0$.

Let us number $(\nu_\ell^-)_{\ell \in \mathbb{N}^*}$ the increasing sequence of $\nu > 1$ satisfying this condition; $\lambda_\ell^- = \nu_\ell^- (\nu_\ell^- - 1)$ the corresponding eigenvalues of Λ^- , and $u_\ell^- = C_\ell^- \sin \phi P_{\nu_\ell^-}^1(\cos \phi)$ the corresponding normalised eigenfunctions of Λ^- .

The three following facts are worth noticing: (i) all eigenvalues of Λ^\pm are simple; (ii) as shown by tables of Legendre functions (see e.g. [1]), there are no $x \in]-1, 1[$ nor $\nu \in]1, 2]$ such that $P_{\nu-1}^1(x) = 0$; hence, $\nu_\ell^- > 2$; (iii) the ν_ℓ^\pm have an asymptotic linear behaviour, as shown by the

Lemma 5.5 *One has $\nu_\ell^\pm \sim \beta \ell$ when $\ell \rightarrow +\infty$.*

Proof: It relies on the asymptotic expansion [1, Eq. 8.6.6] of $P_\nu^\mu(\cos \phi)$ when $\nu \rightarrow +\infty$ with ϕ and $\mu \geq 0$ fixed, taking into account the following equivalence: $\Gamma(n + \alpha)/\Gamma(n) \sim n^\alpha$ when $n \rightarrow +\infty$ and α is fixed [1, Eq. 6.1.45]. \blacksquare

Let us return to the singularities of the modified Laplacians. As $p(\rho, \phi) \in N^{d^\pm}$ belongs to $L_{\pm 1}^2(\omega) \cap C^\infty(\bar{\omega} \setminus V)$, for any neighbourhood V of O , it can be viewed as a C^∞ function of $\rho \in]0, R[$ (for some $R > 0$) with values in \mathcal{H}_\pm . The equation $\Delta^+ p = 0$, resp. $\Delta^- p = 0$ then becomes:

$$\frac{\partial^2 p}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial p}{\partial \rho} - \frac{1}{\rho^2} \Lambda^+ p = 0, \quad (59)$$

$$\text{resp. } \frac{\partial^2 p}{\partial \rho^2} - \frac{1}{\rho^2} \Lambda^- p = 0. \quad (60)$$

Hence p takes its values in $D(\Lambda^\pm)$ defined by (55–56). We denote $D_R = \omega_O \cap \{0 < \rho < R\}$.

Lemma 5.6 *Let $p \in C^\infty(]0, R[; D(\Lambda^-))$ be a solution of (60), and assume $p \in L_{-1}^2(D_R) = L^2(D_R, (\rho \sin \phi)^{-1} \rho d\rho d\phi)$. There exists a sequence $(c_\ell)_{\ell \in \mathbb{N}^*}$ satisfying*

$$p(\rho, \phi) = \sum_{\ell=1}^{+\infty} c_\ell \rho^{\nu_\ell^-} u_\ell^-(\phi), \quad (61)$$

$$|c_\ell| \leq K R^{-\nu_\ell^-} \sqrt{\nu_\ell^-}, \quad (62)$$

where the constant K depends only on p .

Proof: For a fixed ρ , one expands $\phi \mapsto p(\rho, \phi)$ on the Hilbert basis $(u_\ell^-)_{\ell \in \mathbb{N}^*}$:

$$p(\rho, \phi) = \sum_{\ell=1}^{+\infty} p_\ell(\rho) u_\ell^-(\phi), \text{ where } p_\ell(\rho) = \int_0^{\pi/\beta} p(\rho, \phi) u_\ell^-(\phi) \frac{d\phi}{\sin \phi}. \quad (63)$$

Then (60) yields:

$$p_\ell''(\rho) - \nu_\ell^- (\nu_\ell^- - 1) \frac{p_\ell(\rho)}{\rho^2} = 0, \text{ hence: } p_\ell(\rho) = c_\ell \rho^{\nu_\ell^-} + d_\ell \rho^{1-\nu_\ell^-}.$$

By Proposition 3.11, the function $\rho^\sigma u_\ell^-(\phi)$ belongs to $L_{-1}^2(D_R)$ iff $\sigma > -1/2$. As $\nu_\ell^- > 2$, the term $d_\ell \rho^{1-\nu_\ell^-} u_\ell^-$ cannot be in $L_{-1}^2(D_R)$ unless $d_\ell = 0$; on the other hand, $c_\ell \rho^{\nu_\ell^-} u_\ell^- \in L_{-1}^2(D_R)$ for any ℓ .

To obtain the estimate (62), we apply the Schwarz inequality to (63):

$$p_\ell(\rho)^2 \leq \int_0^{\pi/\beta} p(\rho, \phi)^2 \frac{d\phi}{\sin \phi},$$

since the functions u_ℓ^- are normalised. Hence:

$$\int_0^R p_\ell(\rho)^2 d\rho \leq \int_0^R \int_0^{\pi/\beta} p(\rho, \phi)^2 \frac{d\rho d\phi}{\sin \phi} = \|p\|_{L_{-1}^2(D_R)}^2 = \text{cst.}$$

The integral in the left-hand side is $c_\ell^2 R^{2\nu_\ell^-+1} / (2\nu_\ell^- + 1)$, and (62) follows. \blacksquare

Lemma 5.7 *Let $p \in \mathcal{C}^\infty]0, R[; D(\Lambda^+)$ be a solution of (59), and assume $p \in L_1^2(D_R) = L^2(D_R, (\rho \sin \phi) \rho d\rho d\phi)$. There exists a sequence $(c_\ell)_{\ell \in \mathbb{N}^*}$ satisfying*

$$p(\rho, \phi) = \sum_{\ell=1}^{+\infty} c_\ell \rho^{\nu_\ell^+} u_\ell^+(\phi), \text{ if } \beta > \beta_\star, \quad (64)$$

$$p(\rho, \phi) = d_1 \rho^{-1-\nu_1^+} u_1^+(\phi) + \sum_{\ell=1}^{+\infty} c_\ell \rho^{\nu_\ell^+} u_\ell^+(\phi), \text{ if } \beta < \beta_\star, \quad (65)$$

$$|c_\ell| \leq K R^{-\nu_\ell^+} \sqrt{\nu_\ell^+}, \text{ in both cases,} \quad (66)$$

for some constant K depending only on p .

The limiting value $\beta = \beta_\star$ corresponds to the exceptional value for which $\check{\Phi}_R^d$ is not closed. It is defined by $P_{1/2}^0(\cos \pi/\beta_\star) = 0$; its value is $\beta_\star \simeq 1,3771$, or $\pi/\beta_\star \simeq 130^\circ 43'$.

Proof: Similarly to the previous proof, one writes: $p(\rho, \phi) = \sum_{\ell=1}^{+\infty} p_\ell(\rho) u_\ell^+(\phi)$. One finds

that: $p_\ell(\rho) = c_\ell \rho^{\nu_\ell^+} + d_\ell \rho^{-1-\nu_\ell^+}$.

By Proposition 3.11, the function $\rho^\sigma u_\ell^+(\phi)$ belongs to $L_1^2(D_R)$ iff $\sigma > -3/2$: the term $d_\ell \rho^{-1-\nu_\ell^+} u_\ell^+(\phi)$ is $L_1^2(D_R)$, with $d_\ell \neq 0$, iff $\nu_\ell^+ < 1/2$. Tables [1] show that

- if $\beta < \beta_\star$, $\nu_1^+ < 1/2$ and $\nu_\ell^+ > 1/2$, hence $d_\ell = 0$, for $\ell > 1$;
- if $\beta > \beta_\star$, $\nu_\ell^+ > 1/2$, hence $d_\ell = 0$, for $\ell \geq 1$;
- if $\beta = \beta_\star$, $\nu_1^+ = 1/2$, which does correspond to the eigenvalue $\lambda_1^+ = 3/4$ for Λ^+ .

Moreover, $c_\ell \rho^{\nu_\ell^+} u_\ell^+ \in L_1^2(D_R)$ for any ℓ .

The Schwarz estimate of the coefficient $p_\ell(\rho)$ yields:

$$\int_0^R p_\ell(\rho)^2 \rho^2 d\rho \leq \int_0^R \int_0^{\pi/\beta} p(\rho, \phi)^2 \rho^2 \sin \phi d\rho d\phi = \|p\|_{L_1^2(D_R)}^2 = \text{cst.}$$

For $\ell > 1$, the left-hand side is equal to $c_\ell^2 R^{2\nu_\ell^+ + 3} / (2\nu_\ell^+ + 3)$, and (66) follows. \blacksquare

We will call O a *sharp* vertex if its aperture angle is strictly greater than π/β_\star . The consequence of the previous results is that Δ^- has no singularity near O , while Δ^+ has *locally* one singular function if O is sharp, and no singularity if it is not.

Theorem 5.8 *Let $p \in C^\infty(]0, R[; D(\Lambda^-))$ be a solution of (60). If $p \in L_{-1}^2(D_R)$, then $p \in H_{-1}^1(D_{R'})$ for any $R' < R$.*

Proof: In polar coordinates, the fact that a function $f \in L_{-1}^2(D_R)$ belongs to $H_{-1}^1(D_{R'})$ is equivalent to the convergence of the integrals

$$\iint_{D_{R'}} \left(\frac{\partial f}{\partial \rho} \right)^2 \frac{d\rho d\phi}{\sin \phi} \quad \text{and} \quad \iint_{D_{R'}} \frac{1}{\rho^2} \left(\frac{\partial f}{\partial \phi} \right)^2 \frac{d\rho d\phi}{\sin \phi}.$$

Let us differentiate formally the expansion (61):

$$\frac{\partial p}{\partial \rho} = \sum_{\ell=1}^{+\infty} c_\ell \nu_\ell^- \rho^{\nu_\ell^- - 1} u_\ell^-(\phi), \quad \frac{1}{\rho} \frac{\partial p}{\partial \phi} = \sum_{\ell=1}^{+\infty} c_\ell \rho^{\nu_\ell^- - 1} \sqrt{\nu_\ell^- (\nu_\ell^- - 1)} \frac{(u_\ell^-(\phi))'}{\sqrt{\nu_\ell^- (\nu_\ell^- - 1)}}.$$

For a fixed $\rho \leq R' < R$, these expansions are performed on orthonormal families in \mathcal{H}_- , respectively $(u_\ell^-)_{\ell \in \mathbb{N}^*}$ and $\left((u_\ell^-)' / \sqrt{\lambda_\ell^-} \right)_{\ell \in \mathbb{N}^*}$. Thus, establishing their convergence in \mathcal{H}_- amounts to checking that their coefficients are in $l^2(\mathbb{N}^*)$. This follows from the estimate (62), since by Lemma 5.5, $\nu_\ell^- \geq \beta \ell / 2$ for ℓ large enough. Hence for ρ fixed, $\mathbf{grad} p \in [\mathcal{H}_-]^2$ and its norm is

$$\int_0^{\pi/\beta} |\mathbf{grad} p(\rho, \phi)|^2 \frac{d\phi}{\sin \phi} \leq 2K R \sum_{\ell=1}^{+\infty} (\nu_\ell^-)^3 \left(\frac{\rho}{R} \right)^{2\nu_\ell^- - 2} \leq K' \frac{\rho^\beta}{R^\beta - \rho^\beta}.$$

So, by Fubini's theorem,

$$\iint_{D_{R'}} |\mathbf{grad} p(\rho, \phi)|^2 \frac{d\rho d\phi}{\sin \phi} \leq \int_0^{R'} K' \frac{\rho^\beta}{R^\beta - \rho^\beta} d\rho < +\infty,$$

and $p \in H_{-1}^1(D_{R'})$. \blacksquare

Theorem 5.9 *Let $p \in C^\infty(]0, R[; D(\Lambda^+))$ be a solution of (59). If $p \in L_1^2(D_R)$, then*

- if $\beta > \beta_\star$, $p \in H_1^1(D_{R'})$ for $R' < R$.
- if $\beta < \beta_\star$, $p - d_1 \rho^{-1-\nu_1^+} u_1^+(\phi) \in H_1^1(D_{R'})$ for $R' < R$.

Proof: Similarly to above argument, the fact that a function $f \in L_1^2(D_R)$ belongs to $H_1^1(D_{R'})$ is equivalent to the convergence of the integrals

$$\iint_{D_{R'}} \left(\frac{\partial f}{\partial \rho} \right)^2 \rho^2 \sin \phi \, d\rho \, d\phi \quad \text{and} \quad \iint_{D_{R'}} \frac{1}{\rho^2} \left(\frac{\partial f}{\partial \phi} \right)^2 \rho^2 \sin \phi \, d\rho \, d\phi.$$

According to the value of β , we set $f = p$ or $f = p - d_1 \rho^{-1-\nu_1^+} u_1^+(\phi)$; and applying to f a reasoning like above proves its $H_1^1(D_R)$ regularity. \blacksquare

Lemma 5.10 *If $\beta < \beta_*$, there exists $\sigma^+ \in N^{d^+}$ such that*

$$\sigma^+(\rho, \phi) - \eta(\rho) \rho^{-1-\nu_1^+} u_1^+(\phi) \in H_1^1(\omega).$$

Proof: Let $u(\rho, \phi) = \eta \rho^{-1-\nu_1^+} u_1^+(\phi)$ and $f = \Delta^+ u$. f vanishes everywhere except in a shell which stands away from all corners, and is smooth there. Define w as the variational solution in $\mathring{H}_1^1(\omega)$ of $\Delta^+ w = f$; w satisfies $\partial_r w = 0$ on γ_a , and so does u . The difference $\sigma^+ = u - w$ satisfies $\Delta^+ \sigma^+ = 0$ and the boundary conditions; hence $\sigma^+ \in N^{d^+}$. \blacksquare

5.3 Dimensions of the singular spaces.

Definition 5.11 *Let the sets of geometrical singularities be:*

$$\begin{aligned} \mathcal{K}_E &= \{ \text{edges} \}, & \mathcal{K}_O &= \{ \text{vertices} \} \\ \mathcal{K}_{ES} &= \{ j : E_j \text{ is a reentrant edge} \} \\ \mathcal{K}_{OS} &= \{ j : O_j \text{ is a sharp vertex} \}. \end{aligned}$$

Theorem 5.12 *The spaces N^{d^+} and N^{d^-} are finite-dimensional. The dimension of N^{d^-} is equal to the number of reentrant edges. The dimension of N^{d^+} is equal to the sum of the number of reentrant edges and the number of sharp vertices.*

Proof: Let us introduce a cut-off function η_j for each singularity A_j , $j \in \mathcal{K}_E \cup \mathcal{K}_O$. (We take care that their supports are all disjoint.) For any $p \in N^{d^\pm}$, there holds by Lemma 5.1:

$$T_\eta p \stackrel{\text{def}}{=} p \left(1 - \sum_{j \in \mathcal{K}_E \cup \mathcal{K}_O} \eta_j \right) \in \mathcal{C}^\infty(\bar{\omega}).$$

If $p \in N^{d^-}$, $T_\eta p$ has a vanishing trace on γ and belongs to $H_{-1}^1(\omega)$. As a matter of fact, $T_\eta p = r T_\eta P_\theta$, where \mathbf{P} , as the solution of (46), is \mathcal{C}^∞ up to the faces of Γ , and the cut-off suppresses the edge and vertex effects. Thus, $T_\eta \mathbf{P} \in \mathring{\mathbf{D}}(\bar{\Omega}) \subset \mathring{\mathbf{H}}^1(\Omega)$, so $T_\eta P_\theta \in H_-^1(\omega)$ and $T_\eta p \in H_{-1}^1(\omega)$ by Proposition 3.9.

If $j \in \mathcal{K}_O$, $\eta_j p \in H_{-1}^1(\omega)$ by Theorem 5.8. Near an outgoing edge E_j , p is locally H^1 , hence $\eta_j p \in H_{-1}^1(\omega)$. If $j \in \mathcal{K}_{ES}$, there exists $\sigma_j^- \in N^{d^-}$ such that $w_j = \eta_j p - c_j \sigma_j^- \in H_{-1}^1(\omega)$ by Lemma 5.3. Summarising, we have:

$$p = T_\eta p + \sum_{j \in \mathcal{K}_O} \eta_j p + \sum_{j \in \mathcal{K}_E \setminus \mathcal{K}_{ES}} \eta_j p + \sum_{j \in \mathcal{K}_{ES}} \left(c_j \sigma_j^- + w_j \right) = w + \sum_{j \in \mathcal{K}_{ES}} c_j \sigma_j^-,$$

with w in $H_{-1}^1(\omega)$, as the sum of functions in $H_{-1}^1(\omega)$, and in N^{d^-} , as the difference $p - \sum c_j \sigma_j^-$ of elements of N^{d^-} . Thus $w = 0$.

So, the $(\sigma_j^-)_{j \in \mathcal{K}_{ES}}$ are a generating family in N^{d^-} ; on the other hand, they are—obviously—linearly independent. Hence the dimension of N^{d^-} .

The demonstration is very similar for $p \in N^{d+}$. Here $T_\eta p \in H_1^1(\omega)$, and

$$\begin{aligned} p &= T_\eta p + \sum_{j \in \mathcal{K}_{OS}} (c_j \sigma_j^+ + w_j) + \sum_{j \in \mathcal{K}_O \setminus \mathcal{K}_{OS}} \eta_j p + \sum_{j \in \mathcal{K}_E \setminus \mathcal{K}_{ES}} \eta_j p + \sum_{j \in \mathcal{K}_{ES}} (c_j \sigma_j^+ + w_j) \\ &= w + \sum_{j \in \mathcal{K}_{OS}} c_j \sigma_j^+ + \sum_{j \in \mathcal{K}_{ES}} c_j \sigma_j^+, \end{aligned}$$

where w is both in $H_1^1(\omega)$ and N^{d+} , hence $w = 0$. The conclusion follows once more from the obvious linear independence of the $(\sigma_j^+)_{j \in \mathcal{K}_{ES} \cup \mathcal{K}_{OS}}$. \blacksquare

6 Analysis of the static problems.

We examine the simplification of the static problems (1–3) and (4–6) induced by the axial symmetry. As noted in [3, Proposition 2.2], there is a decoupling of meridian and azimuthal components in the divergence, curl and vector Laplacian operators; as a consequence, each of the static problems is decoupled into two problems, concerning the meridian and azimuthal components of the field \mathcal{U} .

We shall also pay special attention to the divergence-free case (i.e. $G = 0$ in (2) or (5)), which is of particular interest in the magnetostatic case.

6.1 The general meridian field problem.

The meridian component $\mathbf{U}_m = U_r \mathbf{e}_r + U_z \mathbf{e}_z$ of \mathcal{U} satisfies $\operatorname{div} \mathbf{U}_m = G$ and $\operatorname{curl} \mathbf{U}_m = \mathbf{F}_\theta$. Introducing $\mathbf{u}_m = r \mathbf{U}_m$, $g = r G$ and $f_\theta = r F_\theta$, one has:

$$\operatorname{div} \mathbf{u}_m = g \text{ in } \omega, \quad \operatorname{curl}_- \mathbf{u}_m = f_\theta \text{ in } \omega, \quad \mathbf{u}_m^n \cdot \boldsymbol{\nu}, \text{ resp. } \mathbf{u}_m^d \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma_b. \quad (67)$$

Any vector field in $\mathbf{H}(\operatorname{curl}; \Omega) \cap \mathbf{H}(\operatorname{div}; \Omega)$ is \mathbf{H}^1 away from the boundary Γ (cf. [13]); hence its r -component vanishes on the axis: $\mathbf{U} \cdot \boldsymbol{\nu} = U_r = 0|_{\gamma_a}$; the same is true for \mathbf{u}_m .

We shall treat in some detail the electrostatic case only. For the magnetostatic case, the superscripts d and n and the boundary conditions *on γ_b only* are to be swapped: $\mathbf{u} \cdot \boldsymbol{\tau} = 0$ is to be replaced with $\mathbf{u} \cdot \boldsymbol{\nu} = 0$, $V = 0$ becomes $\partial_\nu V = 0$, and so on. On the other hand, the boundary conditions on the axis, which stem from the axial symmetry, are the same.

Let us consider the Hodge decomposition (7). Defining $W^n = r A_\theta^n$, V^d and W^n are related to \mathbf{u}_m^d by:

$$\mathbf{u}_m^d = \operatorname{curl} W^n - r \operatorname{grad} V^d. \quad (68)$$

As a function of the source f_θ , W^n is obtained as the solution of the mixed Dirichlet–Neumann problem with modified Laplacian Δ^-

$$-\Delta^- W^n = f_\theta^n \text{ in } \omega, \quad W^n = 0 \text{ on } \gamma_a, \quad \frac{\partial W^n}{\partial \nu} = 0 \text{ on } \gamma_b. \quad (69)$$

Similarly, V^d is the solution of the mixed Dirichlet–Neumann problem with modified Laplacian Δ^+ :

$$-\Delta^+ V^d = G^d \text{ in } \omega, \quad \frac{\partial V^d}{\partial \nu} = 0 \text{ on } \gamma_a, \quad V^d = 0 \text{ on } \gamma_b. \quad (70)$$

The boundary conditions on γ_a are justified by the Remarks following Propositions 3.12 and 3.14; $\partial_\nu W^n = 0$ on γ_b is the trace of (14); the boundary condition (13) only concerns the meridian components of \mathcal{A}^n .

f_θ belongs to $L^2_{-1}(\omega)$, and so does g . As for \mathbf{U}_m^d , it belongs to

$$\check{\mathcal{X}}_m^d = \left\{ \mathbf{U}_m^d \in L^2_1(\omega)^2 : \operatorname{div}_+ \mathbf{U}_m^d \in L^2_1(\omega) \text{ and } \operatorname{curl} \mathbf{U}_m^d \in L^2_1(\omega) \text{ and } \mathbf{U}_m^d \cdot \boldsymbol{\tau} = 0 \text{ and } \gamma_b \right\}.$$

So, by Lemma 3.1, the variable \mathbf{u}_m^d belongs to the space

$$\mathbb{U}^d = \left\{ \mathbf{u}^d \in L^2_{-1}(\omega)^2 : \operatorname{div} \mathbf{u} \in L^2_{-1}(\omega) \text{ and } \operatorname{curl}_- \mathbf{u} \in L^2_{-1}(\omega) \text{ and } \mathbf{u}_m \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma_b \right\}.$$

Finally, by Propositions 3.3 and 3.7, the potentials V^d and W^n belong to

$$\Phi^{d+} = \left\{ V^d \in H^1_1(\omega) : \Delta^+ V^d \in L^2_1(\omega) \text{ and } \frac{\partial V^d}{\partial \nu} = 0 \text{ on } \gamma_a \text{ and } V^d = 0 \text{ on } \gamma_b \right\}. \quad (71)$$

$$\Phi^{n-} = \left\{ W^n \in H^1_{-1}(\omega) : \Delta^- W^n \in L^2_{-1}(\omega) \text{ and } W^n = 0 \text{ on } \gamma_a \text{ and } \frac{\partial W^n}{\partial \nu} = 0 \text{ on } \gamma_b \right\}. \quad (72)$$

Theorem 6.1 *For $(f_\theta, g) \in L^2_{-1}(\omega) \times L^2_{-1}(\omega)$, i.e. $G \in L^2_1(\omega)$, Eq. (67) has a unique solution in \mathbb{U}^d ; and (68) has a unique solution in $\Phi^{d+} \times \Phi^{n-}$. The mappings defined by these equations are isomorphisms between the relevant spaces.*

As a consequence, Δ^+ and Δ^- are isomorphisms, respectively between Φ^{d+} and $L^2_1(\omega)$, and Φ^{n-} and $L^2_{-1}(\omega)$.

Proof: Consider the following diagramme:

$$\begin{array}{ccccc} \check{\Phi}^d & & \check{\mathcal{X}}^d & & \check{\mathcal{L}}^d \\ & \xrightarrow[\text{+ curl}]{-\text{grad}} & & \xrightarrow[\text{curl}]{\operatorname{div}} & \\ & \mathcal{M}^n & & & \check{\mathcal{J}}^n \\ & \downarrow \varpi_\theta & & \downarrow \varpi_m & \downarrow \varpi_\theta \\ \check{\Phi}^d & & \check{\mathcal{X}}_m^d & & L^2_1(\omega) \\ & \xrightarrow[\text{+1/r curl}(r \cdot)]{-\text{grad}} & & \xrightarrow[\text{curl}]{\operatorname{div}_+} & \\ & \mathcal{M}_\theta^n & & & L^2_1(\omega) \\ & \downarrow \mathbb{R} & & \downarrow \mathbb{R} & \downarrow \mathbb{R} \\ \Phi^{d+} & & \mathbb{U}^d & & L^2_1(\omega) \\ & \xrightarrow[\text{curl}]{-r \text{ grad}} & & \xrightarrow[\text{curl}_-]{(1/r) \operatorname{div}} & \\ & \Phi^{n-} & & & L^2_{-1}(\omega) \end{array} \quad (73)$$

The operators and spaces that come with the horizontal arrows are to be understood as in (16). The operators placed on both sides of a vertical arrow, are in tensor product; they operate between the spaces just above and just beneath them.

The first row of (73) is made of isomorphisms, by Proposition 2.1. ϖ_m and ϖ_θ are the meridian and azimuthal projections; they are surjective (onto) morphisms by Lemmas 3.5 and 3.7. \mathbb{R} is as in Lemma 3.1; \mathbb{I} is the identity mapping, or the trace operator in a meridian half-plane, which we (more or less) merge. The sub-diagramme made of the first two rows in (73), and the vertical arrows between them, is obviously commutative. Hence, the second row is made of isomorphisms.

Then, straightforward calculations show that the last two rows, and the vertical arrows between, make up a commutative diagramme. Hence, the third row is made of isomorphisms. The last assertion, too, is straightforward. \blacksquare

6.2 The divergence-free meridian field problem.

To study it—in the magnetostatic case—let us introduce the divergence-free spaces

$$\begin{aligned}\check{\mathcal{X}}_m^{0n} &= \{ \mathbf{B}_m \in L_1^2(\omega)^2 : \operatorname{div}_+ \mathbf{B}_m = 0 \text{ in } \omega \text{ and } \operatorname{curl} \mathbf{B}_m \in L_1^2(\omega) \text{ and } \mathbf{B}_m \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma_b \}, \\ \mathcal{U}^{0n} &= \{ \mathbf{v} \in L_{-1}^2(\omega)^2 : \operatorname{div} \mathbf{v} = 0 \text{ in } \omega \text{ and } \operatorname{curl}_- \mathbf{v} \in L_{-1}^2(\omega) \text{ and } \mathbf{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma_b \}.\end{aligned}$$

and the space of potentials

$$\Phi^{d-} = \left\{ W^d \in H_{-1}^1(\omega) : \Delta^- W^d \in L_{-1}^2(\omega) \text{ and } W^d = 0 \text{ on } \gamma_a \text{ and } \gamma_b \right\}. \quad (74)$$

There holds the simplified result:

Lemma 6.2 *The following mappings are isomorphisms:*

$$\mathbf{curl} : \Phi^{d-} \rightarrow \mathcal{U}^{0n}, \quad \operatorname{curl}_- : \mathcal{U}^{0n} \rightarrow L_{-1}^2(\omega), \quad \Delta^- : \Phi^{d-} \rightarrow L_{-1}^2(\omega),$$

and are linked by $-\Delta^- = \operatorname{curl}_- \mathbf{curl}$.

Proof: Consider this time the diagramme:

$$\begin{array}{ccccc} \check{\mathcal{M}}^d & \xrightarrow{\mathbf{curl}} & \check{\mathcal{X}}^{0n} & \xrightarrow{\mathbf{curl}} & \check{\mathcal{J}}^d \\ \varpi_\theta \downarrow & & \varpi_m \downarrow & & \varpi_\theta \downarrow \\ \check{\mathcal{M}}_\theta^d & \xrightarrow{1/r \mathbf{curl}(r \cdot)} & \check{\mathcal{X}}_m^{0n} & \xrightarrow{\mathbf{curl}} & L_1^2(\omega) \\ \mathbb{R} \downarrow & & \mathbb{R} \downarrow & & \mathbb{R} \downarrow \\ \Phi^{d-} & \xrightarrow{\mathbf{curl}} & \mathcal{U}^{0n} & \xrightarrow{\operatorname{curl}_-} & L_{-1}^2(\omega) \end{array} \quad (75)$$

and show that it is commutative. ■

If we were interested in divergence-free electro“static” problems, similar results would hold, with boundary conditions on γ_b swapped.

6.3 Space decomposition results for the meridian problems.

For both div-curl problems, there holds the following result, which we state in the electrostatic case—for the magnetostatic case, just swap the superscripts d and n .

Lemma 6.3 *In \mathcal{U}^d , the semi-norm $(\|\operatorname{curl}_- \mathbf{u}_m\|_{0,-1,\omega}^2 + \|\operatorname{div} \mathbf{u}\|_{0,-1,\omega}^2)^{1/2}$ is a norm, which is equivalent to the canonical norm; and similarly in Φ^{d+} , resp. Φ^{n-} , the norm $\|\Delta^+ V\|_{0,1,\omega}$, resp. $\|\Delta^- W\|_{0,-1,\omega}$. If \mathcal{U}^d , Φ^{d+} , Φ^{n-} are endowed with these norms, the isomorphisms of Lemma 6.1 are isometric.*

In the divergence-free case, we have the simplified version, which we state in the magnetostatic case:

Lemma 6.4 *In U^{0n} , the L_{-1}^2 norm of the modified curl defines a norm equivalent to the canonical norm; and similarly in Φ^{d-} the L_{-1}^2 norm of the modified Laplacian. If U^{0n} and Φ^{d-} are endowed with these norms, the isomorphisms of Lemma 6.2 are isometric.*

The traces of the spaces $\check{\mathcal{X}}_R^d$, $\check{\mathcal{M}}_R^d$, $\check{\Phi}_R^d$ etc. are closed within U^d , Φ^{d+} , Φ^{n-} , and so on. One finds that:

$$U_R^d = \left\{ \mathbf{u} = (u_r, u_z) \in L_{-3}^2(\omega) \times L_{-1}^2(\omega) : \left(\frac{\partial u_r}{\partial r}, \frac{\partial u_r}{\partial z}, \frac{\partial u_z}{\partial z} \right) \in L_{-1}^2(\omega)^3 \right. \\ \left. \text{and } \frac{\partial u_z}{\partial r} - \frac{u_z}{r} \in L_{-1}^2(\omega) \text{ and } \mathbf{u} \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma_b \right\}. \quad (76)$$

For the vector potential, the trace of $\mathbf{curl} \mathcal{A} \in \check{\mathbf{H}}^1(\Omega)$ is $\mathbf{curl} W \in U_R$, i.e.

$$\frac{\partial^2 W}{\partial r^2} - \frac{1}{r} \frac{\partial W}{\partial r} \in L_{-1}^2(\omega), \quad \frac{\partial^2 W}{\partial r \partial z} \in L_{-1}^2(\omega), \quad \frac{\partial^2 W}{\partial z^2} \in L_{-1}^2(\omega), \quad (77)$$

hence the range of $\check{\mathcal{M}}_R^n$ is

$$\Phi_R^{n-} = \left\{ W \in H_{-1}^1(\omega) : W \text{ satisfies (77) and } W = 0 \text{ on } \gamma_a \text{ and } \frac{\partial W}{\partial \nu} = 0 \text{ on } \gamma_b \right\}. \quad (78)$$

The range of $\check{\Phi}_R^d$ is:

$$\Phi_R^{d+} = \left\{ V \in H_+^2(\omega) : \frac{\partial V}{\partial \nu} = 0 \text{ on } \gamma_a \text{ and } V = 0 \text{ on } \gamma_b \right\}. \quad (79)$$

Similarly, one defines U_R^n , Φ_R^{d-} , Φ_R^{n+} by swapping the boundary conditions on γ_b ; and U_R^{0n} by imposing the extra condition $\text{div} \mathbf{u} = 0$.

The regular potential subspaces enjoy the following property, which is the trace of Propositions 4.12 and 4.17:

Proposition 6.5 *The spaces Φ_R^{d-} and Φ_R^{n-} are the range, by the isometry R of Proposition 3.1, of closed subspaces in $H_-^2(\omega)$; and the norms $\|\Delta^- W\|_{0,-1,\omega}$ and $\|W/r\|_{2,-,\omega}$ are equivalent on these spaces.*

Explicit singular potentials. As we already know the codimensions of the regular spaces, we shall not try to determine their orthogonal complements; instead, we are interested in explicit complements. Let

$$\Phi^{d+} = \Phi_R^{d+} \oplus \Phi_{S_*}^{d+}, \quad \Phi^{d-} = \Phi_R^{d-} \oplus \Phi_{S_*}^{d-}, \quad \Phi^{n-} = \Phi_R^{n-} \oplus \Phi_{S_*}^{n-}$$

be non-orthogonal decompositions of the various potential spaces. Thanks, respectively, to Theorem 4.8 and to the isomorphisms of Lemma 6.2, there holds:

$$U^d = U_R^d \oplus r \mathbf{grad} \Phi_{S_*}^{d+}, \quad U^{0n} = U_R^{0n} \oplus \mathbf{curl} \Phi_{S_*}^{d-}.$$

Moreover, $U_R^d \cap \mathbf{curl} \Phi_{S_*}^{n-} = \{0\}$; so, if we exhibit a sufficient number of linearly independent functions in Φ^{n-} that do not belong to Φ_R^{n-} , we will have additionally obtained

$$\dim \Phi_{S_*}^{n-} = \# [\mathcal{K}_{ES} \cup \mathcal{K}_{OS}] \text{ and } U^d = U_R^d \oplus \mathbf{curl} \Phi_{S_*}^{n-},$$

a decomposition that will be useful in the analysis of time-dependent problems. These different splittings are the analogues, in axisymmetric geometry, of those exhibited by Costabel–Dauge [9] in the 2D Cartesian case.

Using the notations of Section 5, we define for each reentrant edge E_j the functions

$$S_j^{d-}(\rho_j, \phi_j) \equiv \eta_j(\rho_j) \rho_j^{\alpha_j} Y_j^{d-}(\phi_j), \text{ where } Y_j^{d-}(\phi_j) = \sqrt{\frac{2\alpha_j}{\pi}} \sin(\alpha_j \phi_j), \quad (80)$$

$$S_j^{d+}(\rho_j, \phi_j) \equiv \eta_j(\rho_j) \rho_j^{\alpha_j} Y_j^{d+}(\phi_j), \text{ where } Y_j^{d+}(\phi_j) = Y_j^{d-}(\phi_j), \quad (81)$$

$$S_j^{n-}(\rho_j, \phi_j) \equiv \eta_j(\rho_j) \rho_j^{\alpha_j} Y_j^{n-}(\phi_j), \text{ where } Y_j^{n-}(\phi_j) \equiv \sqrt{\frac{2\alpha_j}{\pi}} \cos(\alpha_j \phi_j). \quad (82)$$

It is easy to check that these functions satisfy all the criteria that define respectively Φ^{d-} , Φ^{d+} and Φ^{n-} . On the other hand, they are not in $H^2(\omega_{E_j})$ because of the exponent $\alpha_j < 1$; hence they fail to belong to the regularised subspaces.

For sharp vertex O_j , let us introduce

$$S_j^{d+}(\rho_j, \phi_j) \equiv \eta_j(\rho_j) \rho_j^{\nu_j} Y_j^{d+}(\phi_j), \text{ where } Y_j^{d+}(\phi_j) = C_j^{d+} P_{\nu_j}^0(\cos \phi_j), \quad (83)$$

$$S_j^{n-}(\rho_j, \phi_j) \equiv \eta_j(\rho_j) \rho_j^{1+\nu_j} Y_j^{n-}(\phi_j), \text{ where } Y_j^{n-}(\phi_j) = C_j^{n-} \sin \phi_j P_{\nu_j}^1(\cos \phi_j). \quad (84)$$

We recall that $\nu_j = \nu_{j,1}^+$ is the unique root of $P_{\nu}^0(\cos \pi/\beta_j) = 0$ in the interval $]0, 1/2[$. The C_j are normalisation factors in \mathcal{H}_{\pm} . Thanks to the expression [1, Eq. 8.5.2] of the derivative of the Legendre function, one shows that $S_j^{n-}(\rho_j, \phi_j)$ satisfies the homogeneous Neumann condition on γ_b , and that Y_j^{n-} is an eigenfunction of the operator Δ with mixed boundary conditions, with eigenvalue $\nu_j(\nu_j + 1)$. By Eqs. (53, 54) one has:

$$\Delta^+ \left\{ f(\rho_j) Y_j^{d+}(\phi_j) \right\} = \left[f''(\rho_j) + \frac{2}{\rho} f'(\rho_j) - \frac{\nu_j(\nu_j + 1)}{\rho_j^2} f(\rho_j) \right] Y_j^{d+}(\phi_j), \quad (85)$$

$$\Delta^- \left\{ f(\rho_j) Y_j^{n-}(\phi_j) \right\} = \left[f''(\rho_j) - \frac{\nu_j(\nu_j + 1)}{\rho_j^2} f(\rho_j) \right] Y_j^{n-}(\phi_j), \quad (86)$$

and so $\Delta^+ S_j^{d+} = \Delta^- S_j^{n-} \equiv 0$ near O_j . Elsewhere, these modified Laplacians vanish except in a shell where they are \mathcal{C}^∞ up to the boundary. In addition, thanks to the factor $\sin \phi_j$ in $Y_j^{n-}(\phi_j)$, there holds $\Delta^- S_j^{n-}(\rho_j, 0) = 0$, so $\Delta^- S_j^{n-} \in L^2_{-1}(\omega)$.

By Proposition 3.11, $S_j^{d+} \in H^1_+(\omega)$, but $S_j^{d+} \notin H^2_+(\omega)$; $S_j^{n-} \in H^1_{-1}(\omega)$, but $S_j^{n-} \notin \mathbb{R}[H^2_-(\omega)]$. So we have $S_j^{d+} \in \Phi^{d+}$ and $S_j^{d+} \notin \Phi^{d+}_R$; $S_j^{n-} \in \Phi^{n-}$ and $S_j^{n-} \notin \Phi^{n-}_R$.

Finally, the obvious linear independence of the S_j associated with different geometrical singularities, and the dimensions of the singular spaces, imply that $\left(S_j^{d+} \right)_{j \in \mathcal{K}_{ES} \cup \mathcal{K}_{OS}}$, $\left(S_j^{n-} \right)_{j \in \mathcal{K}_{ES} \cup \mathcal{K}_{OS}}$, $\left(S_j^{d-} \right)_{j \in \mathcal{K}_{ES}}$, are respective bases of $\Phi^{d+}_{S^*}$, $\Phi^{n-}_{S^*}$ and $\Phi^{d-}_{S^*}$.

Explicit singular electric fields. Near a reentrant edge E_j , one has:

$$\begin{aligned} \text{grad } S_j^{d+}(\rho_j, \phi_j) &= \sqrt{2\pi\alpha_j} \eta_j(\rho_j) \rho_j^{\alpha_j-1} \left[\sin(\alpha_j \phi_j) \mathbf{e}_{\rho_j} + \cos(\alpha_j \phi_j) \mathbf{e}_{\phi_j} \right] \\ &\quad + \sqrt{\frac{2\pi}{\alpha_j}} \eta'_j(\rho_j) \rho_j^{\alpha_j} \sin(\alpha_j \phi_j) \mathbf{e}_{\rho_j}, \\ \text{curl } S_j^{n-}(\rho_j, \phi_j) &= \sqrt{2\pi\alpha_j} \eta_j(\rho_j) \rho_j^{\alpha_j-1} \left[\sin(\alpha_j \phi_j) \mathbf{e}_{\rho_j} + \cos(\alpha_j \phi_j) \mathbf{e}_{\phi_j} \right] \\ &\quad + \sqrt{\frac{2\pi}{\alpha_j}} \eta'_j(\rho_j) \rho_j^{\alpha_j} \cos(\alpha_j \phi_j) \mathbf{e}_{\phi_j}. \end{aligned}$$

These two types of singularities have the same ‘‘principal part’’, i.e. they only differ by an \mathbf{H}^1 term. Besides, $(r/a_j) \mathbf{grad} S_j^{d+}$ still has the same principal part, since

$$\left(\frac{r}{a_j} - 1\right) \mathbf{grad} S_j^{d+}(\rho_j, \phi_j) = \rho_j \cos \phi_j' \mathbf{grad} S_j^{d+}(\rho_j, \phi_j) \approx \rho_j^{\alpha_j} \in \mathbf{H}^1(\omega_{E_j}).$$

In absence of divergence constraint, we shall use a practical singular field

$$\begin{aligned} \mathbf{S}_j^d(\rho_j, \phi_j) &\equiv \sqrt{2\pi\alpha_j} \eta_j(\rho_j) \rho_j^{\alpha_j-1} \left[\sin(\alpha_j \phi_j) \mathbf{e}_{\rho_j} + \cos(\alpha_j \phi_j) \mathbf{e}_{\phi_j} \right] \\ &= \frac{1}{2} \left\{ \frac{r}{a_j} \mathbf{grad} S_j^{d+} + \mathbf{curl} S_j^{n-} \right\} + \mathbf{w}_j, \text{ with } \mathbf{w}_j \in \mathbf{U}_R^d. \end{aligned} \quad (87)$$

Let us now construct a singular field near a sharp conical vertex O_j . Using the differentiation formulae for the Legendre functions [1, Eqs. 8.5.2 and 8.6.6], we find:

$$\begin{aligned} r \mathbf{grad} S_j^{d+}(\rho_j, \phi_j) &= C_j^{d+} \eta_j(\rho_j) \rho_j^{\nu_j} \sin \phi_j \left[\nu_j P_{\nu_j}^0(\cos \phi_j) \mathbf{e}_{\rho_j} + P_{\nu_j}^1(\cos \phi_j) \mathbf{e}_{\phi_j} \right] \\ &\quad + C_j^{d+} \eta_j'(\rho_j) \rho_j^{\nu_j+1} \sin \phi_j P_{\nu_j}^0(\cos \phi_j) \mathbf{e}_{\rho_j} \\ \mathbf{curl} S_j^{n-}(\rho_j, \phi_j) &= (\nu_j + 1) C_j^{n-} \eta_j(\rho_j) \rho_j^{\nu_j} \sin \phi_j \left[\nu_j P_{\nu_j}^0(\cos \phi_j) \mathbf{e}_{\rho_j} + P_{\nu_j}^1(\cos \phi_j) \mathbf{e}_{\phi_j} \right] \\ &\quad + C_j^{n-} \eta_j'(\rho_j) \rho_j^{\nu_j+1} \sin \phi_j \sin \phi_j P_{\nu_j}^1(\cos \phi_j) \mathbf{e}_{\phi_j}. \end{aligned}$$

Here, too, the two types of singularities have proportional principal parts, since

$$(\nu_j + 1) C_j^{n-} r \mathbf{grad} S_j^{d+} - C_j^{d+} \mathbf{curl} S_j^{n-} \approx \rho_j^{\nu_j+1},$$

so the r and z components of this field are in $H_{-1}^1(\omega)$, and the field belongs to \mathbf{U}_R^d . In absence of divergence constraint, we shall use the singular field

$$\begin{aligned} \mathbf{S}_j^d(\rho_j, \phi_j) &\equiv \eta_j(\rho_j) \rho_j^{\nu_j} \sin \phi_j \left[\nu_j P_{\nu_j}^0(\cos \phi_j) \mathbf{e}_{\rho_j} + P_{\nu_j}^1(\cos \phi_j) \mathbf{e}_{\phi_j} \right] \\ &= \frac{1}{2} \left\{ \frac{r \mathbf{grad} S_j^{d+}}{C_j^{d+}} + \frac{\mathbf{curl} S_j^{n-}}{(\nu_j + 1) C_j^{n-}} \right\} + \mathbf{w}_j, \text{ with } \mathbf{w}_j \in \mathbf{U}_R^d. \end{aligned} \quad (88)$$

By a dimension argument, we conclude that the $(\mathbf{S}_j^d)_{j \in \mathcal{K}_{ES} \cup \mathcal{K}_{OS}}$ make up a basis of $\mathbf{U}_{S^*}^d$, a complement of \mathbf{U}_R^d within \mathbf{U}^d .

Since the azimuthal component of any field in $\check{\mathbf{H}}(\mathbf{curl}; \Omega)$ is in $\check{\mathbf{H}}^1(\Omega)$ (Proposition 3.9), we infer that the fields $\mathbf{S}_j^d(\rho_j, \phi_j)/r$ span a complement of $\check{\mathcal{X}}_R^d$ within $\check{\mathcal{X}}^d$.

Explicit singular magnetic fields. The space span $\left\{ \mathbf{S}_j^{0n} \right\}_{j \in \mathcal{K}_{ES}}$, where $\mathbf{S}_j^{0n} \stackrel{def}{=} \mathbf{curl} S_j^{d-}$, obviously complements \mathbf{U}_R^{0n} within \mathbf{U}^{0n} . One has explicitly:

$$\begin{aligned} \mathbf{S}_j^{0n}(\rho_j, \phi_j) &= \sqrt{2\pi\alpha_j} \eta_j(\rho_j) \rho_j^{\alpha_j-1} \left[-\cos(\alpha_j \phi_j) \mathbf{e}_{\rho_j} + \sin(\alpha_j \phi_j) \mathbf{e}_{\phi_j} \right] \\ &\quad + \sqrt{\frac{2\pi}{\alpha_j}} \eta_j'(\rho_j) \rho_j^{\alpha_j} \sin(\alpha_j \phi_j) \mathbf{e}_{\phi_j}. \end{aligned} \quad (89)$$

The last term vanishes near E_j , and is of global H^1 regularity. It is necessary to preserve the divergence constraint. If, however, we work within \mathbf{U}^n , i.e. without divergence constraint, or if we just need local expressions, we can use the singular fields

$$\begin{aligned} \mathbf{S}_j^n(\rho_j, \phi_j) &\equiv \sqrt{2\pi\alpha_j} \eta_j(\rho_j) \rho_j^{\alpha_j-1} \left[-\cos(\alpha_j \phi_j) \mathbf{e}_{\rho_j} + \sin(\alpha_j \phi_j) \mathbf{e}_{\phi_j} \right] \\ &= \mathbf{curl} S_j^{d-} + \mathbf{w}_j, \text{ with } \mathbf{w}_j \in \mathbf{U}_R^n. \end{aligned} \quad (90)$$

We remark that \mathbf{S}_j^n is \mathbf{S}_j^d rotated of $\pi/2$: this is similar to the Cartesian geometry.

Like in the electric case, we infer that the fields $\mathbf{S}_j^{0n}(\rho_j, \phi_j)/r$ span a complement of $\check{\mathcal{X}}_R^{0n}$ within $\check{\mathcal{X}}^{0n}$. If we work within $\check{\mathcal{X}}^n$, i.e. without divergence constraint, we can use the \mathbf{S}_j^n : because of their support, they belong to $\check{\mathbf{H}}(\mathbf{curl}, \text{div}; \Omega)$ but not to $\check{\mathbf{H}}^1(\Omega)$, and satisfy the magnetic boundary condition.

6.4 The azimuthal field problem.

The azimuthal component $\mathbf{U}_\theta = U_\theta \mathbf{e}_\theta$ satisfies $\mathbf{curl} \mathbf{U}_\theta = \mathbf{F}_m$, and because of the axial symmetry, $\text{div} \mathbf{U}_\theta = 0$. In the electrostatic case, $\mathcal{U} \times \mathbf{n} = 0$ implies $\mathbf{U}_\theta = 0$ on Γ .

Introducing $u_\theta = r U_\theta$ and $\mathbf{f}_m = r \mathbf{F}_m$, Eq. (1) resp. (4) becomes:

$$\mathbf{curl} u_\theta = \mathbf{f}_m. \quad (91)$$

In the electrostatic case, one has $\mathbf{f}_m \in \mathcal{F}^n = \{\mathbf{f} \in L_{-1}^2(\omega)^2 : \text{div} \mathbf{f} = 0 \text{ and } \omega \text{ and } \mathbf{f} \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma_b\}$, and by Lemma 3.7, $u_\theta \in \mathcal{V}^{d-} = \mathring{H}_{-1}^1(\omega)$. In the magnetostatic case, $\mathbf{f}_m \in \mathcal{F}^d = \{\mathbf{f} \in L_{-1}^2(\omega)^2 : \text{div} \mathbf{f} = 0\}$, and $u_\theta \in \mathcal{V}^{n-} = H_{-1}^1(\omega)$. The following diagramme is commutative and isometric

$$\begin{array}{ccc} \check{\mathcal{X}} & \xrightarrow{\mathbf{curl}} & \check{\mathcal{J}} \\ \varpi_\theta \downarrow & & \varpi_m \downarrow \\ \check{\mathcal{X}}_\theta & \xrightarrow{1/r \mathbf{curl}(r \cdot)} & \check{\mathcal{J}}_m \\ \mathbf{R} \downarrow & & \mathbf{R} \downarrow \\ \mathcal{V}^- & \xrightarrow{\mathbf{curl}} & \mathcal{F} \end{array} \quad (92)$$

We recall the absence of azimuthal singularities; the azimuthal component of any field in $\check{\mathbf{H}}(\mathbf{curl}, \text{div}; \Omega)$ is automatically in $\check{\mathbf{H}}^1(\Omega)$, i.e. $v \in H_{-1}^1(\omega)$.

6.5 Space regularity of the electric and magnetic fields.

Theorem 6.6 *There holds: $\check{\mathcal{X}}_m^d \subset H_1^s(\omega)^2$, i.e. $\mathcal{U}^d \subset H_{-1}^s(\omega)^2$, or $\check{\mathcal{X}}^d \in \check{\mathbf{H}}^s(\Omega)$, iff $s < \min\{\alpha_j, j \in \mathcal{K}_{ES}; \nu_j + 1/2, j \in \mathcal{K}_{OS}\}$.*

Similarly, $\check{\mathcal{X}}_m^{0n} \subset H_1^s(\omega)^2$, i.e. $\mathcal{U}^{0n} \subset H_{-1}^s(\omega)^2$, or $\check{\mathcal{X}}^{0n} \in \check{\mathbf{H}}^s(\Omega)$, iff $s < \min\{\alpha_j, j \in \mathcal{K}_{ES}\}$.

Proof: For any $\mathcal{U}^d \in \check{\mathcal{X}}^d$, its azimuthal component is in $\check{\mathbf{H}}^1(\Omega)$; so the global space regularity will be determined by the singular part of its meridian component. We have:

$$\check{\mathcal{X}}^d = \check{\mathcal{X}}_R^d \oplus \text{span} \left\{ \mathbf{S}_j^d/r, j \in \mathcal{K}_{ES} \cup \mathcal{K}_{OS} \right\}.$$

By Proposition 3.11, the formula (87) proves that the \mathbf{S}_j^d associated to reentrant edges are in $H^s(\omega_{E_j})^2$ iff $s < \alpha_j$. Then, because of their support, $\mathbf{S}_j^d \in H_{-1}^s(\omega)^2$, and $\mathbf{S}_j^d/r \in H_1^s(\omega)^2 = H_+^s(\omega) \times H_-^s(\omega)$. Similarly, for conical vertices, (88) proves that $\mathbf{S}_j^d \in H_{-1}^s(\omega)^2$ and $\mathbf{S}_j^d/r \in H_1^s(\omega)^2 = H_-^s(\omega) \times H_+^s(\omega)$, iff $s < \nu_j + 1/2$.

The conclusions follow. The proof in the magnetic case is similar (and simpler). \blacksquare

7 Analysis of the modified wave equation.

Setting $q = \omega \times]0, T[$, $\sigma_{a/b} = \gamma_{a/b} \times]0, T[$, we consider the evolution problem

$$\begin{cases} u''(t) - \Delta^- u(t) = f(t) \text{ in } q, \\ u = 0 \text{ on } \sigma_a; u = 0, \text{ resp. } \partial_\nu u = 0 \text{ on } \sigma_b, \\ u(0) = u_0, u'(0) = u_1 \text{ in } \omega. \end{cases} \quad (93)$$

We are mainly interested in the decomposition into regular and singular parts, and in the space-time regularity, of the solution to (93). This study, carried out in Subsections 7.1 and 7.2, closely parallels and relies upon the similar work on the standard Laplacian and the standard wave operator by Grisvard [14, Paragraphs 2.5.2 and 5.3]; so many proofs will be sketched or even omitted.

Let $H = L^2_{-1}(\omega)$, $V^- = \mathring{H}^1_{-1}(\omega)$ or $H^1_{-1}(\omega)$, A the unbounded operator $-\Delta^-$ on H , and $D_A = \Phi^{d/n^-}$ its domain. As A is a strictly positive self-adjoint operator with compact inverse, it admits an increasing sequence of positive eigenvalues $(\lambda_m)_{m \in \mathbb{N}}$, repeated according to their multiplicity, and going to infinity. The associated eigenfunctions (normalised in H) are denoted $(w_m)_{m \in \mathbb{N}}$; the families $(w_m)_{m \in \mathbb{N}}$, $(w_m/\sqrt{\lambda_m})_{m \in \mathbb{N}}$ and $(w_m/\lambda_m)_{m \in \mathbb{N}}$ make up Hilbert bases of H , V^- and D_A respectively.

By interpolation, the power A^ϑ , $0 \leq \vartheta \leq 1$ is defined by its domain and its values:

$$D_{A^\vartheta} = \left\{ u \in H : \sum_m \lambda_m^{2\vartheta} (u | w_m)_H^2 < +\infty \right\} \text{ and } A^\vartheta u = \sum_m \lambda_m^\vartheta (u | w_m)_H w_m.$$

The square root of the quantity involved in the definition of D_{A^ϑ} is the canonical norm of this space. In particular, $V^- = D_{A^{1/2}}$.

Theorem 7.1 *Assume $f \in L^1(0, T; V^-)$, $u_0 \in D_A$ and $u_1 \in V^-$. The problem (93) admits a unique solution $u \in C^0(0, T; D_A) \cap C^1(0, T; V^-)$, depending continuously of the data f , u_0 , u_1 in their respective spaces. If, moreover, $f \in C^0(0, T; H)$, then $u \in C^2(0, T; H)$.*

Proof: Assume for the moment that $f \in C^0(0, T; H) \cap L^1(0, T; V^-)$. Then the semi-group theory [10] asserts the existence and uniqueness of $u \in C^0(0, T; \Phi^-) \cap C^1(0, T; V^-)$ solution to (93). This solution is explicitly given by

$$\begin{aligned} u(t) = & \sum_m \left\{ \cos(t\sqrt{\lambda_m}) (u_0 | w_m)_H + \sin(t\sqrt{\lambda_m}) \left(u_1 \left| \frac{w_m}{\sqrt{\lambda_m}} \right|_H \right. \right. \\ & \left. \left. + \int_{s=0}^t \sin((t-s)\sqrt{\lambda_m}) \left(f(s) \left| \frac{w_m}{\sqrt{\lambda_m}} \right|_H \right) ds \right\} w_m. \end{aligned}$$

We notice that u depends continuously on the $L^1(0, T; V^-)$ norm of f . Hence the existence and uniqueness of u for $f \in L^1(0, T; V^-)$ by a density argument. The last statement is straightforward. \blacksquare

7.1 Estimates with parameter for Δ^- .

Given $\xi \geq 0$, we study the variational solution in $V^{d^-} = \mathring{H}^1_{-1}(\omega)$ resp. $V^{n^-} = H^1_{-1}(\omega)$, in the sense of Proposition 3.14, to the Dirichlet, resp. Neumann problem

$$-\Delta^- u + \xi^2 u = f \text{ in } \omega, \quad u = 0 \text{ on } \gamma_a, \quad u, \text{ resp. } \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_b. \quad (94)$$

If $f \in L^2_{-1}(\omega)$, u belongs to $\Phi^- = \Phi^{d-}$, resp. Φ^{n-} ; in this Section, we shall usually omit the index d or n ; and we write $S_j = S_j^{d-}$ or S_j^{n-} , $Y_j = Y_j^{d-}$ or Y_j^{n-} , $\nu_j = \nu_{j,1}^+$...

The relevant sets of geometrical singularities are:

$$\mathcal{K}_S^d = \mathcal{K}_{ES} \text{ (Dirichlet case), } \quad \mathcal{K}_S^n = \mathcal{K}_{ES} \cup \mathcal{K}_{OS} \text{ (Neumann case).}$$

In this section, we write \mathcal{K}_S to cover both cases. u is split onto $\Phi^- = \Phi_R^- \oplus \Phi_{S^*}^-$:

$$u = u_R^*(\xi) + \sum_{j \in \mathcal{K}_S} c_j(\xi) S_j, \text{ with } u_R^* \in \Phi_R^-. \quad (95)$$

It is indeed more convenient to use the following decomposition:

$$u = u_R(\xi) + \sum_{j \in \mathcal{K}_S} c_j(\xi) e^{-\xi \rho_j} S_j, \text{ with } u_R \in \Phi_R^-, \quad (96)$$

which holds with *the same* $c_j(\xi)$, since one easily checks that $(1 - e^{-\xi \rho_j}) S_j \in \Phi_R^-$. The main goal of this Subsection is to obtain estimates of the various terms in (96) as $\xi \rightarrow \infty$.

As a consequence of (45) and (50), we have the following ‘‘very weak’’ integration by parts formulae for Δ^- . If $p \in L^2_{-1}(\omega)$, $\Delta^- p \in L^2_{-1}(\omega)$, and $w \in \Phi_R^{d-}$, resp. $w \in \Phi_R^{n-}$ and $w|_{\gamma_i}$ belongs to the trace space of $\mathbb{R}[\check{H}(\Gamma_i)]$, there holds:

$$\iint_{\omega} \{p \Delta^- w - w \Delta^- p\} \frac{d\omega}{r} = \sum \left\langle p, \frac{\partial w}{\partial \nu} \right\rangle, \quad (97)$$

$$\text{resp. } \iint_{\omega} \{p \Delta^- w - w \Delta^- p\} \frac{d\omega}{r} = - \sum \left\langle \frac{\partial p}{\partial \nu}, w \right\rangle, \quad (98)$$

the duality brackets in the right-hand sides being taken between the suitable spaces, the traces of $\mathbb{R}[\check{H}(\Gamma_i)'] = \mathbb{R}[\check{H}(\Gamma_i)']'$ and $\mathbb{R}[\check{H}(\Gamma_i)]$, resp. of $\mathbb{R}[\check{H}(\Gamma_i)'] = \mathbb{R}[\check{H}(\Gamma_i)']'$ and $\mathbb{R}[\check{H}(\Gamma_i)]$. In particular, the hypotheses of (98) are automatically satisfied when the trace of w vanishes on all sides γ_i except one.

For a given ξ , the mapping $f \mapsto u$ is linear and continuous, and so is the projection on the closed subspace spanned by $e^{-\xi \rho_j} S_j$: so the mapping $f \mapsto c_j(\xi)$ is a linear continuous form on $L^2_{-1}(\omega)$, and there exists $g_j(\xi) \in L^2_{-1}(\omega)$ such that

$$c_j(\xi) = \iint_{\omega} f g_j(\xi) \frac{d\omega}{r}. \quad (99)$$

Obviously, $g_j(\xi) \in N_{\xi}^-$, the orthogonal of $(\Delta^- - \xi^2 \text{I}) \Phi_R^-$ within $L^2_{-1}(\omega)$. As a consequence of the formulae (97) and (98), one has:

Proposition 7.2 *Let $v \in L^2_{-1}(\omega)$; v belongs to N_{ξ}^- iff:*

$$\Delta^- v - \xi^2 v = 0 \text{ in } \omega, \quad v = 0 \text{ on } \gamma_a, \quad v, \text{ resp. } \frac{\partial v}{\partial \nu} = 0 \text{ on } \gamma_b. \quad (100)$$

The boundary condition is understood in the suitable space (see above) on γ_b , and in the strong sense on γ_a .

From this characterisation, we can infer local expressions for the basis functions of N_{ξ}^- :

Lemma 7.3 For any reentrant edge E_j , let $u_j \in L_{-1}^2(\omega)$ be the function

$$u_j(\rho_j, \phi_j) \equiv \eta_j(\rho_j) e^{-\xi \rho_j} \rho_j^{-\alpha_j} Y_j(\phi_j). \quad (101)$$

There exists $v_j \in N_\xi^-$ such that $-w_j \equiv v_j - u_j \in V^-$.

Proof: Similar to that of Lemma 5.3. ■

Lemma 7.4 In the Neumann case, define for any sharp vertex O_j the function

$$u_j(\rho_j, \phi_j) \equiv \eta_j(\rho_j) e^{-\xi \rho_j} \rho_j^{-\nu_j} Y_j(\phi_j) \in L_{-1}^2(\omega). \quad (102)$$

There exists $v_j \in N_\xi^-$ such that $-w_j \equiv v_j - u_j \in V^-$.

Proof: Here, a straightforward calculation shows that $f_j = \Delta^- u_j - \xi^2 u_j$ belongs to $L_{1-p}^p(\omega)$ for $p < 3/(2 + \nu_j) \equiv p_{\max}$. Since $\nu_j < 1/2$, $p_{\max} > 6/5$ and one can apply Proposition 3.14 and define w_j as the variational solution in V^- of $\Delta^- w_j - \xi^2 w_j = f_j$. The difference $v_j = u_j - w_j$ is in $L_{-1}^2(\omega)$ and satisfies (100). ■

Remark 7.1 By a dimension argument, the $(v_j)_{j \in \mathcal{K}_S}$ make up a basis of N_ξ^- . ■

Lemma 7.5 There holds

$$I \equiv \iint_\omega v_j (\Delta^- - \xi^2 \text{l}) S_k \frac{d\omega}{r} = -2 \lambda_j(\xi) \delta_{jk}, \quad (103)$$

where $\lambda_j(\xi)$ satisfies, for ξ large enough,

$$0 < \lambda_{\min} \leq \lambda_j(\xi) \leq \lambda_{\max}, \quad (104)$$

and the constants λ_{\min} and λ_{\max} depend only on the geometry, not on ξ .

Proof: The integral

$$I_1 \equiv \iint_\omega u_j (\Delta^- - \xi^2 \text{l}) S_k \frac{d\omega}{r}$$

vanishes for $j \neq k$ since u_j and S_k have disjoint supports. Moreover, using Proposition 3.15 and the variational definition of w_j , we calculate

$$I_2 \equiv \iint_\omega w_j (\Delta^- - \xi^2 \text{l}) S_k \frac{d\omega}{r} = \iint_\omega S_k (\Delta^- - \xi^2 \text{l}) w_j \frac{d\omega}{r} = \iint_\omega S_k (\Delta^- - \xi^2 \text{l}) u_j \frac{d\omega}{r}, \quad (105)$$

which vanishes again for $j \neq k$. So we are left with the case $j = k$, and we drop the subscript j . It stems from (105) that:

$$I = I_1 - I_2 = \iint_\omega \{u (\Delta^- - \xi^2 \text{l}) S - S (\Delta^- - \xi^2 \text{l}) u\} \frac{d\omega}{r} = \iint_\omega \{u \Delta^- S - S \Delta^- u\} \frac{d\omega}{r}. \quad (106)$$

We shall examine the cases of a reentrant edge and a sharp vertex.

EDGE SINGULARITY. We write (106) as:

$$I = \iint_\omega \{u \Delta S - S \Delta u\} \frac{d\omega}{r} - \iint_\omega \left\{ u \frac{\partial S}{\partial r} - S \frac{\partial u}{\partial r} \right\} \frac{d\omega}{r^2} \equiv I_3 - I_4.$$

In the region where u and S are non-zero, there holds $0 < R_{\min} \leq r \leq R_{\max}$, hence it is enough to estimate the integrals:

$$I_3' \equiv \iint_\omega \{u \Delta S - S \Delta u\} d\omega \quad \text{and} \quad I_4' \equiv \iint_\omega \left\{ u \frac{\partial S}{\partial r} - S \frac{\partial u}{\partial r} \right\} d\omega.$$

Grisvard [14, p. 66] has calculated $I_3' = -2\alpha$; hence, $-2\alpha/R_{\min} \leq I_3 \leq -2\alpha/R_{\max}$.

Now let us estimate I'_4 . In the Dirichlet case, a straightforward calculation yields:

$$u \frac{\partial S}{\partial r} - S \frac{\partial u}{\partial r} = \frac{2\alpha}{\pi} e^{-\xi \rho} \eta(\rho)^2 \rho^{-1} \cos \phi' \sin^2(\alpha \phi) (2\alpha + \xi \rho),$$

and thus

$$I'_4 = \frac{2\alpha}{\pi} \int_{\phi=0}^{\pi/\alpha} \cos(\phi + \phi_0) \sin^2(\alpha \phi) d\phi \int_{\rho=0}^{+\infty} e^{-\xi \rho} \eta(\rho)^2 (2\alpha + \xi \rho) d\rho \equiv k_1 J(\xi),$$

where k_1 depends only on the geometry. Moreover:

$$0 \leq J(\xi) = \int_{s=0}^{+\infty} e^{-s} \eta\left(\frac{s}{\xi}\right)^2 (2\alpha + s) \frac{ds}{\xi} \leq \frac{1}{\xi} \int_{s=0}^{+\infty} e^{-s} (2\alpha + s) ds \equiv k_2/\xi,$$

where k_2 is independent of ξ . So, $I_4 = O(\xi^{-1})$. This is still true in the Neumann case: the only change concerns the angular part of the integrand, which is once more bounded and independent of ξ . The estimate (104) follows.

In fact, one can prove that I defined by (103) is independent of ξ : $I = -2\alpha/a$. This calculation uses a generalised version of (38) and an approximation of ω which avoids the singularity.

CONICAL SINGULARITY. From (86), we calculate

$$u \Delta^- S - S \Delta^- u = \left\{ e^{-\xi \rho} \rho^{-\nu} \eta(\rho) \frac{d^2}{d\rho^2} [\rho^{1+\nu} \eta(\rho)] - \rho^{1+\nu} \eta(\rho) \frac{d^2}{d\rho^2} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] \right\} Y(\phi)^2.$$

Then (106) yields, taking into account the normalisation of $Y(\phi)$:

$$\begin{aligned} I &= \int_{\rho=0}^{+\infty} \left\{ e^{-\xi \rho} \rho^{-\nu} \eta(\rho) \frac{d^2}{d\rho^2} [\rho^{1+\nu} \eta(\rho)] - \rho^{1+\nu} \eta(\rho) \frac{d^2}{d\rho^2} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] \right\} d\rho \\ &= \left[e^{-\xi \rho} \rho^{-\nu} \eta(\rho) \frac{d}{d\rho} [\rho^{1+\nu} \eta(\rho)] \right]_0^{+\infty} - \int_0^{+\infty} \frac{d}{d\rho} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] \frac{d}{d\rho} [\rho^{1+\nu} \eta(\rho)] d\rho \\ &\quad - \left[\rho^{1+\nu} \eta(\rho) \frac{d}{d\rho} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] \right]_0^{+\infty} + \int_0^{+\infty} \frac{d}{d\rho} [\rho^{1+\nu} \eta(\rho)] \frac{d}{d\rho} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] d\rho \\ &= - \left\{ e^{-\xi \rho} \rho^{-\nu} \eta(\rho) \frac{d}{d\rho} [\rho^{1+\nu} \eta(\rho)] - \rho^{1+\nu} \eta(\rho) \frac{d}{d\rho} [e^{-\xi \rho} \rho^{-\nu} \eta(\rho)] \right\} \Big|_{\rho=0} = -2\nu. \end{aligned}$$

This value is constant, so it satisfies (104). ■

Proposition 7.6 *The coefficient $c_j(\xi)$ in (96) is explicitly given by:*

$$c_j(\xi) = -\frac{1}{2\lambda_j(\xi)} \iint_{\omega} f v_j \frac{d\omega}{r}. \quad (107)$$

Proof: Using the decomposition (95), we calculate:

$$I = \iint_{\omega} f v_j \frac{d\omega}{r} = \iint_{\omega} v_j (\Delta^- - \xi^2 \text{I}) u \frac{d\omega}{r} = \iint_{\omega} v_j (\Delta^- - \xi^2 \text{I}) \left\{ u_R^*(\xi) + \sum_k c_k(\xi) S_k \right\} \frac{d\omega}{r}.$$

But $u_R^*(\xi) \in \Phi_R^-$ and $v_j \in N_{\xi}^-$, so $(\Delta^- - \xi^2 \text{I}) u_R^*(\xi)$ and v_j are orthogonal in $L_{-1}^2(\omega)$. Then Lemma 7.5 implies $I = -2\lambda_j(\xi) c_j(\xi)$. ■

In close analogy with Lemma 2.5.7 in [14], and using techniques similar to that of the proof of Lemma 7.5, we obtain:

Lemma 7.7 *The function w_j defined in Lemma 7.3, resp. 7.4, associated with a reentrant edge E_j , resp. a sharp vertex O_j , satisfies for ξ large enough*

$$\|w_j\|_{0,-1,\omega} \leq K \xi^{\alpha_j-1}, \text{ resp. } \|w_j\|_{0,-1,\omega} \leq K \xi^{\nu_j-1/2}.$$

Theorem 7.8 *There exists a constant K such that the different terms in (96) satisfy, for ξ large enough:*

$$\|u_R(\xi)\|_{\Phi_R^-} + \xi \|u_R(\xi)\|_{1,-1,\omega} + \xi^2 \|u_R(\xi)\|_{0,-1,\omega} \leq K \|f\|_{0,-1,\omega}, \quad (108)$$

$$\text{(for a reentrant edge)} \quad |c_j(\xi)| \leq K (1 + \xi)^{\alpha_j-1} \|f\|_{0,-1,\omega}, \quad (109)$$

$$\text{(for a sharp vertex)} \quad |c_j(\xi)| \leq K (1 + \xi)^{\nu_j-1/2} \|f\|_{0,-1,\omega}. \quad (110)$$

Proof: Let us first prove the estimates for $c_j(\xi)$. One has:

$$c_j(\xi) = -\frac{1}{2\lambda_j(\xi)} \iint_{\omega} f v_j \frac{d\omega}{r} = -\frac{1}{2\lambda_j(\xi)} \iint_{\omega} f (u_j - w_j) \frac{d\omega}{r}.$$

Hence by the Schwarz and triangle inequalities

$$|c_j(\xi)| \leq \frac{1}{2\lambda_j(\xi)} \|f\|_{0,-1,\omega} \times \left\{ \|u_j\|_{0,-1,\omega} + \|w_j\|_{0,-1,\omega} \right\}. \quad (111)$$

The estimate for $\|w_j\|_{0,-1,\omega}$ was obtained in the previous Lemma, and simple calculations show that $\|u_j\|_{0,-1,\omega} = O(\xi^{\alpha_j-1})$ or $O(\xi^{\nu_j-1/2})$. Then (109), resp. (110) follows from (111) and the estimate (104) for $\lambda_j(\xi)$.

Let us now consider : $f_R = -(\Delta^- - \xi^2 \text{I})u_R$. To obtain (108), we shall prove the bound

$$\|f_R\|_{0,-1,\omega} \leq K \|f\|_{0,-1,\omega}. \quad (112)$$

One has

$$f_R = f + (\Delta^- - \xi^2 \text{I}) \sum_{j \in \mathcal{K}_S} c_j(\xi) e^{-\xi \rho_j} S_j,$$

so it is enough to bound each $c_j(\xi) (\Delta^- - \xi^2 \text{I}) \{e^{-\xi \rho_j} S_j\}$. By adapting the proof of [14, Lemma 2.5.9], one can show that:

$$\left\| (\Delta^- - \xi^2 \text{I}) \{e^{-\xi \rho_j} S_j\} \right\|_{0,-1,\omega}^2 = O(\xi^{2-2\alpha_j}) \text{ or } O(\xi^{1-2\nu_j}).$$

Then we use (109) or (110) to estimate

$$\left\| c_j(\xi) (\Delta^- - \xi^2 \text{I}) \{e^{-\xi \rho_j} S_j\} \right\|_{0,-1,\omega}^2 = O(1) \|f\|_{0,-1,\omega}.$$

The estimate (112) follows. On the other hand, there holds:

$$\|(\Delta^- - \xi^2 \text{I})u_R\|_{0,-1,\omega}^2 = \|\Delta^- u_R\|_{0,-1,\omega}^2 - 2\xi^2 (\Delta^- u_R | u_R)_{0,-1,\omega} + \xi^4 \|u_R\|_{0,-1,\omega}^2.$$

According to (37), the scalar product in the second term, with its minus sign, is equal to $\|\mathbf{grad} u_R\|_{0,-1,\omega}^2$, which is a norm equivalent to $\|u_R\|_{1,-1,\omega}^2$. As for the first term, it is a norm equivalent to the canonical norm of Φ_R^- (Lemmas 6.4 or 6.3). Hence, by (112):

$$\|u_R(\xi)\|_{\Phi_R^-}^2 + 2\xi^2 \|u_R(\xi)\|_{1,-1,\omega} + \xi^4 \|u_R(\xi)\|_{0,-1,\omega} \leq K_1 \|f_R\|_{0,-1,\omega}^2 \leq K_2 \|f\|_{0,-1,\omega}^2,$$

which is (108). ■

7.2 Space-time regularity of the solution to the wave-like problem.

We define σ_M and σ_m as the maximum and minimum of the set $\{\alpha_j, j \in \mathcal{K}_{ES}\}$ in the Dirichlet case, and $\{\alpha_j, j \in \mathcal{K}_{ES}; \nu_j + 1/2, j \in \mathcal{K}_{OS}\}$ in the Neumann case.

Lemma 7.9 *Let $\vartheta > (1 + \sigma_M) / 2$; then for any $s < 2\vartheta$:*

$$D_{A^\vartheta} \subset E \equiv \mathbb{R} [H_-^s(\omega)] + \text{span} \{S_j, j \in \mathcal{K}_S\}. \quad (113)$$

Proof: The regularity of the S_j in the scale $\mathbb{R} [H_-^s(\omega)]$ is given by Proposition 3.11. Eqs. (80) and (82) show that the edge functions belong to $H^s(\omega)$ —or $\mathbb{R} [H_-^s(\omega)]$ by a support argument—iff $s < 1 + \alpha_j$. Similarly, the vertex functions S_j belong to $\mathbb{R} [H_-^s(\omega)]$, $1 < s < 2$, iff $S_j/r \in H_-^s(\omega) = \left\{ w \in H_1^s(\Omega) : w|_{\gamma_a} = 0 \right\}$. Since

$$\frac{S_j}{r} = \frac{\rho_j^{1+\nu_j} Y_j(\phi_j)}{\rho_j \sin \phi_j} = \rho_j^{\nu_j} P_{\nu_j}^1(\cos \phi_j), \quad (114)$$

this holds iff $s < \nu_j + 3/2$: the condition $S_j/r|_{\gamma_a} = 0$ is ensured by the Legendre function.

Hence for $1 + \sigma_M < s < 2\vartheta$, the space E is a direct sum, which we equip with the product topology; and it is enough to prove (113) in this case. We follow Lemmas 5.3.2 and 5.3.3 of [14]. First we show that

$$\|(A + tI)^{-1}\|_{H \rightarrow E} = O\left(t^{s/2-1}\right) \quad (115)$$

as $t \rightarrow +\infty$. Let $f \in H$ and $u = (A + tI)^{-1}f$; the decomposition (96) and the estimates (108–109) hold, with $\xi = \sqrt{t}$. One also has the decomposition (95)

$$u = \left[u_R + \sum_{j \in \mathcal{K}_S} c_j(t) \left(e^{-\rho_j \sqrt{t}} - 1 \right) S_j \right] + c_j(t) S_j. \quad (116)$$

As the bracket belongs to $\Phi_R^- \subset \mathbb{R} [H_-^2(\omega)] \subset \mathbb{R} [H_-^s(\omega)]$ (Proposition 6.5), Eq. (116) coincides with the decomposition of u in the direct sum E . Then we use Sobolev injections and interpolation arguments to show that the $\mathbb{R} [H_-^s(\omega)]$ norm of the bracket is $O(t^{s/2-1})$; as $c_j(t) S_j$ decays faster, (115) is proven.

Finally, consider $u \in D_{A^\vartheta}$ and write $u = A^{-\vartheta} A^\vartheta u$, i.e.

$$u = \frac{\sin \pi \vartheta}{\pi} \int_0^{+\infty} t^{-\vartheta} (A + tI)^{-1} A^\vartheta u dt.$$

By (115), the norm of the integrand in E is $O(t^{s/2-1-\vartheta}) \|A^\vartheta u\|_H$, and the integral converges for $s < 2\vartheta$. \blacksquare

The solution to (93) belongs to Φ^- at any time, hence we have the decomposition

$$u = u_R^*(t) + \sum_{j \in \mathcal{K}_S} c_j(t) S_j, \text{ with } u_R^*(t) \in \Phi_R^-. \quad (117)$$

It stems from the continuity of projections onto closed subspaces that

$$u_R^* \in \mathcal{C}^0(0, T; \Phi_R^-), \quad c_j(t) \in \mathcal{C}^0(0, T; \mathbb{R}).$$

Moreover, we have the more precise result of

Theorem 7.10 *The different terms in (117) satisfy*

$$u_R^*(t) \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} \left(0, T; \mathbb{R} \left[H_-^{1+\sigma_M+\delta}(\omega) \right] \right), \quad c_j(t) \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R}), \quad \text{for } \varepsilon > \delta > 0;$$

$$u(t) \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} \left(0, T; \mathbb{R} \left[H_-^{1+\sigma_m-\varepsilon'}(\omega) \right] \right), \quad \text{for } \varepsilon, \varepsilon' > 0.$$

Proof: By Theorem 7.1, $u \in \mathcal{C}^0(0, T; D_A) \cap \mathcal{C}^1(0, T; D_{A^{1/2}})$. By convexity, it follows that $u \in \mathcal{C}^{0,\sigma}(0, T; D_{A^{1-\sigma/2}})$ for $0 \leq \sigma \leq 1$. Now, let $\sigma < 1 - \sigma_M$ and $1 + \sigma_M < s < 2 - \sigma$. Applying Lemma 7.9, we split u on the direct sum $E = \mathbb{R} [H_-^s(\omega)] + \text{span} \{S_j, j \in \mathcal{K}_S\}$.

As $\Phi_R^- \subset \mathbb{R} [H_-^s(\omega)] \subset \mathbb{R} [H_-^s(\omega)]$, the components of u in the two direct sums $\Phi_R^- \oplus \text{span} \{S_j\}$ and $\mathbb{R} [H_-^s(\omega)] \oplus \text{span} \{S_j\}$ are the same. This shows $u_R^*(t) \in \mathcal{C}^{0,\sigma}(0, T; \mathbb{R} [H_-^s(\omega)])$ and $c(t) \in \mathcal{C}^{0,\sigma}(0, T; \mathbb{R})$.

Finally, we notice that u_R^* and all the S_j belong to $\mathbb{R} [H_-^{1+\sigma_m-\varepsilon'}(\omega)]$, hence $u(t) \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} \left(0, T; \mathbb{R} [H_-^{1+\sigma_m-\varepsilon'}(\omega)] \right)$. \blacksquare

8 Analysis of the time-dependent Maxwell equations.

8.1 Reduction to two-dimensional problems and basic regularity results.

We now examine the simplification of the time-dependent Maxwell problem (18–25) induced by the axial symmetry. Similarly to the static problems (see Section 6) there is a decoupling of meridian and azimuthal components: namely, the problem (18–25) is decoupled into two sub-systems:

- the “first system” links the meridian electric field and the azimuthal magnetic field;
- the “second system” links the azimuthal electric and meridian magnetic fields.

Like in Section 6, it is convenient to introduce the product by r of the “natural” variables

$$\mathbf{u} = r \mathcal{E}, \quad \mathbf{v} = r \mathcal{B}, \quad \mathbf{f} = (r/\varepsilon_0) \mathcal{J}, \quad g = r \varrho/\varepsilon_0.$$

The following forms for the two systems are obtained through simple calculations.

The first system. The evolution and constraint equations are:

$$\frac{\partial \mathbf{u}_m}{\partial t} - c^2 \mathbf{curl} v_\theta = -\mathbf{f}_m \text{ in } q, \quad (118)$$

$$\frac{\partial v_\theta}{\partial t} + \mathbf{curl}_- \mathbf{u}_m = 0 \text{ in } q, \quad (119)$$

$$\mathbf{div} \mathbf{u}_m = g \text{ in } q, \quad (120)$$

$$\mathbf{u}_m \cdot \boldsymbol{\nu} = 0 \text{ on } \sigma_a, \quad \mathbf{u}_m \cdot \boldsymbol{\tau} = 0 \text{ on } \sigma_b. \quad (121)$$

The compatibility condition between \mathbf{f}_m and g reads

$$\mathbf{div} \mathbf{f}_m + \frac{\partial g}{\partial t} = 0 \text{ in } q. \quad (122)$$

As for the initial data, they are

$$\mathbf{u}_m(0) = \mathbf{u}_{m0} = r \mathbf{E}_{0m}, \quad v_\theta(0) = v_{\theta0} = r B_{0\theta} \text{ in } \omega, \quad (123)$$

they satisfy

$$\mathbf{div} \mathbf{u}_{m0} = g(0) \text{ in } \omega, \quad \mathbf{u}_{m0} \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma_a, \quad \mathbf{u}_{m0} \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma_b. \quad (124)$$

The second system. The evolution and constraint equations read:

$$\frac{\partial u_\theta}{\partial t} - c^2 \operatorname{curl}_\perp \mathbf{v}_m = -f_\theta \text{ in } q, \quad (125)$$

$$\frac{\partial \mathbf{v}_m}{\partial t} + \operatorname{curl} u_\theta = 0 \text{ in } q, \quad (126)$$

$$\operatorname{div} \mathbf{v}_m = 0 \text{ in } q, \quad (127)$$

$$\mathbf{v}_m \cdot \boldsymbol{\nu} = 0 \text{ on } \sigma, \quad (128)$$

$$u_\theta = 0 \text{ on } \sigma. \quad (129)$$

There is no compatibility condition for this problem. The initial data are:

$$\mathbf{v}_m(0) = \mathbf{v}_{m0} = r \mathbf{B}_{0m}, \quad u_\theta(0) = u_{\theta0} = r E_{0\theta} \text{ in } \omega, \quad (130)$$

they satisfy

$$\operatorname{div} \mathbf{v}_{m0} = 0 \text{ in } \omega, \quad \mathbf{v}_{m0} \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma, \quad u_{\theta0} = 0 \text{ on } \gamma. \quad (131)$$

Basic regularity results. Combining Proposition 2.2 with the results of Sections 3 and 6, we obtain the following regularity result.

$$(\mathbf{u}_m, v_\theta) \in \mathcal{C}^0(0, T; \mathbb{U}^d \times H_{-1}^1(\omega)) \cap \mathcal{C}^1(0, T; L_{-1}^2(\omega)^2 \times L_{-1}^2(\omega)), \quad (132)$$

$$(\mathbf{v}_m, u_\theta) \in \mathcal{C}^0(0, T; \mathbb{U}^{0n} \times \overset{\circ}{H}_{-1}^1(\omega)) \cap \mathcal{C}^1(0, T; L_{-1}^2(\omega)^2 \times L_{-1}^2(\omega)). \quad (133)$$

The rest of this Section is devoted to the improvement of this result, as announced in [12]. For the azimuthal components, there holds by interpolation:

$$(u_\theta, v_\theta) \in \mathcal{C}^{0,1-\sigma}(0, T; \overset{\circ}{H}_{-1}^\sigma(\omega) \times H_{-1}^\sigma(\omega)), \text{ for } 0 < \sigma < 1. \quad (134)$$

We now focus on the meridian components and their splitting, valid at any time, into a regular and a singular part, the latter being chosen within the explicit singular spaces of Subsection 6.3. Notice that *we have slightly changed the notation* with respect to Section 7, setting here $\mathcal{K}_S = \mathcal{K}_{ES} \cup \mathcal{K}_{OS}$.

$$\mathbf{u}_m(t) = \mathbf{u}_R(t) + \sum_{i \in \mathcal{K}_S} \kappa_i(t) \mathbf{S}_i^d, \quad (135)$$

$$\mathbf{v}_m(t) = \mathbf{v}_R(t) + \sum_{j \in \mathcal{K}_{ES}} \delta_j(t) \mathbf{S}_j^{0n}. \quad (136)$$

By the continuity of projections, there holds: $(\mathbf{u}_R, \mathbf{v}_R) \in \mathcal{C}^0(0, T; \mathbb{U}_R^d \times \mathbb{U}_R^{0n})$ and, for $(i, j) \in \mathcal{K}_S \times \mathcal{K}_{ES}$, $(\kappa_i, \delta_j) \in \mathcal{C}^0(0, T; \mathbb{R} \times \mathbb{R})$.

8.2 Regularity in time of the singular coefficients and global space-time regularity of the fields.

The first system. We consider the Hodge decomposition, valid at any time:

$$\mathbf{u}_m(t) = \operatorname{curl} W(t) - r \operatorname{grad} V(t). \quad (137)$$

If we assume: $\forall t, V(t) \in \Phi^{d+}$ and $W(t) \in \Phi^{n-}$, the results of Subsection 6.1 allow us to state that they are uniquely defined and satisfy:

$$(V, W) \in \mathcal{C}^0(0, T; \Phi^{d+} \times \Phi^{n-}) \cap \mathcal{C}^1(0, T; \mathbb{V}^{d+} \times \mathbb{V}^{n-}). \quad (138)$$

Combining (137) with (120–121) shows that, for all t , $V(t)$ is the variational solution to:

$$-\Delta^+ V(t) = \frac{\varrho(t)}{\varepsilon_0} \text{ in } \omega, \quad V(t) = 0 \text{ on } \gamma_b, \quad \frac{\partial V(t)}{\partial \nu} = 0 \text{ on } \gamma_a. \quad (139)$$

Obviously, the time regularity of ϱ and V are related:

Proposition 8.1 *If $\varrho \in C^m(0, T; [\mathbf{V}^{d+}]')$ resp. $W^{s,p}(0, T; [\mathbf{V}^{d+}]')$ or $C^{0,\sigma}(0, T; L^2_1(\omega))$, then $V \in C^m(0, T; \mathbf{V}^{d+})$ resp. $W^{s,p}(0, T; \mathbf{V}^{d+})$ or $C^{0,\sigma}(0, T; \Phi^{d+})$.*

Let us now look for the equation satisfied by $W(t)$. Plugging (137) in (119–118) yields:

$$\frac{\partial v_\theta}{\partial t} - \Delta^- W = 0, \quad (140)$$

$$\mathbf{curl} \left(\frac{\partial W}{\partial t} - c^2 v_\theta \right) = -\mathbf{f}_m + \frac{\partial}{\partial t} (r \mathbf{grad} V). \quad (141)$$

The right-hand side in (141) is divergence-free, thanks to (122) and (139). Hence, there exists a function χ such that $\mathbf{curl} \chi = -\mathbf{f}_m + \partial_t (r \mathbf{grad} V)$. But the left-hand side of (141), by (138) and (132), is at any time in $L^2_{-1}(\omega)$. Hence, it has a unique potential in $H^1_{-1}(\omega)$, and we can choose $\chi = \partial_t W - c^2 v_\theta$. Combining this equation with (140) yields:

$$\frac{\partial^2 W}{\partial t^2} - c^2 \Delta^- W = \frac{\partial \chi}{\partial t} \stackrel{def}{=} \psi. \quad (142)$$

Proposition 8.2 *Assume $\mathbf{j}_m \in W^{1,1}(0, T; \check{\mathbf{L}}^2(\Omega))$ —hence $\mathbf{f}_m \in W^{1,1}(0, T; L^2_{-1}(\omega)^2)$. Then there exists a strong solution $W(t) \in C^1(0, T; \mathbf{V}^{n-}) \cap C^0(0, T; \Phi^{n-})$ to the evolution equation (142), supplemented with the boundary and initial conditions*

$$\begin{cases} W = 0 \text{ on } \sigma_a, \quad \partial_\nu W = 0 \text{ on } \sigma_b, \\ W(0) = W_0, \quad W'(0) = W_1 \text{ in } \omega, \end{cases} \quad (143)$$

where the initial conditions satisfy:

$$W_0 \in \Phi^{n-}, \quad W_1 \in H^1_{-1}(\omega), \quad \mathbf{curl} W_0 - r \mathbf{grad} V(0) = \mathbf{u}_{m0}, \quad W_1 = c^2 v_{\theta 0} + \chi(0). \quad (144)$$

Proof: $\mathbf{j}_m \in W^{1,1}(0, T; \check{\mathbf{L}}^2(\Omega))$ implies $\partial_t \varrho \in W^{1,1}(0, T; \check{H}^{-1}(\Omega))$ by (24) and $\partial_t V \in W^{1,1}(0, T; \mathbf{V}^{d+})$ by Proposition 8.1. Hence, $\mathbf{curl} \chi = -\mathbf{f}_m + \partial_t (r \mathbf{grad} V) \in W^{1,1}(0, T; L^2_{-1}(\omega)^2)$, $\chi \in W^{1,1}(0, T; H^1_{-1}(\omega))$, and finally $\psi \in L^1(0, T; H^1_{-1}(\omega))$. Hence the existence of W by Theorem 7.1. Now (144) is clear, since χ and $\partial_t (r \mathbf{grad} V)$, as $W^{1,1}$ functions of time, are continuous up to $t = 0$, like \mathbf{u}_m and v_θ . ■

Conversely, checking that V and W defined respectively by (139) and (142–143) satisfy (137), provided (144) holds, is straightforward. Thus, we can apply the results of Section 7. We set:

$$V(t) = V_R(t) + \sum_{i \in \mathcal{K}_S} \kappa_i^d(t) \{ \lambda_i^d S_i^{d+} \} \quad (145)$$

$$W(t) = W_R(t) + \sum_{i \in \mathcal{K}_S} \kappa_i^n(t) \{ \lambda_i^n S_i^{n-} \} \quad (146)$$

As stated in Subsection 6.3, we can choose the constants λ_i^d and λ_i^n so as to have

$$\begin{aligned} -r \mathbf{grad} \{ \lambda_i^d S_i^{d+} \} + \mathbf{curl} \{ \lambda_i^n S_i^{n-} \} &= 2 \mathbf{S}_i^d + 2 \mathbf{w}_i^+, \quad \mathbf{w}_i^+ \in \mathbf{U}_R^d, \\ -r \mathbf{grad} \{ \lambda_i^d S_i^{d+} \} - \mathbf{curl} \{ \lambda_i^n S_i^{n-} \} &= 2 \mathbf{w}_i^- \in \mathbf{U}_R^d. \end{aligned}$$

Combining (137) with (145–146), we find

$$\begin{aligned} \mathbf{u}_m(t) &= \mathbf{curl} W_R(t) - r \mathbf{grad} V_R(t) + \sum_{i \in \mathcal{K}_S} \left\{ \left(\kappa_i^d(t) - \kappa_i^n(t) \right) \mathbf{w}_i^- + \left(\kappa_i^d(t) + \kappa_i^n(t) \right) \mathbf{w}_i^+ \right\} \\ &\quad + \sum_{i \in \mathcal{K}_S} \left(\kappa_i^d(t) + \kappa_i^n(t) \right) \mathbf{S}_i^d. \end{aligned}$$

Comparing this equation to (135), we infer:

$$\mathbf{u}_R(t) = \mathbf{curl} W_R(t) - r \mathbf{grad} V_R(t) + \sum_{i \in \mathcal{K}_S} \left\{ \left(\kappa_i^d(t) - \kappa_i^n(t) \right) \mathbf{w}_i^- + \kappa_i(t) \mathbf{w}_i^+ \right\} \quad (147)$$

$$\kappa_i(t) = \kappa_i^d(t) + \kappa_i^n(t), \quad \forall i \in \mathcal{K}_S. \quad (148)$$

Theorem 8.3 *Assume that the sources enjoy the following space-time regularity:*

$$\varrho \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; L_1^2(\omega)), \quad \forall \varepsilon > 0, \quad \text{and } \mathbf{f}_m \in W^{1,1} (0, T; L_{-1}^2(\omega)^2).$$

Then the following regularity results hold:

$$\kappa_i \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R}), \quad \forall \varepsilon > 0, \quad (149)$$

$$\mathbf{u}_m \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega)^2), \quad \forall \varepsilon, \varepsilon' > 0. \quad (150)$$

Here, σ_M and σ_m are meant as in the Neumann case.

Proof: The projection onto closed subspaces, such as Φ_R^{d+} , $\text{span } S_i^{n-}$, etc., is smooth. Hence, the assumed regularity of ϱ , together with Proposition 8.1, yields:

$$V_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \Phi_R^{d+}), \quad \kappa_i^d \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R}),$$

and Theorem 7.10 implies:

$$W_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R} [H_-^{1+\sigma_M+\delta}(\omega)]), \quad \kappa_i^n \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R}).$$

So, under the above hypotheses, $\kappa_i \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R})$.

Moreover, we know from Theorem 6.6 that for all t , $\mathbf{u}_m(t) \in H_{-1}^{\sigma_m-\varepsilon'}(\omega)$; and this space regularity is optimal. Since $\Phi_R^{d+} \subset H_+^2(\omega) \subset H_+^{1+\sigma_m-\varepsilon'}(\omega)$, one has

$$-r \mathbf{grad} V_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R} [H_1^{\sigma_m-\varepsilon'}(\omega)^2]) = \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega)^2),$$

by Proposition 3.10. The same Proposition yields:

$$W_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \mathbb{R} [H_-^{1+\sigma_m-\varepsilon'}(\omega)]) \subset \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; H_{-1}^{1+\sigma_m-\varepsilon'}(\omega)),$$

thus $\mathbf{curl} W_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega)^2)$. Then (150) follows from (147). \blacksquare

Corollary 8.4 *Under the hypotheses of the above Theorem, there holds:*

$$\mathcal{E} \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; \check{\mathbf{H}}^{\sigma_m-\varepsilon'}(\Omega)) \quad (151)$$

Proof: It stems from (150) that $\mathbf{E}_m \in \mathcal{C}^{0,1-\sigma_M-\varepsilon} (0, T; H_1^{\sigma_m-\varepsilon'}(\omega)^2)$, where $H_1^{\sigma_m-\varepsilon'}(\omega)^2 = H_-^{\sigma_m-\varepsilon'}(\omega) \times H_+^{\sigma_m-\varepsilon'}(\omega)$ is the meridian component of $\check{\mathbf{H}}^{\sigma_m-\varepsilon'}(\Omega)$. On the other hand, the azimuthal component u_θ satisfies (134); taking $\sigma = \sigma_m - \varepsilon'$ yields $u_\theta \in \mathcal{C}^{0,1-\sigma_m+\varepsilon'} (0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega))$ or $E_\theta \in \mathcal{C}^{0,1-\sigma_m+\varepsilon'} (0, T; H_-^{\sigma_m-\varepsilon'}(\omega))$. The conclusion follows. \blacksquare

The second system. According to Subsection 6.2, we set at any time

$$\mathbf{v}_m(t) = \mathbf{curl} \varphi(t), \quad (152)$$

with $\varphi(t) \in \Phi^{d-}$. It stems from the regularity (133) of \mathbf{v}_m , and Section 6.2 that

$$\varphi \in \mathcal{C}^0(0, T; \Phi^{d-}) \cap \mathcal{C}^1(0, T; \mathbb{V}^{d-}). \quad (153)$$

Let us look for the equation satisfied by φ . It follows from (126) that $\mathbf{curl}(\partial_t \varphi + u_\theta) = 0$. Since $u_\theta \in \mathbb{V}^{d-}$, we infer $u_\theta = -\partial_t \varphi$, and (125) becomes:

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \Delta^- \varphi = f_\theta. \quad (154)$$

Given (153), Theorem 7.1 implies:

Proposition 8.5 *If $j_\theta \in L^1(0, T; H_-^1(\omega))$ and $j_\theta|_\sigma = 0$ —hence $f_\theta \in L^1(0, T; \mathring{H}_{-1}^1(\omega))$ — φ is the strong solution in $\mathcal{C}^0(0, T; \Phi^{d-}) \cap \mathcal{C}^1(0, T; \mathbb{V}^{d-})$ to the evolution equation (154), supplemented with the initial and boundary conditions:*

$$\begin{cases} \varphi = 0 \text{ on } \sigma, \\ \varphi(0) = \varphi_0, \varphi'(0) = \varphi_1 \text{ in } \omega, \end{cases} \quad (155)$$

where the initial conditions satisfy:

$$\varphi_0 \in \Phi^{d-}, \quad \varphi_1 \in \mathbb{V}^{d-}, \quad \mathbf{curl} \varphi_0 = \mathbf{v}_{m0}, \quad \varphi_1 = -u_{\theta 0}. \quad (156)$$

The proof is similar to Proposition 8.2, and simpler. Conversely, checking that φ solution to (154–155) satisfies (152), provided (156) holds, is straightforward. To apply the results of Section 7, we set

$$\varphi(t) = \varphi_R(t) + \sum_{j \in \mathcal{K}_{ES}} \delta_j(t) S_j^{d-}. \quad (157)$$

We recall that $\mathbf{curl} S_j^{d-} = \mathbf{S}_j^{0n}$. Comparing (152) and (157), one sees that the singular coefficients $\delta_j(t)$ are indeed the same as in (136), and that $\mathbf{v}_R(t) = \mathbf{curl} \varphi_R(t)$.

Theorem 8.6 *Assume that the current f_θ belongs to $W^{1,1}(0, T; \mathbb{V}^{d-})$. Then the following regularity results hold:*

$$\delta_j \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathbb{R}), \quad \forall \varepsilon > 0, \quad (158)$$

$$\mathbf{v}_m \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega)^2\right), \quad \forall \varepsilon, \varepsilon' > 0 \quad (159)$$

Here, σ_M and σ_m are meant as in the Dirichlet case.

Proof: By Theorem 7.10:

$$\varphi_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; \mathbb{R}\left[H_-^{1+\sigma_M+\delta}(\omega)\right]\right), \quad \delta_j \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathbb{R}).$$

Moreover, we know from Theorem 6.6 that for all t , $\mathbf{v}_m(t) \in H_{-1}^{\sigma_m-\varepsilon'}(\omega)$; again, this space regularity is optimal. But:

$$\varphi_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; \mathbb{R}\left[H_-^{1+\sigma_m-\varepsilon'}(\omega)\right]\right) \subset \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; H_{-1}^{1+\sigma_m-\varepsilon'}(\omega)\right),$$

by Proposition 3.10; hence $\mathbf{curl} \varphi_R \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; H_{-1}^{\sigma_m-\varepsilon'}(\omega)^2\right)$. As δ_j is an element of $\mathcal{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathbb{R})$, this implies (159). \blacksquare

Corollary 8.7 *Under the hypotheses of the above Theorem, there holds:*

$$\mathcal{B} \in \mathcal{C}^{0,1-\sigma_M-\varepsilon}\left(0, T; \check{\mathbf{H}}\sigma_m - \varepsilon'(\Omega)\right) \quad (160)$$

Proof: Similar to Corollary 8.4. \blacksquare

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