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On the harmonic Boltzmannian waves in laser–plasma interaction

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Abstract
We study the permanent regimes of the reduced Vlasov–Maxwell system for laser–plasma interaction. A non-relativistic and two different relativistic models are investigated. We prove the existence of solutions where the distribution function is Boltzmannian and the electromagnetic variables are time-harmonic and circularly polarized.

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1. Introduction

The in-depth understanding of laser–plasma interaction is not only of paramount importance for the eventual success of inertial confinement fusion research, but also interesting for magnetic confinement fusion research, since tokamak plasmas can be heated by electromagnetic waves. The complex kinetic phenomena involved in this interaction, and the instabilities they may generate [1], need to be studied by kinetic models [2], even though hydrodynamic models [3] are more affordable to simulate complex, high-dimensional geometries. However, the use of the full 3D Vlasov–Maxwell system is of course impossible in most practical situations. Therefore, the *reduced Vlasov–Maxwell system for laser–plasma interaction* (hereafter called the 'laser–plasma system'; see (1)–(3) below) was introduced in [2]. The model has been shown to capture some essential features of this interaction [2, 1], and it has been successfully used for deriving relevant physical models in novel situations [4].

The laser–plasma system has been the object of several mathematical investigations [5–7]. In this framework, it is interesting to find classes of exact solutions which may serve as 'reference solutions', to which other solutions may be compared in order to study the dynamic of the interaction. Reference solutions for the Vlasov–Poisson system are, for instance, the Bernstein–Greene–Kruskal or *BGK modes* [8], given by a distribution function of the form $f(W)$, where W is the energy of one particle. When the function f is convex, as in the Maxwellian case $f(W) \propto \mathrm{e}^{-W/\theta}$, such solutions represent fundamental equilibrium states; their existence and stability are well known [9]. For a general f, BGK solutions may represent various wave phenomena; they have been the object of many investigations in the physical community [10], and references therein. The mathematical theory is still less developed; interesting existence and (in)stability results have appeared recently [10, 11]. For both Vlasov–Poisson and Vlasov–Maxwell systems, there are also linearized solutions leading to the dispersion relations of the various types of waves; e.g. for electromagnetic waves $\omega^2 = \omega_{\mathrm{p}}^2 + k^2$, where (ω, k) are the pulsation and wave number, and ω_{p} is the plasma pulsation.

In this paper, we shall introduce a class of exact solutions to the laser–plasma system which generalizes, at the same time, Maxwellian equilibria and linear electromagnetic waves. Indeed, we investigate the existence of quasi-static solutions where the distribution function is at any time proportional to the Boltzmann factor; this static character can be reconciled with the electromagnetic character of the system by assuming a harmonic time dependence of the electromagnetic field and a circular polarization. This ansatz was already used in [4], but in a different physical and mathematical context. The latter work investigates the existence of solitons in an electron–positron plasma, where no charge separation occurs. Here we are dealing with a general ion–electron plasma, and we are looking for space periodic solutions.

The paper is organized as follows. We recall the mathematical results known about the laser–plasma system and introduce the quasi-static model in section 2. Then, in section 3 we solve (in the space periodic setting) the so-called *Boltzmann problem*, which consists in finding the equilibrium density given the electromagnetic potentials, and we estimate its solutions. In section 4 we construct a fixed point application and we study its properties. The existence of Boltzmaniann equilibria then follows by applying the Schauder fixed point theorem. Several extensions of the model are briefly discussed in section 5, and we conclude in section 6.

2. The harmonic Boltzmannian model

The reduced Vlasov–Maxwell system for laser–plasma interaction describes the evolution of the distribution function of a population of electrons in a one space dimensional plasma interacting with a laser wave. In a first approach, we assume that the ions are at rest and their density is given—which is physically acceptable at the time scale of a laser wave. After a suitable rescaling [5], this system can be cast in the following form:

$$\frac{\partial f}{\partial t} + \frac{p}{\gamma_1}\frac{\partial f}{\partial x} - \left(E(t,x) + \frac{\boldsymbol{A}(t,x)}{\gamma_2}\cdot\frac{\partial \boldsymbol{A}}{\partial x}\right)\frac{\partial f}{\partial p} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = \rho_{\mathrm{b}}(x) - \rho(t,x), \qquad \frac{\partial E}{\partial t} - j(t,x) = 0, \tag{2}$$

$$\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{\partial^2 \boldsymbol{A}}{\partial x^2} + \tilde{\rho}(t,x)\boldsymbol{A}(t,x) = 0, \tag{3}$$

where $f(t,x,p)$ is the electron distribution function (p denotes the x-component of the momentum vector); E is the x-component if the electric field; $\boldsymbol{A} = (0, A_y, A_z)$ is the vector potential of the laser wave; $\rho_{\mathrm{b}}(x)$ is the (static) background ion density; γ_1, γ_2 are Lorentz factors. We distinguish three cases:

(i) the non-relativistic case (NR), $\gamma_1 = \gamma_2 = 1$;
(ii) the quasi-relativistic case (QR), $\gamma_1 = (1+p^2)^{1/2}$, $\gamma_2 = 1$;
(iii) the fully relativistic case (FR), $\gamma_1 = \gamma_2 = (1+p^2+|\boldsymbol{A}|^2)^{1/2}$, which is the original model of [2].

The moments ρ, $\tilde{\rho}$, j are given by

$$\rho(t,x) := \int_{\mathbb{R}} f(t,x,p)\,\mathrm{d}p, \quad \tilde{\rho}(t,x) := \int_{\mathbb{R}} \frac{f(t,x,p)}{\gamma_2}\,\mathrm{d}p, \quad j(t,x) := \int_{\mathbb{R}} \frac{p}{\gamma_1} f(t,x,p)\,\mathrm{d}p. \tag{4}$$

We supplement the system (1)–(3) with initial conditions

$$\begin{aligned} &f(0,x,p) = f_0(x,p), && (x,p)\in\mathbb{R}^2, \\ &(E,\boldsymbol{A},\partial_t\boldsymbol{A})(0,x) = (E_0,\boldsymbol{A}_0,\boldsymbol{A}_1), && x\in\mathbb{R}. \end{aligned} \tag{5}$$

In [5] it was proved that, for suitable initial conditions, (1)–(5) has a unique classical solution, which is global in time in the QR case, and local in time in the NR case. In the latter case, the classical solution can be extended to a global weak solution with f continuous and $\boldsymbol{A}$ continuously differentiable in all their variables. The FR model was studied in [6]. It was shown that (1)–(5) admits a unique global classical solution preserving the total energy. The stationary solutions of these models in a bounded domain have been analysed in [7].

All three models admit space periodic solutions. If the initial data are L-periodic in x and satisfy the neutrality condition

$$\int_0^L \int_{\mathbb{R}} f_0(x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_0^L \rho_{\mathrm{b}}(x)\,\mathrm{d}x =: M, \tag{6}$$

then, by using the continuity equation $\partial_t\rho + \partial_x j = 0$, we deduce that the system remains globally neutral at any time $t > 0$,

$$\int_0^L \int_{\mathbb{R}} f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_0^L \int_{\mathbb{R}} f_0(x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_0^L \rho_{\mathrm{b}}(x)\,\mathrm{d}x. \tag{7}$$

By uniqueness of the solution one gets also that $(f(t), E(t), \boldsymbol{A}(t))$ are L-periodic in space for any $t > 0$. From now on, we work in the framework of periodic functions: all differential equations will be implicitly supplemented with L-periodic boundary conditions.

From (7) we deduce the existence of a unique function $V = V(t,x)$, satisfying $\partial_x^2 V(t,x) = \rho_{\mathrm{b}}(x) - \rho(t,x)$, $V(t,0) = 0$ and $(V,\partial_x V)(t,x) = (V,\partial_x V)(t,x+L)$, for all $(t,x)\in[0,+\infty)\times\mathbb{R}$. The field E derives from the potential V, i.e., $E = \partial_x V$.

The purpose of this paper is to study the existence of particular solutions of (1)–(3) corresponding to local Boltzmannian equilibria. These are defined by $f(t,x,p) \propto \mathrm{e}^{-W(t,x,p)/\theta}$, where $W(t,x,p)$ is the energy of one particle being at the phase space point (x,p) at time t, and θ is the scaled temperature. As it is well known, such functions are solutions to the Vlasov equation (1) iff W is independent of time. Thus, we assume that V does not depend on t, and that $\boldsymbol{A}$ is time-harmonic and circularly polarized, i.e.,

$$A_y(t,x) + \mathrm{i}A_z(t,x) = a(x)\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \text{with } a\ priori \quad a(x)\in\mathbb{C}.$$

Then the energy $W(x,p)$ is given, according to the relativistic character, by

$$\begin{aligned} &W(x,p) = \tfrac{1}{2}(p^2 + |a(x)|^2) + V(x), && \text{in the NR case,} \\ &W(x,p) = \sqrt{1+p^2} + \tfrac{1}{2}|a(x)|^2 + V(x), && \text{in the QR case,} \\ &W(x,p) = \sqrt{1+p^2+|a(x)|^2} + V(x), && \text{in the FR case.} \end{aligned}$$

Imposing the constraint (6) yields

$$f(x,p) = M\frac{e^{-W(x,p)/\theta}}{\int_0^L\int_{\mathbb{R}} e^{-W(y,q)/\theta}\,dq\,dy}, \qquad \forall (x,p)\in\mathbb{R}^2. \tag{8}$$

By direct computation we check that in all three cases f solves the Vlasov equation (1). We then observe that $j(x) = \int_{\mathbb{R}} \frac{p}{\gamma_1} f(x,p)\,dp = 0$, for $x \in \mathbb{R}$, and thus the system (1)–(3) reduces to

$$V''(x) = \rho_b(x) - \rho(x), \qquad x \in \mathbb{R}, \tag{9}$$

$$-\omega^2 a(x) - a''(x) = -\tilde{\rho}(x)a(x), \qquad x\in\mathbb{R}, \tag{10}$$

with $\rho = \int_{\mathbb{R}} f\,dp$, $\tilde{\rho} = \int_{\mathbb{R}} \frac{f}{\gamma_2}\,dp$ and f given by (8).

Of course, we are interested in solutions such that $a \not\equiv 0$, otherwise we find a Vlasov–Poisson equilibrium. If such a solution exists, a appears as an eigenfunction of the operator $A_{\tilde{\rho}} := -\frac{d^2}{dx^2} + \tilde{\rho}(x)$, associated with the eigenvalue ω^2. It is well known that these eigenvalues are real and generically simple; in particular, the lowest eigenvalue is always simple. As the coefficients of $A_{\tilde{\rho}}$ are real, we infer that both $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are eigenfunctions; thus, generically, they must be proportional. In other words, $a(x) = \mathsf{a}(x)\,e^{i\varphi}$, where a is a *real* eigenfunction and $\varphi\in\mathbb{R}$. Then, $|a(x)|^2 = \mathsf{a}(x)^2$, and $W, f, \rho, \tilde{\rho}$ are defined in terms of a; while we may take $\varphi = 0$ by rotating the axes Oy, Oz. This means that, without loss of generality, we may restrict our search to *real* functions a solution to (10).

We now rewrite the model (9), (10) in a form which will prove more convenient for analysis. We shall denote by the subscript # the spaces of L-periodic functions, e.g.: $L^1_\#(\mathbb{R}) := \left\{g \in L^1_{loc}(\mathbb{R}) : \forall x, g(x+L) = g(x)\right\}$, $C^0_\#(\mathbb{R}) := \{w \in C^0(\mathbb{R}) : \forall x, w(x+L) = w(x)\}$. First, we introduce the operator $\Phi : L^1_\#(\mathbb{R}) \to C^0_\#(\mathbb{R})$ given by

$$\Phi[g] = w \in C^0_\#(\mathbb{R}), \quad -w''(x) = g(x), \quad x\in(0,L), \quad w(0)=w(L)=0,$$

for any $g \in L^1_\#(\mathbb{R})$. Then, we consider the function $\psi : \mathbb{R}\to\mathbb{R}$ given by

$$e^{-\psi(x)/\theta} = \int_{\mathbb{R}} \exp -\frac{W(x,p)-V(x)}{\theta}\,dp, \quad x\in\mathbb{R},$$

namely, according to the relativistic character

$$\text{NR:}\quad e^{-\psi(x)/\theta} = e^{-a(x)^2/2\theta}\int_{\mathbb{R}} e^{-p^2/2\theta}\,dp, \tag{11}$$

$$\text{QR:}\quad e^{-\psi(x)/\theta} = e^{-a(x)^2/2\theta}\int_{\mathbb{R}} \exp -\frac{\sqrt{1+p^2}}{\theta}\,dp, \tag{12}$$

$$\text{FR:}\quad e^{-\psi(x)/\theta} = \int_{\mathbb{R}} \exp -\frac{\sqrt{1+p^2+a(x)^2}}{\theta}\,dp. \tag{13}$$

Note that there is a constant $C(\theta)\in\mathbb{R}$ such that

$$\psi(x) = \frac{a(x)^2}{2} + C(\theta), \text{ in the NR and QR cases.} \tag{14}$$

In the FR case, by observing that

$$\tfrac{1}{2}(\sqrt{1+p^2} + |a(x)|) \leqslant \sqrt{1+p^2+a(x)^2} \leqslant \sqrt{1+p^2} + |a(x)|,$$

we obtain

$$C_2(\theta)\,e^{-|a(x)|/\theta} \leqslant e^{-\psi(x)/\theta} \leqslant C_1(\theta)\,e^{-|a(x)|/2\theta},$$

with

$$C_1(\theta) := \int_{\mathbb{R}} \exp -\frac{\sqrt{1+p^2}}{2\theta}\,\mathrm{d}p > \int_{\mathbb{R}} \exp -\frac{\sqrt{1+p^2}}{\theta}\,\mathrm{d}p =: C_2(\theta).$$

Finally, one gets

$$\frac{|a(x)|}{2} \leqslant \psi(x) + \theta \ln C_1(\theta) \leqslant |a(x)| + \theta \ln \frac{C_1(\theta)}{C_2(\theta)}. \tag{15}$$

The density ρ can be expressed in function of ψ, and the system (8)–(10) can be recast as

$$\begin{aligned} &f(x,p) = K\,\mathrm{e}^{-W(x,p)/\theta}, \\ &\rho(x) = K\,\mathrm{e}^{-\frac{\psi(x)+V(x)}{\theta}}, \quad x \in \mathbb{R}, \quad V = \Phi[\rho - \rho_{\mathrm{b}}], \end{aligned} \tag{16}$$

$$\tilde{\rho}(x) = K \int_{\mathbb{R}} \frac{1}{\gamma_2}\,\mathrm{e}^{-W(x,p)/\theta}\,\mathrm{d}p, \tag{17}$$

$$-\omega^2 a(x) - a''(x) = -\tilde{\rho}(x)a(x), \tag{18}$$

where the constant $K = M\big(\int_0^L\int_{\mathbb{R}} \mathrm{e}^{-W(y,q)/\theta}\,\mathrm{d}q\,\mathrm{d}y\big)^{-1}$ is such that $\int_0^L \rho(x)\,\mathrm{d}x = M$. We call this system the *Boltzmann–Helmholtz equations*; they can be seen as a sort of nonlinear eigenvalue problem.

3. The Boltzmann problem

For the moment we suppose that the function ψ is given and we solve the so-called *Boltzmann problem* (16). The proof of the following proposition is immediate and left to the reader.

Proposition 1. *For any function $g \in L^1_\#(\mathbb{R})$ we have*

$$\|\Phi[g]\|_{L^\infty(\mathbb{R})} \leqslant L\|g\|_{L^1(0,L)}.$$

If the function g satisfies $\int_0^L g(x)\,\mathrm{d}x = 0$, then $\Phi[g] \in C^1_\#(\mathbb{R})$ and we have

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x}\Phi[g] \right\|_{L^\infty(\mathbb{R})} \leqslant \|g\|_{L^1(0,L)}.$$

Proposition 2. *Let $\psi \in L^\infty_\#(\mathbb{R})$, $u_{\mathrm{b}} \in L^1_\#(\mathbb{R})$, $u_{\mathrm{b}} \geqslant 0$, $M = \int_0^L u_{\mathrm{b}}(x)\,\mathrm{d}x$ and $\theta > 0$. Then there is a unique function $u \in L^1_\#(\mathbb{R})$ such that*

$$u = M \frac{\exp -\frac{\psi + \Phi[u-u_{\mathrm{b}}]}{\theta}}{\int_0^L \exp -\frac{\psi(y)+\Phi[u-u_{\mathrm{b}}](y)}{\theta}\,\mathrm{d}y}. \tag{19}$$

Moreover, it satisfies

$$0 \leqslant u \leqslant \inf_{C\in\mathbb{R}} \frac{M}{L} \exp \frac{1}{\theta}\left(\frac{1}{L}\int_0^L (\psi(y) - C)\,\mathrm{d}y - \inf_{\mathbb{R}}(\psi - C) + 4LM\right) =: u_\psi, \tag{20}$$

and if $\psi \in W^{1,\infty}(\mathbb{R})$ then $u \in W^{1,\infty}(\mathbb{R})$ and

$$\mathrm{Lip}\, u \leqslant \frac{\mathrm{Lip}\,\psi + 2M}{\theta} u_\psi. \tag{21}$$

Proof. One readily checks that (19) is equivalent to the minimization of the functional

$$J[v] := \int_0^L \left\{\theta\sigma(v(x)) + \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}x}\Phi[v - u_{\mathrm{b}}]\right|^2 + \psi(x)v(x)\right\}\mathrm{d}x,$$

under the constraint $\int_0^L v(x)\,\mathrm{d}x = M$, where $\sigma(s) = s\ln s, s > 0$ and $\sigma(0) = 0$. This problem is a variant of that considered in [9, 12] and its well posedness follows from a similar argument. The functional J is strictly convex, l.s.c. and bounded from below on the set

$$\mathcal{K}(L,M) = \left\{ v \in L^1_\#(\mathbb{R}) : v \geqslant 0, \int_0^L v(x)\,\mathrm{d}x = M \right\}.$$

Indeed, by applying the Jensen inequality with the convex function σ, the measure $\mathrm{d}\mu = \mathrm{e}^{-\psi(x)/\theta}\left(\int_0^L \mathrm{e}^{-\psi(y)/\theta}\,\mathrm{d}y\right)^{-1}\mathrm{d}x$ and the function $v/\mathrm{e}^{-\psi/\theta}$, one gets

$$J[v] \geqslant \int_0^L \{\theta\sigma(v(x)) + \psi(x)v(x)\}\,\mathrm{d}x \geqslant \theta M \ln\left[M \left(\int_0^L \mathrm{e}^{-\psi(y)/\theta}\,\mathrm{d}y \right)^{-1} \right],$$

saying that $\inf_{v\in\mathcal{K}(L,M)} J[v] > -\infty$. Take a minimizing sequence $(u_n)_n$. By using the Dunford–Pettis criterion we can assume (after a suitable extraction) that $(u_n)_n$ converges weakly in $L^1(0,L)$ towards a function $u \in \mathcal{K}(L,M)$. Since J is convex we can pass to the limit by involving the semi-continuity of J and we obtain that $J[u] = \inf_{v\in\mathcal{K}(L,M)} J[v]$. Writing the Euler–Lagrange equation we obtain

$$\theta(1 + \ln u) + \Phi[u - u_\mathrm{b}] + \psi - \alpha = 0,$$

where α enters as the Lagrange multiplier associated with the constraint $\int_0^L u(x)\,\mathrm{d}x = M$ and thus we deduce (19). By using now the Jensen inequality with the convex function $t \mapsto \mathrm{e}^{-t}$, the measure $\mathrm{d}\mu = L^{-1}\,\mathrm{d}x$ and the function $(\psi + \Phi[u - u_\mathrm{b}])/\theta$ we obtain

$$\exp\left[-\frac{1}{L}\int_0^L \frac{\psi + \Phi[u-u_\mathrm{b}]}{\theta}\,\mathrm{d}y \right] \leqslant \int_0^L \exp\left(-\frac{\psi + \Phi[u-u_\mathrm{b}]}{\theta} \right)\frac{\mathrm{d}y}{L}.$$

Therefore, by using proposition 1 we deduce

$$\begin{aligned}\left(\int_0^L \exp\left(-\frac{\psi + \Phi[u-u_\mathrm{b}]}{\theta} \right)\mathrm{d}y \right)^{-1} &\leqslant \frac{1}{L}\exp\frac{1}{L}\int_0^L \exp\left(-\frac{\psi + \Phi[u-u_\mathrm{b}]}{\theta} \right)\mathrm{d}y \\ &\leqslant \frac{1}{L}\exp\frac{1}{\theta}\left(\frac{1}{L}\int_0^L \psi\,\mathrm{d}y + 2LM \right),\end{aligned}$$

which implies

$$0 \leqslant u(x) \leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{L}\int_0^L \psi(y)\,\mathrm{d}y - \psi(x) + 4LM \right).$$

This inequality is unchanged by replacing ψ with $\psi - C$, for any $C \in \mathbb{R}$; one thus infers (20). Assume now that $\psi \in W^{1,\infty}(\mathbb{R})$. By taking the derivative with respect to x in (19) one gets by using proposition 1

$$|u'(x)| = |u(x)| \left| \frac{\psi'(x) + \frac{\mathrm{d}}{\mathrm{d}x}\Phi[u-u_\mathrm{b}]}{\theta} \right| \leqslant \|u\|_{L^\infty(\mathbb{R})} \frac{\|\psi'\|_{L^\infty(\mathbb{R})} + 2M}{\theta},$$

and (21) follows immediately. □

4. The fixed point application

For any $\mathfrak{a} > 0$ we define the fixed point application $\mathcal{F}_\mathfrak{a} : W^{1,\infty}_\#(\mathbb{R}) \to W^{1,\infty}_\#(\mathbb{R})$, $\mathcal{F}_\mathfrak{a} a = \tilde{a}$ for any $a \in W^{1,\infty}_\#(\mathbb{R})$, where

- ψ is given, according to the case, by (11), (12) or (13);

- ρ is the unique solution to the Boltzmann problem

$$\rho = K \exp\left(-\frac{\psi + \Phi[\rho - \rho_b]}{\theta}\right), \qquad \int_0^L \rho(x)\,\mathrm{d}x = M;$$

- $\tilde{\rho} = \rho$ in the NR and QR cases, while in the FR case

$$\tilde{\rho}(x) = \frac{M}{\int_0^L \exp\left(-\frac{\psi(y)+\Phi[\rho-\rho_b](y)}{\theta}\right)\mathrm{d}y} \int_{\mathbb{R}} \frac{\exp\left(-\frac{\sqrt{1+p^2+a(x)^2}+\Phi[\rho-\rho_b](x)}{\theta}\right)}{\sqrt{1+p^2+a(x)^2}}\,\mathrm{d}p;$$

- λ is the first eigenvalue of the operator $A_{\tilde{\rho}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{\rho}$ with L-periodic boundary conditions, i.e.,

$$\lambda = \inf_{b\in H^1_\#(\mathbb{R}), b\neq 0} \frac{\int_0^L b'(x)^2 + \tilde{\rho}(x) b(x)^2\,\mathrm{d}x}{\int_0^L b(x)^2\,\mathrm{d}x}; \tag{22}$$

- $\tilde{a}$ is the corresponding eigenfunction of $A_{\tilde{\rho}}$:

$$-\tilde{a}''(x) + \tilde{\rho}(x)\tilde{a}(x) = \lambda \tilde{a}(x), \qquad x \in (0, L), \tag{23}$$

$$\tilde{a}(0) = \tilde{a}(L), \qquad \tilde{a}'(0) = \tilde{a}'(L), \tag{24}$$

such that $\tilde{a} > 0$ and $\int_0^L \tilde{a}(x)^2\,\mathrm{d}x = \mathfrak{a}^2$.

Remark 1. It is well known that the first eigenvalue of $A_{\tilde{\rho}}$ with L-periodic boundary conditions is simple [13] and that the eigenfunction vanishes nowhere. Therefore, $\tilde{a} = \mathcal{F}_{\mathfrak{a}} a$ is well defined.

The properties of the application $\mathcal{F}_{\mathfrak{a}}$ are summarized up below.

Proposition 3. *Assume that $\rho_b \in L^1_\#(\mathbb{R})$, $\rho_b \geqslant 0$, $\int_0^L \rho_b(x)\,\mathrm{d}x = M$ and let $\mathfrak{a}, \theta$ be positive real numbers. For any $a \in W^{1,\infty}_\#(\mathbb{R})$ such that $\int_0^L a(x)^2\,\mathrm{d}x \leqslant \mathfrak{a}^2$ construct $\psi, \rho, \tilde{\rho}, \lambda$ and $\tilde{a} = \mathcal{F}_{\mathfrak{a}} a$ as above.*

(i) There are constants $\rho_\star, a_\star$ depending on $\mathfrak{a}, L, M, \theta$ such that

$$\|\tilde{\rho}\|_{L^\infty(\mathbb{R})} \leqslant \|\rho\|_{L^\infty(\mathbb{R})} \leqslant \rho_\star, \qquad \|\rho'\|_{L^\infty(\mathbb{R})} \leqslant \frac{\|a\|_{L^\infty(\mathbb{R})}\|a'\|_{L^\infty(\mathbb{R})} + 2M}{\theta}\rho_\star,$$

$$\|\tilde{\rho}'\|_{L^\infty(\mathbb{R})} \leqslant \frac{\|a\|_{L^\infty(\mathbb{R})}\|a'\|_{L^\infty(\mathbb{R})}(1+\theta) + 2M}{\theta}\rho_\star, \quad 0 \leqslant \lambda \leqslant \rho_\star, \quad \|\tilde{a}\|_{W^{2,\infty}(\mathbb{R})} \leqslant a_\star.$$

(ii) $\mathcal{F}_{\mathfrak{a}}$ is continuous with respect to the topology of $C^0_\#(\mathbb{R})$ on the set $\mathcal{C} = \{a \in C^0_\#(\mathbb{R}) : \|a\|_{L^2(0,L)} \leqslant \mathfrak{a}$ and $\|a\|_{L^\infty(\mathbb{R})} + \|a'\|_{L^\infty(\mathbb{R})} \leqslant a_\star\}$.

Proof. (i) Take $a \in W^{1,\infty}_\#(\mathbb{R})$ such that $\|a\|_{L^2(0,L)} \leqslant \mathfrak{a}$. In the NR and QR cases, we deduce from (14) and (20) the bound

$$0 \leqslant \rho \leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{L}\int_0^L \frac{a(x)^2}{2}\,\mathrm{d}x + 4LM\right) \leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{2L}\mathfrak{a}^2 + 4LM\right).$$

In the FR case, combining (15) and (20) yields

$$0 \leqslant \rho \leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{L}\int_0^L (\psi(y) + \theta\ln C_1(\theta))\,\mathrm{d}y - \inf(\psi + \theta \ln C_1(\theta)) + 4LM\right)$$

$$\leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{L}\int_0^L \left(|a(x)| + \theta\ln\frac{C_1(\theta)}{C_2(\theta)}\right)\mathrm{d}y + 4LM\right)$$

$$\leqslant \frac{M}{L}\exp\frac{1}{\theta}\left(\frac{1}{\sqrt{L}}\mathfrak{a} + \theta\ln\frac{C_1(\theta)}{C_2(\theta)} + 4LM\right).$$

We easily check that in all three cases we have $|\psi'(x)| \leqslant |a(x)||a'(x)|, x \in \mathbb{R}$ and thus, by proposition 2 we deduce

$$\|\rho'\|_{L^\infty(\mathbb{R})} \leqslant \frac{\|a\|_{L^\infty(\mathbb{R})}\|a'\|_{L^\infty(\mathbb{R})} + 2M}{\theta}\|\rho\|_{L^\infty(\mathbb{R})}.$$

The estimate for $\tilde{\rho}$ follows, since in all three cases $\gamma_2 \geqslant 1$ and $0 \leqslant \tilde{\rho}(x) \leqslant \rho(x)$. By taking the derivative with respect to x in the expression of $\tilde{\rho}$ we obtain by direct computation

$$\|\tilde{\rho}'\|_{L^\infty(\mathbb{R})} \leqslant \left(\|a\|_{L^\infty(\mathbb{R})}\|a'\|_{L^\infty(\mathbb{R})}\left(1 + \frac{1}{\theta}\right) + \frac{2M}{\theta}\right)\|\rho\|_{L^\infty(\mathbb{R})}.$$

We now estimate the eigenvalue λ and the eigenfunction $\tilde{a}$. Equation (22) shows that $\lambda \geqslant 0$ and, by taking $b = 1$, $\lambda \leqslant \rho_\star$. Then, from (23) we deduce

$$\int_0^L \{\tilde{a}'(x)^2 + \tilde{\rho}(x)\tilde{a}(x)^2\}\,\mathrm{d}x = \lambda\int_0^L \tilde{a}(x)^2\,\mathrm{d}x,$$

and hence $\|\tilde{a}\|_{H^1(0,L)} \leqslant \sqrt{\lambda + 1}\mathfrak{a}$. By using the Sobolev inclusion $H^1(0, L) \subset L^\infty(0, L)$ one gets easily that $\|\tilde{a}\|_{L^\infty(\mathbb{R})} + \|\tilde{a}'\|_{L^\infty(\mathbb{R})} + \|\tilde{a}''\|_{L^\infty(\mathbb{R})} \leqslant a_\star(\mathfrak{a}, L, M, \theta)$.

(ii) Take a sequence $(a^n)_n \subset \mathcal{C}$ which converges towards $a \in \mathcal{C}$ with respect to the topology of $C^0_\#(\mathbb{R})$. For any n let $\psi^n, \rho^n, \tilde{\rho}^n, \lambda^n, \tilde{a}^n = \mathcal{F}_\mathfrak{a} a^n$ constructed as in the definition of the fixed point application. Similarly let $\psi, \rho, \tilde{\rho}, \lambda, \tilde{a} = \mathcal{F}_\mathfrak{a} a$. The sequence $(\psi^n)_n$ is bounded in $W^{1,\infty}(\mathbb{R})$ and therefore, by the Arzelà–Ascoli theorem we can extract a subsequence converging in $C^0_\#(\mathbb{R})$. Obviously the limit function is ψ and by the uniqueness of the limit we deduce that the whole sequence $(\psi^n)_n$ converges towards ψ in $C^0_\#(\mathbb{R})$. In the same manner, since $\sup_n \left(\|\rho^n\|_{L^\infty(\mathbb{R})} + \|\frac{\mathrm{d}}{\mathrm{d}x}\rho^n\|_{L^\infty(\mathbb{R})}\right) < +\infty$ we deduce that $\rho^n \to \rho$, $\tilde{\rho}^n \to \tilde{\rho}$ in $C^0_\#(\mathbb{R})$.

The fact that $\lim_{n\to+\infty}\lambda^n = \lambda$ stems from general spectrum continuity theorems [14], or can be directly deduced from (22). Finally, as $\sup_n \|\tilde{a}^n\|_{W^{2,\infty}(\mathbb{R})} < +\infty$, we can extract a subsequence $(\tilde{a}^{n_k})_k$ converging in $C^1_\#(\mathbb{R})$ towards some function $\tilde{\tilde{a}}$. By passing to the limit with respect to k in the weak formulation of $\tilde{a}^{n_k}$ we obtain that the limit $\tilde{\tilde{a}}$ satisfies

$$-\tilde{\tilde{a}}''(x) + \tilde{\rho}(x)\tilde{\tilde{a}}(x) = \lambda\tilde{\tilde{a}}(x), \qquad x \in (0, L), \quad \tilde{\tilde{a}}(0) = \tilde{\tilde{a}}(L), \quad \tilde{\tilde{a}}'(0) = \tilde{\tilde{a}}'(L).$$

Moreover since $\tilde{a}^n \geqslant 0$, $\|\tilde{a}^n\|_{L^2(0,L)} = \mathfrak{a}$ for any n, we have $\tilde{\tilde{a}} \geqslant 0$ and $\|\tilde{\tilde{a}}\|_{L^2(0,L)} = \mathfrak{a}$ and thus $\tilde{\tilde{a}} = \tilde{a} = \mathcal{F}_\mathfrak{a} a$. By the uniqueness of the limit we have $\lim_{n\to+\infty}\tilde{a}^n = \tilde{a}$ in $C^1_\#(\mathbb{R})$. □

We are now in position to prove our main result by using the fixed point method.

Theorem 4. *Assume that $\rho_\mathfrak{b} \in L^1_\#(\mathbb{R})$, $\rho_\mathfrak{b} \geqslant 0$, $\int_0^L \rho_\mathfrak{b}(x)\,\mathrm{d}x = M$ and let θ be a positive real number. For any $\mathfrak{a} > 0$ there is at least one classical solution $(\rho, a) \in C^1_\#(\mathbb{R}) \times C^2_\#(\mathbb{R})$ for the Boltzmann–Helmholtz equations (16)–(18) satisfying $\rho \geqslant 0$, $\int_0^L \rho(x)\,\mathrm{d}x = M$, $a \geqslant 0$, $\int_0^L a(x)^2\,\mathrm{d}x = \mathfrak{a}^2$.*

Proof. Consider $\tilde{\mathcal{F}}_\mathfrak{a} = \mathcal{F}_\mathfrak{a}|_\mathcal{C}$. The set $\mathcal{C}$ is convex and compact in $C^0_\#(\mathbb{R})$; by proposition 3 we know that $\tilde{\mathcal{F}}_\mathfrak{a}(\mathcal{C}) \subset \mathcal{C}$ and that $\tilde{\mathcal{F}}_\mathfrak{a}$ is continuous with respect to the topology of $C^0_\#(\mathbb{R})$. By the Schauder fixed point theorem we deduce that there is a fixed point $a \in \mathcal{C}$. By construction we have $a \geqslant 0$, $\int_0^L a(x)^2\,\mathrm{d}x = \mathfrak{a}^2$. Consider now $\psi, \rho, \tilde{\rho}, \lambda$ as in the definition of $\tilde{\mathcal{F}}_\mathfrak{a} a$. Obviously $\lambda \geqslant 0$, $\rho \geqslant 0$, $\int_0^L \rho(x)\,\mathrm{d}x = M$ and we easily check that $(\rho, a) \in C^1_\#(\mathbb{R}) \times C^2_\#(\mathbb{R})$. Observe that $\lambda > 0$. Indeed, we have

$$\lambda = \frac{\int_0^L \{a'(x)^2 + \tilde{\rho}(x)a(x)^2\}\,\mathrm{d}x}{\int_0^L a(x)^2\,\mathrm{d}x} \geqslant \frac{\int_0^L \tilde{\rho}(x)a(x)^2\,\mathrm{d}x}{\int_0^L a(x)^2\,\mathrm{d}x}.$$

If $\lambda = 0$ then $\tilde{\rho}(x)a(x)^2 = 0$ for any x, and since by construction $\tilde{\rho} > 0$ we deduce that $a = 0$ which contradicts $\int_0^L a(x)^2\,\mathrm{d}x = \mathfrak{a}^2 > 0$. Consider now $\omega = \sqrt{\lambda} > 0$ and thus (ρ, a) is a solution of (16)–(18). □

5. Extensions

One could investigate the existence of 'nonlinear harmonics' of the 'fundamental mode' given by theorem 4, i.e., solutions to (18) where ω^2 is not the first eigenvalue of $A_{\tilde{\rho}}$, but one of higher rank. Unfortunately, it appears impossible to generalize the construction of $\mathcal{F}_{\mathfrak{a}}$ to these eigenvalues. The reason is that, with periodic boundary conditions (unlike the Dirichlet, Neumann or Fourier b.c.), these eigenvalues may be double for some 'exceptional' densities $\tilde{\rho}$. For instance, if $\rho_{\mathrm{b}} = \mathrm{cst}$ and $a = \mathrm{cst}$, then $\tilde{\rho} = \mathrm{cst}$ and all eigenvalues except the first one are double. There is apparently no way of defining a continuous mapping $\tilde{\rho} \mapsto \tilde{a}$ in the neighbourhood of the exceptional densities. Nevertheless, the existence of harmonics is very likely, as the eigenvalues are generically simple.

Another interesting extension is the case where the ion density is no longer given, but is also proportional to the Boltzmann factor. Let us denote by the subscript 1, resp. 2, the quantities relative to the electrons, resp. ions; we introduce a new parameter μ representing the electron/ion mass ratio. Then, we have $f_1 \propto \mathrm{e}^{-W_1/\theta_1}$ and $f_2 \propto \mathrm{e}^{-W_2/\theta_2}$. The energy W_1 of one electron is given in section 2; that of one ion is, according to the relativistic character

$$W_2(x, p) = \tfrac{1}{2}\mu(p^2 + |a(x)|^2) - V(x), \quad \text{(NR)},$$
$$W_2(x, p) = \mu^{-1}\sqrt{1 + (\mu p)^2} + \tfrac{1}{2}\mu|a(x)|^2 - V(x), \quad \text{(QR)},$$
$$W_2(x, p) = \mu^{-1}\sqrt{1 + \mu^2(p^2 + |a(x)|^2)} - V(x), \quad \text{(FR)}.$$

We arrive at the following system:

$$V''(x) = \rho_2(x) - \rho_1(x), \qquad x \in \mathbb{R}, \tag{25}$$

$$-\omega^2 a(x) - a''(x) = -(\tilde{\rho}_1(x) + \mu\tilde{\rho}_2(x))a(x), \quad x \in \mathbb{R}. \tag{26}$$

The arguments of sections 3 and 4 can be extended without bad surprises to this two-species model. However, the solutions corresponding to the first eigenvalue are not very interesting: one easily checks that all the functions $\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2, a$ are constant, and $V \equiv 0$.

6. Concluding remarks

In this paper, we have shown the existence of quasi-equilibrium solutions to the laser–plasma system, where the distribution function is Boltzmannian and the electromagnetic variables are time-harmonic, at least at the fundamental frequency. The existence of solutions at higher frequencies is probable, both for one-species and two-species models. These solutions appear as generalizations of Vlasov–Poisson equilibria, but are clearly different from them as an electromagnetic wave is present. They can be viewed as a simple case of nonlinear interaction between the electron plasma oscillations and the laser wave. The implicit relation (through the spectrum of the operator $A_{\tilde{\rho}}$) between the frequency ω and the space period L yields in the linear limit the dispersion relation for electromagnetic waves.

Quasi-equilibria can serve as references for analysing the dynamics of laser–plasma interaction, e.g. Raman and Brillouin scattering, which are among the most challenging issues to deal with in order to achieve controlled inertial confinement fusion. Indeed, from a dynamical point of view, it should be noted that quasi-equilibria may be unstable, unlike the Vlasov–Poisson equilibria which are nonlinearly stable, even under 1D Vlasov–Maxwell perturbations [5]. These solutions may also serve as benchmarks for testing numerical codes, even though the numerical solution of the Boltzmann problem appears quite difficult when u_{b} and/or ψ feature large variations.

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