# Jordan-Hölder sequences and self-adjoint $(a, b)$-modules

Piotr P. Karwasz *

April 6, 2011

**Abstract**

Given the lack of uniqueness of the Jordan-Hölder composition series in the theory of $(a, b)$-modules we are interested whether the particularities of certain $(a, b)$-modules can be transmitted to their composition series. This article will focus on the properties of Jordan-Hölder composition series of self-adjoint $(a, b)$-modules. In particular we will prove that a self-adjoint composition series always exists for self-adjoint $(a, b)$-modules.

**Mathematics Subject Classification (2010):** 32S25, 32S40, 32S50

## 1 Introduction

The Brieskorn module of a germ of holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with an isolated singularity at the origin introduced by E Brieskorn in [Bri70] can be formally completed for the operation $b := \mathrm{df} \wedge \mathrm{d}^{-1}$. The result is called $(a, b)$-module and can be defined in an abstract way as:

**Definition 1.1.** *An $(a, b)$-module is a free $\mathbb{C}[[b]]$-module $E$ of finite rank over the ring of formal power series in $b$, endowed with a $\mathbb{C}$-linear endomorphism '$a$' which satisfies:*

$$ab - ba = b^2$$

We recall some basic classification of this object: a sub-$(a, b)$-module $F$ of $E$ is a sub-$\mathbb{C}[[b]]$-module of $E$ stable by '$a$' and the $(a, b)$-structure passes onto the quotient $\mathbb{C}[[b]]$-module $E/F$ which satisfies all the properties of an $(a, b)$-module except that it has possibly a $b$-torsion. The sub-$(a, b)$-module $F$ is called **normal** if $E/F$ is free on $\mathbb{C}[[b]]$.

Since the completion of Brieskorn modules generates **regular** $(a, b)$-modules ([Bar93]), i.e. $(a, b)$-modules that can be embedded as a sub-$(a, b)$-module into an $(a, b)$-module $E$ satisfying $aE \subset bE$, we'll limit our inquiries to this subclass of objects.

All regular $(a, b)$-modules of rank 1 are generated by an element $e_\lambda$ satisfying $ae_\lambda = \lambda be_\lambda$ for a complex number $\lambda$. We will refer to them as **elementary** $(a, b)$-modules of parameter $\lambda$ and note them with $E_\lambda$.

*Institut Élie Cartan Nancy UMR 7502
Nancy-Université, B.P. 70230, 54506 Vandœuvre-lès-Nancy, France
E-mail: piotr.karwasz@iecn.u-nancy.fr

A basic result shows that a regular $(a, b)$-module $E$ admits Jordan-Hölder composition series
$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E$$
with the $F_i$ normal in $E$ and for a regular $(a, b)$-module the quotients $F_i/F_{i-1}$ are **elementary** $(a, b)$-modules $E_\lambda$ of parameter $\lambda$.

The isomorphism classes of such quotients vary from a Jordan-Hölder composition series to another and a quotient $E_\lambda$ may appear as $E_{\lambda+j}$ in another sequence, with $j \in \mathbb{Z}$.

At a first approach we studied the behaviour of such sequences under the duality functor (cf. [Bar97]):

**Definition 1.2.** *Let $E$ be an $(a, b)$-module and $E_0$ the elementary $(a, b)$-module of parameter $0$, then we may define upon the $\mathbb{C}[[b]]$-module $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_0)$ an $(a, b)$-module structure given by:*

$$(a \cdot \varphi)(x) = a\varphi(x) - \varphi(ax)$$

*This module is called the **dual** $(a, b)$-module and noted $E^*$.*

However, as it was proven by R. Belgrade in [Bel01], an $(a, b)$-module $E$ associated to a Brieskorn module is isomorphic to the **conjugate** of the dual, i.e. the **adjoint** $(a, b)$-module. Even if both approaches give similar results concerning the symmetry of Jordan-Hölder composition series, we will prefer the study of self-adjoint $(a, b)$-modules for the greater interest they play in the theory of singularities.

In the context of $(a, b)$-modules, the conjugate itself is defined in a way borrowed from the complex vector spaces:

**Definition 1.3.** *Let $E$ be an $(a, b)$-module, we call **conjugate** $(a, b)$-module and note it $\breve{E}$, the set $E$ endowed with the $(a, b)$-structure given by reversing the signs of both 'a' and 'b': $a \cdot_{\breve{E}} v = -a \cdot_E v$ and $b \cdot_{\breve{E}} v = -b \cdot_E v$.*

*In particular we call **adjoint** the conjugate of the dual $(a, b)$-module $E^*$ and we call self-adjoint an $(a, b)$-module $E$ which is isomorphic to $\breve{E}^*$.*

When working with isomorphisms between an $(a, b)$-module $E$ and its adjoint $\breve{E}^*$, it is often useful to look at it as a $\mathbb{C}[[b]]$-bilinear map between $E \times \breve{E}$ and $E_0$. Such a perspective brought us ([Kar]) to give the following definition of $(a, b)$-bilinear map and $(a, b)$-hermitian forms:

**Definition 1.4.** *Let $E$, $F$ and $G$ be $(a, b)$-modules and $\Phi$ a $\mathbb{C}[[b]]$-bilinear map between $E \times F$ and $G$. We say that $\Phi$ is an $(a, b)$-bilinear map (or form if $G = E_0$) if:*
$$a\Phi(v, w) = \Phi(av, w) + \Phi(v, aw)$$
*for every $v \in E$ and $w \in F$.*

*An $(a, b)$-bilinear map between $E \times \breve{E}$ and $E_0$ is called an $(a, b)$-hermitian form if moreover it satisfies*

$$\Phi(v, w) = S(b)e_0 \Rightarrow \Phi(w, v) = S(-b)e_0$$

*with $v, w \in E$ and $S(b) \in \mathbb{C}[[b]]$. $(a, b)$-anti-hermitian forms satisfy the same equation with $S(-b)$ replaced by $-S(-b)$.*

*An $(a,b)$-bilinear map between $E \times F$ and $E_0$ is* ***non-degenerate*** *if it induces an isomorphism between $E$ and $F^*$.*

*An $(a,b)$-module that admits a non-degenerate hermitian form will be called* ***hermitian****.*

The hermitian $(a,b)$-bilinear forms on an $(a,b)$-module $E$ induce, as in the case of complex vector spaces, an isomorphism $\Phi : E \to \breve{E}^*$ which is equal to its image $\breve{\Phi}^*$ under the adjonction functor.

We should remark that not every self-adjoint $(a,b)$-module is hermitian and there are even examples of $(a,b)$-modules that only admit an anti-hermitian non-degenerate form. In the general case every regular self-adjoint $(a,b)$-module can be decomposed into a hermitian part and an anti-hermitian part (not necessarily in a unique way) ([Kar]).

## 2 Self-adjoint composition series

In view of Belgrade's result we will work with the adjonction functor. All proofs however should be valid with only minor modifications if we work with the duality functor.

Consider a regular $(a,b)$-module $E$ of rank $n \in \mathbb{N}$ and a Jordan-Hölder decomposition of itself:

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = E.$$

with $F_j/F_{j-1} \simeq E_{\lambda_j}$, the elementary $(a,b)$-module of parameter $\lambda_j$. We say that the sequence is **self-adjoint** if $\lambda_{n-j+1} = -\lambda_j$ for all $1 \leqslant j \leqslant n$ and the $(a,b)$-module $F_{n-j}/F_j$ is self-adjoint for all $0 \leqslant j \leqslant [n/2]$.

We shall prove the following theorem for regular hermitian $(a,b)$-modules and we will extend it successively to all self-adjoint $(a,b)$-modules.

**Theorem 2.1.** *Let $E$ be a regular hermitian $(a,b)$-module then it has a Jordan-Hölder sequence*

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = E$$

*which is self-adjoint.*

Before proving the theorem we shall introduce a couple of lemmas.

**Lemma 2.2.** *Let $E$ be a regular hermitian $(a,b)$-module and $\Phi : E \to \breve{E}^*$ a hermitian isomorphism.*

*If there exists $F_1$ normal sub-$(a,b)$-module isomorphic to $E_\lambda$ such that*

$$\Phi(F_1)(F_1) = 0,$$

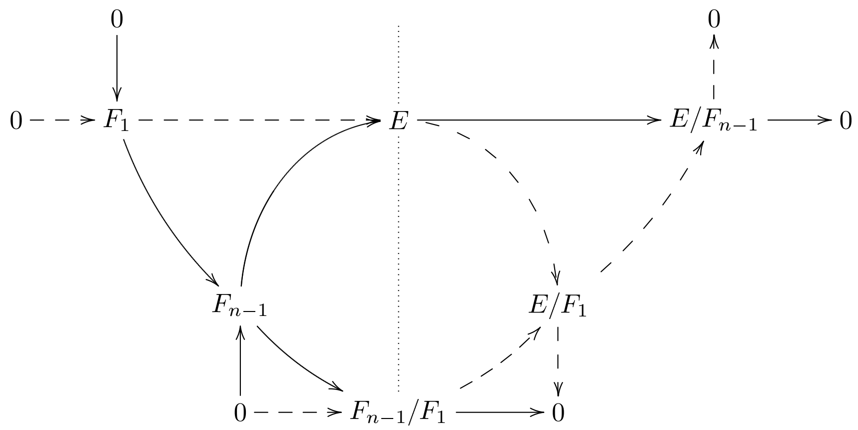
*then there exists a normal sub-$(a,b)$-module $F_{n-1}$ of rank $n-1$ such that $E/F_{n-1} \simeq \breve{F_1}^*$ and $F_{n-1}/F_1$ is hermitian.*

*Proof.* Let $e_\lambda$ be the generator of $F_1$ and $H$ the hermitian form associated to $\Phi$ by $H(x,y) = \Phi(y)(x)$ and consider the annihilator of this form under $H$:

$$F_{n-1} := \{x \in E | H(e_\lambda, x) = 0\}.$$

We remark that the condition $H(e_\lambda, e_\lambda) = 0$ gives us $F_1 \subset F_{n-1}$ and $F_{n-1}$ is normal, because it is the kernel of a morphism.

Figure 1: Modules in symmetric positions with respect to the dotted line are each other's adjoint.




Let us consider the following exact sequence:

$$0 \to F_1 \to E \to E/F_1 \to 0$$

from which we can pass to the adjoint sequence:

$$0 \to \breve{(E/F_1)}^* \to \breve{E}^* \xrightarrow{\pi} \breve{F}_1^* \to 0.$$

Since $\pi$ is the restriction morphism of forms on $E$ to the sub-$(a,b)$-module $F_1$, the kernel of $\pi$, $K := \operatorname{Ker}\pi$ can be described as follows:

$$K = \{\varphi \in \breve{E}^* | \varphi(F_1) = 0\}.$$

The adjoint sequence being exact, we can identify from now on $\breve{(E/F_1)}^*$ with $K$, i.e. sub-$(a,b)$-module of $\breve{E}^*$ whose elements annihilate $F_1$.

If we consider the restriction of the map $\Phi$ to $F_{n-1}$

$$\Phi|_{F_{n-1}} : F_{n-1} \to \breve{E}^*$$

and the fact that by definition $\Phi(x)(e_\lambda) = 0$ for all $x \in F_{n-1}$, we obtain that $\Phi(F_{n-1}) \subset \breve{(E/F_1)}^*$.

On the other side for all $\varphi \in \breve{(E/F_1)}^*$ the element $y = \Phi^{-1}(\varphi)$ verifies $\Phi(y)(e_\lambda) = 0$, therefore we have also $\breve{(E/F_1)}^* \subset \Phi(F_{n-1})$. It follows that $\Phi(F_{n-1}) = \breve{(E/F_1)}^*$ and since $\Phi$ is an isomorphism, $F_{n-1}$ is isomorphic to its image by $\Phi$: $\breve{(E/F_1)}^*$.

Let us look now at the following exact sequence:

$$0 \to (F_{n-1}/F_1) \to (E/F_1) \to (E/F_{n-1}) \to 0$$

and its adjoint sequence:

$$0 \to \widetilde{(E/F_{n-1})}^* \xrightarrow{i} \widetilde{(E/F_1)}^* \xrightarrow{\pi} \widetilde{(F_{n-1}/F_1)}^* \to 0.$$

$\pi$ designates the restriction application on the forms of $\widetilde{(E/F_1)}^*$. $\operatorname{Ker}\pi$ is thus the forms of $\widetilde{(E/F_1)}^*$ that annihilate $\widetilde{(F_{n-1}/F_1)}^*$ or with the convention of the previous paragraph, the forms of $\check{E}^*$ that annihilate $F_{n-1}$ and $F_1 \subset F_{n-1}$:

$$\operatorname{Ker}\pi = \{\varphi \in \check{E}^* \text{ s.t. } \varphi(F_{n-1}) = 0\}$$

We note that the hermitianity of $\Phi$ gives us

$$\Phi(e_\lambda)(F_{n-1}) = \Phi\widetilde{(F_{n-1})}(e_\lambda) = 0$$

and therefore we have $\Phi(F_1) \subset \operatorname{Ker}\pi$. An easy calculation shows that $\operatorname{Ker}\pi$ is of rank 1. Since $\Phi(F_1)$ is normal, of rank 1 and included into $\operatorname{Ker}\pi$, they must be equal.

We obtain $\widetilde{(E/F_{n-1})}^* \simeq \operatorname{Ker}\pi \simeq F_1$. Now we know that $\Phi$ sends $F_{n-1}$ onto $(E/F_1)^*$ and $F_1$ onto $\operatorname{Ker}\pi$, so starting with the following exact sequence:

$$0 \to \operatorname{Ker}\pi \hookrightarrow (E/F_1)^* \xrightarrow{\pi} (F_{n-1}/F_1)^* \to 0$$

we can obtain another by substituting $\operatorname{Ker}\pi$ with $F_1$ and $(E/F_1)^*$ with $F_{n-1}$:

$$0 \to F_1 \to F_{n-1} \to \widetilde{(F_{n-1}/F_1)}^* \to 0.$$

or in other terms $\widetilde{(F_{n-1}/F_1)}^* \simeq (F_{n-1}/F_1)$. Note that the isomorphism is given by $x \to \Phi(x)|_{F_{n-1}}$ and is therefore hermitian.

The proof may be summarized by the graph of interwoven exact sequences presented in figure 1. □

*Remark* 2.3. If $ae_\lambda = \lambda b e_\lambda$ and $2\lambda \notin \mathbb{N}$, then $H(e_\lambda, e_\lambda) = 0$. In fact $H(e_\lambda, e_\lambda) \in E_0$ must satisfy:

$$aH(e_\lambda, e_\lambda) = H(ae_\lambda, e_\lambda) + H(e_\lambda, -ae_\lambda) = \\ H(\lambda b e_\lambda, e_\lambda) + H(e_\lambda, -\lambda b e_\lambda) = 2\lambda b H(e_\lambda, e_\lambda)$$

which has non-trivial solutions in $E_0$ only if $2\lambda \in \mathbb{N}$. The double inversion of signs in the second factor are due to the hermitian nature of the form.

**Lemma 2.4.** *If $E$ is a regular hermitian $(a,b)$-module and there exist $\lambda \in \mathbb{C}$ such that $E$ contains two distinct normal elementary sub-$(a,b)$-modules $F$ and $G$ of parameters*

$$f = g = \lambda \ mod\ \mathbb{Z}$$

*then there exists $F_1 \subset F_{n-1}$ two normal sub-$(a,b)$-modules of rank 1 and $n-1$ respectively such that $\widetilde{(E/F_{n-1})}^* \simeq F_1$ and $F_{n-1}/F_1$ is hermitian.*

*Proof.* We will denote by $H$ an hermitian form on $E$. Let $e_f$ and $e_g$ be generators of $F$ and $G$ and suppose without loss of generality that $f - g \geqslant 0$. We will show that there exists a normal elementary sub-$(a,b)$-module $F_1$ of $E$ whose generator $e \in E$ satisfies $H(e,e) = 0$.

By the fundamental property $ab - ba = b^2$ of $(a,b)$-modules we have $ab^{f-g}e_g = (g+f-g)b \cdot b^{f-g}e_g$. Let's pose $e_1 = b^{f-g}e_g$. Consider now the complex vector space:

$$V := \{\alpha e_f + \beta e_1 | \alpha, \beta \in \mathbb{C}\}$$

Note that every $v \in V$ satisfies $av = fbv$. The $b$-linearity of $H$ and the definition of the action of $a$ gives us:

$$(a - 2fb)H(v,v) = 0$$

which has in $E_0$ only solutions of the form $\alpha b^{2f}e_0$, $\alpha \in \mathbb{C}$. There exists therefore an application $B$ from $V \times V$ to $\mathbb{C}$ such that:

$$H(v,w) = B(v,w)b^{2f}e_0 \quad \forall v,w \in V$$

The bilinearity and hermitianity of $H$ imply that $B$ is in fact a $\mathbb{C}$-bilinear symmetric or anti-symmetric form on a 2 dimensional complex vector space, depending whether $2f$ is even or odd. In the anti-symmetric case every vector will be isotropic, in the symmetric case we have:

$$B(e_f + xe_1, e_f + xe_1) = a_0 + a_1 x + a_2 x^2$$

for some complex numbers $a_i$. The vector space $V$ has therefore an isotropic vector $e \neq 0$ such that $B(e,e) = 0$, and hence $H(e,e) = 0$.

By eventually dividing $e$ by a certain power of $b$, operation that does not change the relation $H(e,e) = 0$, we can assume that $e \notin bE$, hence the module $F_1$ generated by $e$ is normal.

We can now conclude by applying lemma 2.2 □

**Lemma 2.5.** *Let $E$ be a regular $(a,b)$-module and:*

$$0 \subsetneq \dots F_{i-1} \subsetneq F_i \subsetneq F_{i+1} \subsetneq \dots E$$

*be a Jordan-Hölder composition series with $F_i/F_{i-1} \simeq E_{\lambda_i}$ for all $i$ and suppose there is a $j$ such that $\lambda_{j+1} \neq \lambda_j$ mod $\mathbb{Z}$.*

*Then we can find another Jordan-Hölder composition series that differs only in the $j$-th term $F'_j$ such that $F'_j/F_{j-1} \simeq E_{\lambda'_j}$ and $F_{j+1}/F'_j \simeq E_{\lambda'_{j+1}}$ with $\lambda_j = \lambda'_{j+1}$ mod $\mathbb{Z}$ and $\lambda_{j+1} = \lambda'_j$ mod $\mathbb{Z}$, i.e. we can permute the quotients up to an integer shift of the parameters.*

*Proof.* Let consider $G := F_{j+1}/F_{j-1}$ and the canonical projection $\pi : E \to E/F_{j-1}$. $G$ is a rank two module. Using the classification of regular $(a,b)$-modules of rank 2 given by D. Barlet in [Bar93] we see that the only two possibilities for $G$ are:

$$G \simeq E_{\lambda_j} \oplus E_{\lambda_{j+1}}$$

in which case we take $F'_j = \pi^{-1}(E_{\lambda_{j+1}})$ or

$$G \simeq E_{\lambda_{j+1}+1, \lambda_j}$$

generated by $y$ and $t$ satisfying:

$$\begin{aligned} ay &= \lambda_j by \\ at &= \lambda_{j+1} bt + y \end{aligned}$$

that has also another set of generators: $t$ and $x := y + (\lambda_{j+1} - \lambda_j + 1)bt$ which satisfy:

$$\begin{array}{rcl} ax & = & (\lambda_j + 1)bx \\ at & = & (\lambda_j - 1)bt + x. \end{array}$$

In this case we take $F'_j = \pi^{-1}(< x >)$. □

**Lemma 2.6.** *Let $\lambda$ be either $0$ or $1/2$ and $E$ be a regular hermitian $(a, b)$-module. Suppose that there is an unique normal elementary sub-$(a, b)$-module of parameter equal to $\lambda$ modulo $\mathbb{Z}$ and suppose moreover that every Jordan-Hölder sequence contains at least $2$ elementary quotients of parameter equal to $\lambda$ modulo $\mathbb{Z}$.*

*Then there exists $F_1 \subset F_{n-1}$ two normal sub-$(a, b)$-modules of rank $1$ and $n - 1$ respectively such that $\widecheck{(E/F_{n-1})}^* \simeq F_1$ and $F_{n-1}/F_1$ is hermitian.*

*Proof.* Let $F_1 \simeq E_\mu$ be the elementary sub-$(a, b)$-module of the hypothesis and $\{F_i\}$ a J-H sequence beginning with $F_1$ and such that $E/F_{n-1}$ is of parameter $\mu'$ equal to $\lambda$ mod $\mathbb{Z}$. We can find such a sequence by using repeatedly the previous lemma.

Consider the exact sequence:

$$0 \to F_{n-1} \to E \to (E/F_{n-1}) \to 0$$

and the adjoint sequence:

$$0 \to \widecheck{(E/F_{n-1})}^* \xrightarrow{i} \check{E}^* \xrightarrow{\pi} \check{F}^*_{n-1} \to 0.$$

The image of $i$ is a normal elementary sub-$(a, b)$-module of $\check{E}^*$ of parameter equal to $-\lambda$ mod $\mathbb{Z}$ (since $\widecheck{(E/F_{n-1})}^* \simeq E_{-\mu'}$). But $\lambda = -\lambda$ mod $\mathbb{Z}$ and $E \simeq \check{E}^*$ so by the uniqueness of $F_1$ given in the hypothesis $\mathrm{Im}\left(\widecheck{(E/F_{n-1})}^*\right) = \Phi(F_1)$, thus $\widecheck{(E/F_{n-1})}^* \simeq F_1$.

By replacing $\check{E}^*$ by $E$ and $\widecheck{(E/F_{n-1})}^*$ by $F_1$ in the sequence we obtain:

$$0 \to F_1 \xrightarrow{i} E \to \check{F}^*_{n-1} \to 0$$

which is exact and $i$ is the inclusion of sub-$(a, b)$-modules, so $\check{F}^*_{n-1} \simeq (E/F_1)$ or equivalently $F_{n-1} \simeq \widecheck{(E/F_1)}^*$. Note that the first isomorphism is given by $\Phi^{-1}$, while the second by the restriction of $\Phi$.

Consider the following sequence and its adjoint:

$$0 \to F_{n-1}/F_1 \to E/F_1 \to E/F_{n-1} \to 0$$
$$0 \to \widecheck{(E/F_{n-1})}^* \to \widecheck{(E/F_1)}^* \to \widecheck{(F_{n-1}/F_1)}^* \to 0$$

by replacing $\widecheck{(E/F_{n-1})}^*$ and $\widecheck{(E/F_1)}^*$ with $F_1$ and $F_{n-1}$ we obtain:

$$0 \to F_1 \xrightarrow{\varphi} F_{n-1} \xrightarrow{\pi} \widecheck{(F_{n-1}/F_1)}^* \to 0$$

for the uniqueness of $F_1$, $\varphi$ can only be (up to multiplication by a complex number) the inclusion $F_1 \subset F_{n-1}$ and hence $\widecheck{(F_{n-1}/F_1)}^* \simeq (F_{n-1}/F_1)$. Note that $\pi$ is the restriction of $\Phi$ to $F_{n-1}$, so the isomorphism is hermitian. □

We can now prove the theorem.

*Proof of theorem 2.1.* We will prove the theorem by induction on the rank of the $(a, b)$-module. For rank 0 and 1 the theorem is obvious.

Suppose we proved the theorem for every rank $< n$ and let's prove it for rank $n$. Let find $F_1 \subset F_{n-1}$ of rank 1 and $n-1$ such that $\widetilde{(E/F_{n-1})}^* \simeq F_1$ and $F_{n-1}/F_1$ is hermitian. We can have different cases which are exhaustive:

(i) We can find $G$, a normal elementary sub-$(a, b)$-module of $E$ of parameter $\lambda$ not equal to 0 or $1/2$ mod $\mathbb{Z}$. Then $\Phi(G)(G) = 0$ by remark 2.3 and we can apply lemma 2.2.

We still need to prove the induction step for $(a, b)$-modules whose only normal elementary sub-$(a, b)$-modules have parameter $\lambda = 0$ or $\lambda = 1/2$ modulo $\mathbb{Z}$.

(ii) For $\lambda = 0$ or $\lambda = 1/2$ there are two distinct normal elementary sub-$(a, b)$-modules of parameter equal to $\lambda$ mod $\mathbb{Z}$. We apply lemma 2.4.

The $(a, b)$-modules that were not included in the previous points have an unique normal elementary sub-$(a, b)$-module of parameter equal to $1/2$ modulo $\mathbb{Z}$ and an unique normal elementary sub-$(a, b)$-module with an integer value of the parameter.

(iii) There is only one normal elementary sub-$(a, b)$-module of parameter equal to $\lambda$ mod $\mathbb{Z}$, where $\lambda = 0$ or $1/2$, but at least two quotients of a J-H sequence are of parameter equal to $\lambda$ mod $\mathbb{Z}$. We apply lemma 2.6.

Only modules of rank at most 2 (one for each possible value of $\lambda$) still need to be checked.

(iv) The rank of $E$ is 2 and one quotient of a J-H sequence is equal to 0 mod $\mathbb{Z}$, the other equal to $1/2$ mod $\mathbb{Z}$. By the classification of rank 2 modules this case is impossible. In fact with the notations of [Bar93]:

$$\begin{aligned} \widetilde{(E_\lambda \oplus E_\mu)}^* &\simeq E_{-\lambda} \oplus E_{-\mu} \\ \check{E}^*_{\lambda,\mu} &\simeq E_{1-\lambda,1-\mu} \end{aligned}$$

so if $\lambda = 0$ mod $\mathbb{Z}$ and $\mu = 1/2$ mod $\mathbb{Z}$ the $(a, b)$-module is not self-adjoint.

By induction hypothesis $F_{n-1}/F_1$ has a J-H composition series that verifies the theorem and by taking the inverse image by the canonical morphism $F_{n-1} \to F_{n-1}/F_1$ and adding 0 and $E$ we find a J-H sequence of $E$ that satisfies the theorem. □

Since for an anti-hermitian form $A$ we have $A(e, e) = 0$ for every $e \in E$, by using an anti-hermitian version of lemma 2.2 alone and proceeding by induction, we can prove theorem 2.1 in the anti-hermitian case.

We wish now to extend the result to all regular self-adjoint $(a, b)$-modules. We have proven in [Kar] that every regular $(a, b)$-module $E$ can be decomposed into a direct sum of hermitian or anti-hermitian $(a, b)$-modules. We can hence prove the following theorem:

**Theorem 2.7.** *Let $E$ be a self-adjoint regular $(a,b)$-module. Then it admits a self-adjoint Jordan-Hölder composition series.*

*Proof.* Let decompose $E$ into

$$E = \bigoplus_{i=1}^{m} H_i$$

where $m$ is an integer, while the $H_i$ are either indecomposable self-adjoint or of the form $G \oplus \check{G}^*$, where $G$ is indecomposable non self-adjoint $(a,b)$-module.

Each term of this sum admits a self-adjoint composition series. In fact if $H_i$ is indecomposable self-adjoint, then it is hermitian or anti-hermitian. We can therefore apply the previous theorem 2.1.

On the other hand if $H_i$ is the sum $G \oplus \check{G}^*$ of a module and its adjoint, we can easily find a self-adjoint Jordan-Hölder composition series. Take in fact any Jordan-Hölder series of $G$,

$$0 = G_0 \subsetneq \cdots \subsetneq G_n = G.$$

and consider the adjoint series

$$0 = (\widetilde{G/G_n})^* \subsetneq (\widetilde{G/G_{n-1}})^* \subsetneq \ldots (\widetilde{G/G_0})^* = \check{G}^*.$$

Then the following composition series of $G \oplus \check{G}^*$ is self-adjoint:

$$0 = G_0 \subsetneq G_1 \subsetneq \ldots G = G \oplus (\widetilde{G/G_n})^* \subsetneq G \oplus (\widetilde{G/G_{n-1}})^* \subsetneq \\ \cdots \subsetneq G \oplus (\widetilde{G/G_0})^* = G \oplus \check{G}^*.$$

We will now prove the theorem on induction on $m$. The case $m = 1$ was already proven.

Suppose now $m \geqslant 2$ and let $E' := H_1$ and $F := \sum_{i=2}^{m} H_i$. We have therefore $E = E' \oplus F$, and $E'$ and $F$ are both self-adjoint. By the remark above we can find a self-adjoint composition series of $E'$:

$$0 = E'_0 \subsetneq \cdots \subsetneq E'_r = E'$$

while by induction hypothesis, we can find a self-adjoint composition series of $F$:

$$0 = F_0 \subsetneq \cdots \subsetneq F_s = F.$$

Then the following composition series is self-adjoint:

$$0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \subsetneq E'_{[r/2]} \oplus F_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \oplus F_{[s/2]} [\cdots] \\ E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]} \subsetneq E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]+1} \subsetneq \cdots \subsetneq E'_{[(r+1)/2]} \oplus F \\ \subsetneq E'_{[(r+1)/2]+1} \oplus F \subsetneq \cdots \subsetneq E' \oplus F,$$

where depending on the parity of $r$ and $s$, $[\cdots]$ stands for

(i) the $=$ sign if $r$ and $s$ are both even.

(ii) the $\subsetneq$ sign if one is even and the other odd.

(iii) the subsequence

$$\subsetneq E'_{[r/2]} \oplus F_{[(s+1)/2]} \subsetneq$$

This case needs a short verification. If $r$ and $s$ are odd, then the two central quotients of the series are isomorphic to $E'_{[(r+1)/2]}/E'_{[r/2]}$ and $F_{[(s+1)/2]}/F_{[s/2]}$. Since $E'_i$ and $F_i$ are self-adjoint series both quotients are self-adjoint $(a, b)$-modules of rank 1. They are therefore isomorphic to $E_0$.

□

# References


[Bar93] Daniel Barlet. Theory of $(a, b)$-modules. i. In *Complex analysis and geometry*, Univ. Ser. Math., pages 1–43. Plenum, New York, 1993.

[Bar97] D. Barlet. Théorie des $(a, b)$-modules. II. Extensions. In *Complex analysis and geometry (Trento, 1995)*, volume 366 of *Pitman Res. Notes Math. Ser.*, pages 19–59. Longman, Harlow, 1997.

[Bel01] R. Belgrade. Dualité et spectres des $(a, b)$-modules. *J. Algebra*, 245(1):193–224, 2001.

[Bri70] Egbert Brieskorn. Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscripta Math.*, 2:103–161, 1970.

[Kar] Piotr P. Karwasz. Hermitian $(a, b)$-modules and Saito's "higher residue pairings". available at `http://www.iecn.u-nancy.fr/ karwasz/research`.