

Bernstein-Gel'fand-Gel'fand reciprocity and indecomposable projective modules for classical algebraic supergroups

Caroline GRUSON ¹ and Vera SERGANOVA ²

Abstract: We prove a BGG type reciprocity law for the category of finite dimensional modules over algebraic supergroups satisfying certain conditions. The equivalent of a standard module in this case is a virtual module called Euler characteristic due to its geometric interpretation. In the orthosymplectic case, we also describe indecomposable projective modules in terms of those Euler characteristics.

Key words: Finite dimensional representations of algebraic supergroups, Flag variety, BGG reciprocity law.

INTRODUCTION

In many representation theories, there exist reciprocity laws. Roughly speaking, if the category in question has enough projective modules, one defines in a natural way a family of so-called standard modules such that every projective indecomposable module has a filtration with standard quotients. The reciprocity law states that the multiplicity of a standard module in the projective cover of a simple module equals the multiplicity of this simple module in the standard module. Those standard modules are usually easy to describe, in particular, their characters can be described by simple formulae.

For instance, Brauer discovered such a law in the case of finite groups representations in positive characteristic, [4]. Another example is a result of Humphreys, [12] for representations of semi-simple Lie algebras in positive characteristic. In 1976 ([1]) Bernstein, Gel'fand and Gel'fand introduced the category \mathcal{O} of highest weight modules for a semi-simple Lie algebra in characteristic 0, and proved a reciprocity law in this category. Irving, [13], and Cline, Parshall and Scott, [10], introduced a general notion of highest weight category and proved a generalized BGG reciprocity. Using this general approach, it is easy to prove similar results for the category \mathcal{O} of highest weight modules for classical simple Lie superalgebras. For the category of finite-dimensional representations of classical Lie superalgebras of type I Zou proved BGG reciprocity in [26]. For superalgebras of type II the question remained open, in particular since it was unclear how to define a standard object.

The first part of this paper (Section 2) is devoted to the generalized BGG reciprocity for algebraic supergroups G with reductive even part and symmetric root decomposition. In those cases, the irreducible representations are parametrized by a highest weight, and if λ is a highest weight, we denote by L_λ the corresponding irreducible representation. Every L_λ has an indecomposable projective cover P_λ in the category of finite-dimensional representations of G , [23]. However, in this situation there is no direct analogue of the so-called standard modules. Hence we introduce a family of virtual modules $\mathcal{E}(\mu)$, living in the Grothendieck group of the category: we call those modules Euler characteristics

¹U.M.R. 7502 du CNRS, Institut Elie Cartan, Universite Henri Poincaré (Nancy 1), BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France. E-mail: Caroline.Gruson@iecn.u-nancy.fr

²Department of Mathematics, University of California, Berkeley, CA, 94720-3840 USA. E-mail: serganov@Math.Berkeley.EDU

because they come from the cohomology of line bundles on flag supervarieties. It turns out that in the Grothendieck ring the $[P_\lambda]$ -s are linear combinations of $\mathcal{E}(\mu)$ -s and we denote the coefficient of $\mathcal{E}(\mu)$ in $[P_\lambda]$ by $a(\lambda, \mu)$. The reciprocity law (Theorem 1) states that $a(\lambda, \mu)$ is exactly the multiplicity of L_λ in $\mathcal{E}(\mu)$. The key argument in the proof is a $\mathbb{Z}/2\mathbb{Z}$ -graded analogue of the Bott reciprocity result, [2], see Proposition 1.

All the constructions above depend on the choice of a Borel subgroup in G : in the super case, this choice is not unique up to conjugation, and the result is true for every possible choice. In particular, in the case of $GL(m, n)$ our result generalizes Zou's result. It is also to be noted that the weights λ (labeling L_λ and P_λ) and μ (labeling $\mathcal{E}(\mu)$) do not belong to the same set. For instance, in the orthosymplectic case (Section 4) the μ -s must have tailless weight diagrams. Finally, let us emphasize on the fact that this category has infinite cohomological dimension and the subgroup generated by $[P_\lambda]$ -s is a proper subgroup in the whole Grothendieck group.

The rest of the paper deals with the computation of the coefficients $a(\lambda, \mu)$ for the orthosymplectic supergroup $SOSP(m, 2n)$. The first computation of those coefficients in the $GL(m, n)$ case was made in [21]. In [3] J. Brundan used another method, relating this representation theory with the one of \mathfrak{gl}_∞ . He interpreted the translation functors for $\mathfrak{gl}(m, n)$ as linear operators of \mathfrak{gl}_∞ acting on $\Lambda^n(W) \otimes \Lambda^m(W^*)$, where W is the standard representation of \mathfrak{gl}_∞ . Later on, in [5, 6, 7] Brundan and Stroppel introduced weight diagrams, which give a clear picture of the translation functors action. Thus the category of finite dimensional $GL(m, n)$ -modules is very well understood now, including the projective modules.

We adopt Brundan's categorification approach. Here we have to separate in two cases depending on the parity of m . If m is odd, the translation functors can be seen as the Chevalley generators of the Lie algebra $\mathfrak{gl}_{\infty/2}$ with Dynkin diagram

$$\circ - \circ - \circ - \dots,$$

and if m is even, the Lie algebra is $\mathfrak{gl}_{\infty/2} \oplus \mathfrak{gl}_{\infty/2}$ (see Section 5 and 7). We compute the coefficients $a(\lambda, \mu)$ in Section 8 (see Theorem 2, Theorem 3 and Theorem 4) via a comparison between the action of translation functors on P_λ -s and $\mathcal{E}(\mu)$ -s. We start with a typical λ (in this case P_λ , $\mathcal{E}(\lambda)$ and L_λ coincide) and then obtain an arbitrary P_λ by application of translation functors.

Thus, using the results of this paper one can express the character of any projective indecomposable module in terms of $\mathcal{E}(\mu)$. That however does not imply automatically an expression of irreducible characters in the same terms. In the case of $GL(m, n)$ this difficulty can be resolved by allowing infinite linear combinations of Euler characteristics $\mathcal{E}(\mu)$ -s, i.e. by completing the Grothendieck ring. In the case of $SOSP(m, 2n)$ the problem of calculating irreducible characters was solved in [11] by calculating Euler characters of vector bundles over an adequate variety (generalized grassmannian) related to the highest weight and using an induction on the rank of the supergroup. It seems that this difference between general linear and orthosymplectic cases is related to the fact that in the latter case the set of dominant weights has a minimal element with respect to the standard order.

There remain several open questions such as an interpretation of indecomposable projectives in terms of canonical bases and the construction of the analogue of Khovanov's diagram algebra, see [5],[6],[7] and [8]. It would be also quite interesting to understand how formulae for characters of the projective modules obtained in this paper are related to the results of [9].

We thank Jonathan Brundan, Catharina Stroppel and Elizaveta Vishlyakova for fruitful discussions. This work was partially supported by NSF grant n. 0901554.

1. NOTATIONS AND CONTEXT

Let G be a connected algebraic supergroup with reductive even part G_0 and \mathfrak{g} denote its Lie superalgebra. Then \mathfrak{g}_0 is a reductive Lie algebra and \mathfrak{g} is a semisimple \mathfrak{g}_0 -module. We denote by \mathfrak{h}_0 a Cartan subalgebra of \mathfrak{g}_0 and by \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , by H_0 and H we denote the corresponding algebraic subgroups. By W we denote the Weyl group $W(\mathfrak{g}_0, \mathfrak{h}_0)$.

In order to prove the BGG reciprocity we need the following assumptions on \mathfrak{g}

- $\mathfrak{h} = \mathfrak{h}_0$, and therefore $H = H_0$;
- $\mathfrak{g}_1 \simeq \mathfrak{g}_1^*$ as a \mathfrak{g}_0 -module.

Recall that $H = H_0$ is an algebraic torus. Let Λ denote the free abelian group of characters of H . One has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}.$$

The finite subset $\Delta \subset \Lambda$ is called the set of roots of \mathfrak{g} . Our assumptions imply that $\dim \mathfrak{g}_\alpha = (1, 0)$ or $(0, m_\alpha)$ for any root $\alpha \in \Delta$. In the former case we say that α is even and in the latter that α is odd. So we have a decomposition $\Delta = \Delta_0 \cup \Delta_1$ defined by the parity of roots. Furthermore, our assumptions imply that $\Delta = -\Delta$. It is not difficult to show that one can define a parity function $p : \Lambda \rightarrow \mathbb{Z}_2$ satisfying the condition $p(\lambda + \alpha) = p(\lambda) + p(\alpha)$ for all $\lambda \in \Lambda$ and $\alpha \in \Delta$. In general the choice of p is not unique.

As in the case of reductive Lie algebras, we define a decomposition $\Delta = \Delta^+ \cup \Delta^-$ of roots and the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

where

$$\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is called a Borel subalgebra of \mathfrak{g} and the corresponding algebraic subgroup B is called a Borel subgroup of G . Recall that the Borel subgroups in G are not always mutually conjugate. It is easy to see that B is the semi-direct product of the algebraic torus H and the unipotent supergroup N . We set

$$\rho = \frac{1}{2} \left(\sum_{\alpha \in \Delta_0^+} \alpha - \sum_{\alpha \in \Delta_1^+} m_\alpha \alpha \right).$$

By \mathcal{C} we denote the category of finite-dimensional G -modules, which is isomorphic to the category of \mathfrak{g} -modules, semisimple over \mathfrak{h} , with weights in Λ , see [23]. It was shown in [16] that any simple module in \mathcal{C} is a quotient of a Verma module with highest weight $\lambda \in \Lambda$ by a maximal submodule. A weight λ is called *dominant* if this quotient is finite-dimensional. Thus, every dominant weight defines two simple modules, one is obtained from another by application of the functor Π of change of parity. In order to avoid parity chasing, we introduce a parity function $p : \Lambda \rightarrow \mathbb{Z}_2$ and define the category \mathcal{F} as the full subcategory of \mathcal{C} consisting of modules such that the parity of any weight space coincides with the parity of the corresponding weight. It is not hard to see that $\mathcal{C} = \mathcal{F} \oplus \Pi(\mathcal{F})$.

Define the standard order on Λ by setting: $\lambda \leq \mu$ iff $\mu - \lambda = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ where all n_α are non-negative integers.

For any dominant weight λ , we will denote by L_λ the simple \mathfrak{g} -module in the category \mathcal{F} with highest weight λ .

It is well-known (see, for example, [23]) that in the category \mathcal{F} every simple module L_λ has an indecomposable projective cover which we denote by P_λ .

Denote by Λ^+ the set of all weights λ such that $\langle \lambda, \check{\beta} \rangle$ is a positive integer for any simple root β of Δ_0^+ , where $\check{\beta} \in [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ such that $\langle \beta, \check{\beta} \rangle = 2$.

Let $\mathcal{R} = \mathbb{Z}[e^\mu]$ for all $\mu \in \Lambda$. Let $M \in \mathcal{F}$ and m_μ denote the multiplicity of the weight μ in M we define the character

$$Ch(M) = \sum_{\mu \in \Lambda} m_\mu e^\mu \in \mathcal{R}.$$

Denote by $\mathcal{K}(\mathcal{F})$ the Grothendieck ring of the category \mathcal{F} . We will denote by $[M]$ the class of a module M in $\mathcal{K}(\mathcal{F})$ and by $[M : L]$ the multiplicity of an irreducible module L in the module M . Clearly, $Ch : \mathcal{K}(\mathcal{F}) \rightarrow \mathcal{R}$ is a homomorphism of rings. Note that

$$Ch(L_\lambda) = e^\lambda + \sum_{\mu < \lambda} m_\mu e^\mu,$$

thus the $Ch(L_\lambda)$ are linearly independent. Due to our convention about parity, for any two modules M and N in \mathcal{F} , $Ch(M) = Ch(N)$ if and only if $[M] = [N]$ in $\mathcal{K}(\mathcal{F})$. Hence Ch is injective. For classical Lie superalgebras the image of Ch is described in [25].

Let \mathfrak{g} be a finite-dimensional Kac–Moody superalgebra such that the quotient of $[\mathfrak{g}, \mathfrak{g}]$ by the center is simple. In other words, $\mathfrak{g} = \mathfrak{sl}(m, n)$, where $1 \leq m < n$, $\mathfrak{gl}(n, n)$ with $n \geq 2$, $\mathfrak{osp}(m, 2n)$, $D(2, 1; \alpha)$, G_3 or F_4 (see [15]). Let G denote a connected algebraic supergroup with Lie algebra \mathfrak{g} . For existence of such supergroup see, for instance, [23]. Then G satisfies our assumptions.

Recall that \mathfrak{g} is equipped with a non degenerate invariant bilinear form and the restriction of this bilinear form to \mathfrak{h} is also non degenerate. Thus, we have a non-degenerate form on Λ .

Following [16] we call a weight λ *typical* if $\langle \lambda + \rho, \alpha \rangle \neq 0$ for any isotropic root α . (Recall that an isotropic root is automatically odd). It follows from [16] that $P_\lambda = L_\lambda$ if and only if λ is typical.

As follows from [16] a typical λ is dominant iff $\lambda + \rho \in \Lambda^+$. For a general λ , the latter statement is true only for $\mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2, 2n)$ and a special choice of a Borel subgroup.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} and $\mathcal{Z}(\mathfrak{g})$ be its center. For every weight λ , we write χ_λ for the corresponding central character. A central character χ is dominant if there exists a dominant λ such that $\chi = \chi_\lambda$.

The category \mathcal{F} splits into direct sum of blocks $\oplus \mathcal{F}_\chi$ consisting of modules admitting the generalized central character χ . For any $M \in \mathcal{F}$ we denote by M_χ the projection of M to the block \mathcal{F}_χ .

2. GEOMETRIC INDUCTION AND BGG RECIPROCITY

In this section we assume that G satisfies the assumptions of Section 1.

Let B be a Borel subgroup of G with Lie algebra \mathfrak{b} , and let V be a B -module. Denote by \mathcal{V} the induced vector bundle $G \times_B V$ on the flag supervariety G/B . In [11] we defined

$$\Gamma_i(G/B, V) := H^i(G/B, \mathcal{V}^*)^*.$$

Recall that $H^i(G/B, \mathcal{V}^*)$ is a G -module, see [17], [18], and it is not difficult to see that $\Gamma_i(G/B, V) := H^i(G/B, \mathcal{V}^*)^*$ is an object of \mathcal{F} if V satisfies the parity condition about weights.

It is also possible to define $H^i(G/B, \mathcal{V}^*)$ following [14] or using the Zuckerman functor approach, see, for instance, [24]. Using one of those approaches one can avoid the rather technical question of existence of G/B .

Denote by C_λ the one-dimensional B -module with weight $\lambda \in \Lambda$ and by $\mathcal{E}(\lambda)$ the class of the *Euler characteristic* of the sheaf \mathcal{C}_λ^* belonging to the category \mathcal{F} , namely

$$\mathcal{E}(\lambda) := \sum_{\mu} \sum_{i=0}^{\dim(G/B)_0} (-1)^i [\Gamma_i(G/B, C_\lambda) : L_\mu] [L_\mu].$$

One can easily generalize Proposition 1 in [11] or Theorem 12 in [24] and obtain the following character formula for $\mathcal{E}(\lambda)$

$$(1) \quad Ch \mathcal{E}(\lambda) = D \sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)},$$

where

$$D_0 = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}), D_1 = \prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2})^{m_\alpha}, D = \frac{D_1}{D_0}.$$

Note that (1) implies the following

Lemma 1. (a) For all $w \in W$, one has

$$\mathcal{E}(\lambda) = \varepsilon(w) \mathcal{E}(w(\lambda + \rho) - \rho).$$

In particular, if $\langle \lambda + \rho, \check{\beta} \rangle = 0$ for some even root β , then $\mathcal{E}(\lambda) = 0$.

(b) The set

$$\{Ch \mathcal{E}(\lambda), \lambda + \rho \in \Lambda^+\}$$

is linearly independent in \mathcal{R} .

Proof. (a) follows immediately from (1). To prove (b) we observe that any W -orbit in Λ with trivial stabilizer meets Λ^+ in exactly one point. Hence the set $\{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}, \lambda + \rho \in \Lambda^+\}$ is linearly independent in \mathcal{R} . Therefore (1) implies (b). \square

We continue with the following analogue of Bott's reciprocity result.

Proposition 1. Let \mathfrak{n} denote the maximal nilpotent ideal of \mathfrak{b} and $M^{\mathfrak{h}}$ denote the set of \mathfrak{h} -invariants in an \mathfrak{h} -module M . Then, for any B -module V and any dominant weight λ , we have

$$[H^i(G/B, \mathcal{V}) : L_\lambda] = \dim H^i(\mathfrak{n}, P_\lambda^* \otimes V)^{\mathfrak{h}}.$$

Proof. For every $M \in \mathcal{F}$ we have

$$[M : L_\lambda] = \dim Hom_{\mathfrak{g}}(P_\lambda, M).$$

Consider an injective resolution $0 \rightarrow R_0 \rightarrow R_1 \rightarrow \dots$ of V in the category of B -modules. By definition $H^i(G/B, \mathcal{V})$ is given by the i -th cohomology of the complex

$$0 \rightarrow H^0(G/B, \mathcal{R}_0) \rightarrow H^0(G/B, \mathcal{R}_1) \rightarrow \dots$$

Since $Hom_G(P_\lambda, \cdot)$ is an exact functor, $Hom_G(P_\lambda, H^i(G/B, \mathcal{V}))$ is given by the i -th cohomology of the complex

$$0 \rightarrow Hom_G(P_\lambda, H^0(G/B, \mathcal{R}_0)) \rightarrow Hom_G(P_\lambda, H^0(G/B, \mathcal{R}_1)) \rightarrow \dots$$

Using Frobenius reciprocity we have

$$Hom_G(P_\lambda, H^0(G/B, \mathcal{R}_j)) = Hom_B(P_\lambda, R_j).$$

Thus, we obtain that

$$Hom_G(P_\lambda, H^i(G/B, \mathcal{V})) = Ext_B^i(P_\lambda, V).$$

Now we recall that B is the semidirect product of the torus H and the unipotent supergroup N . Therefore it is easy to see that

$$Ext_B^i(P_\lambda, V) = Ext_{\mathfrak{n}}^i((P_\lambda, V)^{\mathfrak{h}}) = H^i(\mathfrak{n}, P_\lambda^* \otimes V)^{\mathfrak{h}}.$$

□

By dualizing we obtain the following corollary.

Corollary 1. *Let \mathfrak{n} denote the maximal nilpotent ideal of \mathfrak{b} and $M^{\mathfrak{h}}$ denote the set of \mathfrak{h} -invariants in an \mathfrak{h} -module M . Then, for any weight ν and any dominant weight λ , we have*

$$[\Gamma_i(G/B, C_\nu) : L_\lambda] = \dim H^i(\mathfrak{n}, P_\lambda \otimes C_{-\nu})^{\mathfrak{h}}.$$

Denote by $b(\nu, \lambda)$ the coefficients in the decomposition

$$\mathcal{E}(\nu) = \sum_{\lambda} b(\nu, \lambda) [L_\lambda].$$

Lemma 1 implies

$$(2) \quad b(\nu, \lambda) = \varepsilon(w)b(w(\nu + \rho) - \rho, \lambda).$$

Corollary 2. *The coefficient $b(\nu, \lambda)$ is equal to the constant term in the formal expression $D^{-1}e^{-\nu-\rho}Ch(P_\lambda)$.*

Proof. - By proposition 1

$$b(\nu, \lambda) = \sum_i (-1)^i \dim H^i(\mathfrak{n}, P_\lambda \otimes C_{-\nu})^{\mathfrak{h}}.$$

Using the complex which computes the Lie superalgebra cohomology we obtain

$$\begin{aligned} \sum_i (-1)^i Ch H^i(\mathfrak{n}, P_\lambda \otimes C_{-\nu}) &= \sum_i (-1)^i Ch(P_\lambda \otimes C_{-\nu} \otimes \Lambda^i(\mathfrak{n}^*)) = \\ &= e^{-\nu} Ch P_\lambda \sum_i (-1)^i Ch \Lambda^i(\mathfrak{n}^*). \end{aligned}$$

To finish the proof, use

$$\sum_i (-1)^i Ch \Lambda^i(\mathfrak{n}^*) = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})^{m_\alpha}} = D^{-1}e^{-\rho}.$$

□

Corollary 3. - *One has:*

$$\text{Ch } P_\lambda = D \sum_{\nu+\rho \in \Lambda^+} b(\nu, \lambda) \sum_{w \in W} \varepsilon(w) e^{w(\nu+\rho)}.$$

Proof. - The statement follows from Corollary 2 and (2) since Corollary 2 implies

$$\text{Ch } P_\lambda = D \sum_{\nu \in \Lambda} b(\nu, \lambda) e^\nu.$$

□

Theorem 1. *We have the following identity in $\mathcal{K}(\mathcal{F})$*

$$[P_\lambda] = \sum_{\mu+\rho \in \Lambda^+} a(\lambda, \mu) \mathcal{E}(\mu).$$

Moreover, the following analogue of BGG reciprocity holds

$$a(\lambda, \mu) = b(\mu, \lambda).$$

Proof. - The statement follows from Corollary 3 and (1). □

Example. Let $G = GL(m, n)$ and B be the subgroup of upper triangular matrices. Then $\Lambda^+ - \rho$ coincides with the set of dominant weights. Moreover, it is well-known (see for example [20]) that for any $\lambda \in \Lambda^+$, $\Gamma_i(G/B, C_\lambda) = 0$ if $i > 0$. Moreover,

$$\Gamma_0(G/B, C_\lambda) \simeq K_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L_\lambda^0,$$

where $\mathfrak{g}^+ = \mathfrak{g}_0 + \mathfrak{b}$ and L_λ^0 is the irreducible \mathfrak{g}_0 -module of highest weight λ with trivial action of \mathfrak{b}_1 . The module K_λ was first considered in [16] and is usually called a Kac module. It was proven in [26] that every indecomposable projective module P_λ has a filtration by Kac modules K_μ and that the multiplicity of K_μ in P_λ equals the multiplicity of L_λ in K_μ . A combinatorial algorithm for calculating $a(\lambda, \mu)$ in this case was obtained by Brundan, [3]. We will explain it in Section 4 after introducing weight diagrams.

3. CLASSICAL SUPERGROUPS $GL(m, n)$ AND $SOSP(m, 2n)$

In this section we collect all necessary facts about roots and weights for the classical supergroups. So we assume that $G = GL(m, n)$ or $SOSP(m, 2n)$.

The lattice Λ of all integral weights is

$$\Lambda = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^n \mathbb{Z}\delta_j.$$

We define a parity homomorphism $p : \Lambda \rightarrow \mathbb{Z}_2$ by $p(\varepsilon_i) = 0, p(\delta_j) = 1$ for all $1 \leq i \leq m, 1 \leq j \leq n$. The invariant form on Λ is given by

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, (\varepsilon_i, \delta_j) = 0, (\delta_i, \delta_j) = -\delta_{ij}.$$

Now we recall the description of Δ (see [15]).

The even roots Δ_0 of $\mathfrak{gl}(m, n)$ are all the vectors of the form $\varepsilon_i - \varepsilon_j$ and $\delta_i - \delta_j$ with $i \neq j$. The odd roots Δ_1 of $\mathfrak{gl}(m, n)$ are all the vectors of the form $\varepsilon_i - \delta_j$ and $\delta_i - \varepsilon_j$.

The even roots Δ_0 of $\mathfrak{osp}(2m, 2n)$ are all the vectors of the form $\pm\varepsilon_i \pm \varepsilon_j, \pm\delta_i \pm \delta_j$ (the signs can be chosen independently) with $i \neq j$ and $2\delta_i$. The odd roots Δ_1 of $\mathfrak{osp}(2m, 2n)$ are all the vectors of the form $\pm\varepsilon_i \pm \delta_j$.

The even roots Δ_0 of $\mathfrak{osp}(2m+1, 2n)$ are all the vectors of the form $\pm\varepsilon_i \pm \varepsilon_j$, $\pm\delta_i \pm \delta_j$ with $i \neq j$, $\pm\varepsilon_i$ and $\pm 2\delta_i$. The odd roots Δ_1 of $\mathfrak{osp}(2m+1, 2n)$ are all the vectors of the form $\pm\varepsilon_i \pm \delta_j$ and $\pm\delta_i$.

From now on we fix a Borel subalgebra of \mathfrak{g} , we make the same choice as in [11]. Below is the list of the simple roots for our choice of Borel subalgebras.

- If $\mathfrak{g} = \mathfrak{gl}(m, n)$, $m \geq n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n,$$

$$\rho = \frac{m-n-2}{2}\varepsilon_1 + \frac{m-n-4}{2}\varepsilon_2 + \dots + \frac{-m-n}{2}\varepsilon_m + \frac{m+n}{2}\delta_1 + \dots + \frac{m-n+2}{2}\delta_n;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ and $m \geq n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \dots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2} \sum_{i=1}^m \varepsilon_i + \frac{1}{2} \sum_{j=1}^n \delta_j + \sum_{i=1}^{m-n} (m-n-i+1)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ and $m < n$, the simple roots are

$$\delta_1 - \delta_2, \dots, \delta_{n-m} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+1}, \dots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2} \sum_{i=1}^m \varepsilon_i + \frac{1}{2} \sum_{j=1}^n \delta_j + \sum_{j=1}^{n-m} (n-m-j)\delta_j;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $m > n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \dots, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m,$$

$$\rho = \sum_{i=1}^{m-n} (m-n-i)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $m \leq n$, the simple roots are

$$\delta_1 - \delta_2, \dots, \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, \dots, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m,$$

$$\rho = \sum_{i=1}^{n-m} (n-m-i+1)\delta_i.$$

Finally, we give a description of Λ^+ . Let

$$\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots + b_n\delta_n.$$

Then λ is integral iff $a_i, b_j \in \mathbb{Z}$ for $G = GL(m, n)$ or $SOSP(2m, 2n)$, and $a_i, b_j \in \frac{1}{2} + \mathbb{Z}$ for $G = SOSP(2m+1, 2n)$. Furthermore, $\lambda + \rho \in \Lambda^+$ if

$$a_1 > a_2 > \dots > a_m, \quad b_1 > b_2 > \dots > b_n \quad \text{if } G = GL(m, n);$$

$$a_1 > a_2 > \dots > a_m \geq \frac{1}{2}, \quad b_1 > b_2 > \dots > b_n \geq \frac{1}{2} \quad \text{if } G = SOSP(2m+1, 2n);$$

$$a_1 > a_2 > \dots > a_{m-1} > |a_m| \text{ and } b_1 > b_2 > \dots > b_n > 0, \quad \text{if } G = SOSP(2m, 2n).$$

Every $\lambda \in \Lambda^+ - \rho$ is dominant. If $G = GL(m, n)$, the set of dominant weights coincides with $\Lambda^+ - \rho$. In the orthosymplectic case we will formulate the condition of dominance in the next section.

4. WEIGHT DIAGRAMS

We recall the definitions and notations for weight diagrams, introduced in [5] for $GL(m, n)$, and in [11] for $SOSP(m, 2n)$. Note that our notations slightly differ from those in [5], for translation see [19].

Let $\mathbb{T} \subset \mathbb{R}$ be a discrete set, $X = (x_1, \dots, x_m) \in \mathbb{T}^m$, $Y = (y_1, \dots, y_n) \in \mathbb{T}^n$. A diagram $f_{X,Y}$ is a function defined on \mathbb{T} whose values are multisets with elements $<, >, \times$ according to the following algorithm.

- Put the symbol $>$ in position t for all i such that $x_i = t$.
- Put the symbol $<$ in position t for all i such that $y_i = t$.
- If there are both $>$ and $<$ in the same position replace them by the symbol \times , repeat if possible.

Thus, $f_{X,Y}(t)$ may contain at most one of the two symbols $>, <$. We represent $f_{X,Y}$ by the picture with \circ standing in position t whenever $f(t)$ is an empty set.

Let $G = GL(m, n)$. Let λ be a dominant integral weight such that

$$\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots + b_n\delta_n.$$

Set $\mathbb{T} = \mathbb{Z}$,

$$X_\lambda = (a_1, \dots, a_m), Y_\lambda = (-b_1, \dots, -b_n).$$

The diagram $f_\lambda = f_{X_\lambda, Y_\lambda}$ is called the *weight diagram* of λ .

A diagram is the weight diagram of some dominant weight if and only if $f(t)$ is empty or is a single element set since both sequences a_1, \dots, a_m and b_1, \dots, b_m are strictly decreasing and hence do not have repetitions.

Now let $G = SOSP(2m, 2n)$. Set $\mathbb{T} = \mathbb{Z}_{\geq 0}$. For a dominant weight λ such that $\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots + b_n\delta_n$ let

$$X_\lambda = (a_1, \dots, a_{m-1}, |a_m|), Y_\lambda = (b_1, \dots, b_n), f_\lambda = f_{X_\lambda, Y_\lambda}.$$

The position 0 is called the *tail position*. If the tail position is empty we put $[+]$ or $[-]$ before the diagram if $a_m > 0$ or $a_m < 0$ respectively.

A diagram f_λ is the weight diagram of a dominant λ if and only if

- for any $t \neq 0$, $f_\lambda(t)$ is empty or a single element set;
- the multiset $f_\lambda(0)$ does not contain $<$, contains $>$ with multiplicity at most 1 (it may contain any number of \times).

A diagram f_λ and a weight λ are called *tailless* if $f(0)$ does not contain \times . A weight λ is tailless iff $\lambda + \rho \in \Lambda^+$.

Finally, let $G = SOSP(2m+1, 2n)$. Let $\mathbb{T} = \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and define X_λ, Y_λ and f_λ as in the case $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$. The dominance condition is equivalent to the following condition on a weight diagram f

- $f(t)$ is empty or a single element set for any $t \neq \frac{1}{2}$;
- $f(\frac{1}{2})$ may contain at most one of $<$ or $>$ and any number of \times .

The position $\frac{1}{2}$ is called the *tail position*. It is possible that two dominant weights have the same weight diagram. That may happen if $f(\frac{1}{2})$ does not contain $>$ or $<$ and has at least one \times . For example, the diagram with two \times at $\frac{1}{2}$ corresponds to $(\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}, \frac{1}{2})$ and to $(-\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}, \frac{1}{2})$. So if the weight diagram has at least one \times and no $<, >$ at the position $\frac{1}{2}$ we put an indicator (which we sometimes refer to as "sign") (\pm) before the weight diagram. Its value is $+$ if the corresponding weight has the form

$$\lambda + \rho = (a_1, \dots, a_{m-s}, \frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}|b_1, \dots, b_{n-s}, \frac{1}{2}, \dots, \frac{1}{2}),$$

and $-$ if the corresponding weight has the form

$$\lambda + \rho = (a_1, \dots, a_{m-s}, -\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2} | b_1, \dots, b_{n-s}, \frac{1}{2}, \dots, \frac{1}{2}),$$

where s is the number of crosses at the position $\frac{1}{2}$.

A weight λ is *tailless* if $f_\lambda(\frac{1}{2})$ has at most one symbol, and if this symbol is \times the indicator is $+$. As in the case $G = SOSP(2m, 2n)$ a weight is tailless iff $\lambda + \rho \in \Lambda^+$.

Thus, tailless diagrams are exactly the diagrams of weights in $\Lambda^+ - \rho$.

Recall that the number of \times in the diagram is called the degree of atypicality of the corresponding weight. The symbols $>$, $<$ are called the *core symbols* of the diagram. The *core* of a weight λ is the diagram f_λ with all \times and the sign indicator removed except the case when $G = SOSP(2m, 2n)$ and there are no \times . As follows from [11] two weights admit the same central character (i.e. the corresponding simple modules belong to the same block) if the cores of the two weights are the same. Thus, if two weights have the same central character, then they have the same degree of atypicality. Hence the degree of atypicality of a central character χ is well defined. We denote it by $at(\chi)$. A weight (resp. central character) is typical if its degree of atypicality is zero. If χ is typical, then \mathcal{F}_χ is semisimple and has only one simple object up to isomorphism.

Let λ be dominant and tailless. Following Brundan and Stroppel [8] we define the cap diagram of f_λ by the following rules. The left end of a cap is at a \times and the right end is at an empty position. We start from the rightmost \times , make the cap by joining it to the next free position on the right (the end is not free any more), and then repeat for the next \times to the left, and so on until there is no \times left. There is no empty position under any cap. One can see that for each weight diagram there is a unique cap diagram.

Example. For the following weight diagram,

$$\circ \quad \times \quad > \quad < \quad \times \quad \circ \quad \circ$$

the caps are the following:

$$\circ \quad \times \quad > \quad < \quad \times \quad \circ \quad \circ$$

Denote by $\mathcal{P}(\lambda)$ the set of all the weights μ with weight diagram f_μ which can be obtained from f_λ by moving some of the \times -s from the left end to the right end of a cap. Note that the cardinality of $\mathcal{P}(\lambda)$ is 2^k , where k is the total number of \times .

If $G = GL(m, n)$, then all diagrams are tailless. The following result was proven in [3].

Proposition 2. - Let $G = GL(m, n)$. Then

$$[P(\lambda)] = \sum_{\mu \in \mathcal{P}(\lambda)} \mathcal{E}(\mu).$$

5. CATEGORIFICATION OF $\mathfrak{gl}_{\infty/2}$ IN ORTHOSYMPLECTIC TERMS

In this section we assume that $G = SO(2m+1, 2n)$. We denote by $\mathcal{K}(\mathcal{E})$ the subgroup of $\mathcal{K}(\mathcal{F})$ (resp. $\mathcal{K}(P)$) generated by the Euler characteristics $\mathcal{E}(\lambda)$ for all $\lambda + \rho \in \Lambda^+$ (resp. by the classes of indecomposable projective modules P_λ for all dominant λ).

5.1. Representation of $\mathfrak{gl}_{\infty/2}$. We denote by $\mathfrak{gl}_{\infty/2}$ the infinite dimensional Lie algebra defined over \mathbb{Z} with Dynkin diagram

$$\circ - \circ - \circ - \circ - \dots,$$

and by V its standard representation with basis $v_{1/2}, v_{3/2}, \dots$. We note $w_{1/2}, w_{3/2}, \dots$ the basis in V^* such that $\langle w_i, v_j \rangle = \delta_{ij}(-1)^{i+1/2}$. Let $E_{i,j}$ be the element of $\mathfrak{gl}_{\infty/2}$ which acts on V the following way: $E_{i,j}(v_k) = \delta_{jk}v_i$ and on V^* : $E_{i,j}(w_k) = (-1)^{i+j}\delta_{ik}w_j$.

Consider the representation (defined in the natural way) $\Lambda^m(V^*) \otimes \Lambda^n(V)$ of $\mathfrak{gl}_{\infty/2}$. Then, if λ is a tailless dominant weight for $\mathfrak{osp}(2m+1, 2n)$, such that $\lambda + \rho = \sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j$ with $a_i, b_j \in 1/2 + \mathbb{Z}$, we associate to λ the following vector in $\Lambda^m(V^*) \otimes \Lambda^n(V)$:

$$x_\lambda := w_{a_1} \wedge \dots \wedge w_{a_m} \otimes v_{b_1} \wedge \dots \wedge v_{b_n}.$$

We denote the weight of the action of $\mathfrak{gl}_{\infty/2}$ on v_i by γ_i (hence, the weight of the action on w_i is $-\gamma_i$), so that every vector x_λ is equipped with a $\mathfrak{gl}_{\infty/2}$ -weight which we denote by $\gamma(\lambda)$.

Remark. - The weights with the same core have the same $\mathfrak{gl}_{\infty/2}$ -weight. Therefore for every dominant weights λ, μ , one has

$$\chi_\lambda = \chi_\mu \Leftrightarrow \gamma(\lambda) = \gamma(\mu).$$

Define the map $\varphi : \mathcal{K}(\mathcal{E}) \rightarrow \Lambda^m(V^*) \otimes \Lambda^n(V)$ by $\varphi(\mathcal{E}(\lambda)) = x_\lambda$. Obviously, φ is an isomorphism of abelian groups.

5.2. Interpretation of the translation functors in this setting. Since the weight of the $\mathfrak{gl}_{\infty/2}$ -action encodes the central character, every block in \mathcal{F} can be parametrised by a $\mathfrak{gl}_{\infty/2}$ -weight γ , hence $\mathcal{F} = \bigoplus_\gamma \mathcal{F}_\gamma$. The translation functors consist in tensoring with the standard module E of $\mathfrak{osp}(2m+1, 2n)$ and then projecting in the appropriate block.

For any $M \in \mathcal{F}_\gamma$, we set:

$$T_{a,a+1}(M) = (M \otimes E)_{\gamma + \gamma_a - \gamma_{a+1}},$$

$$T_{a+1,a}(M) = (M \otimes E)_{\gamma + \gamma_{a+1} - \gamma_a}.$$

The functors $T_{a,a+1}$ and $T_{a+1,a}$, being exact, induce linear operators in the Grothendieck group $\mathcal{K}(\mathcal{F})$, and we keep the same notations for them. Note also that $T_{a,a+1}$ and $T_{a+1,a}$ are adjoint, i.e.

$$(3) \quad \text{Hom}_{\mathfrak{g}}(M, T_{a,a+1}(N)) = \text{Hom}_{\mathfrak{g}}(T_{a+1,a}(M), N)$$

Since, as can be read in [11] (Lemma 1(b) and Corollary 1) we have

$$\Gamma_i(G/B, C_\lambda \otimes E) = \Gamma_i(G/B, C_\lambda) \otimes E,$$

we see that $\mathcal{K}(\mathcal{E})$ is invariant under both $T_{a,a+1}$ and $T_{a+1,a}$.

Lemma 2. - For any $a \geq \frac{1}{2}$ one has

$$\varphi \circ T_{a+1,a} = E_{a+1,a} \circ \varphi$$

and

$$\varphi \circ T_{a,a+1} = E_{a,a+1} \circ \varphi.$$

Proof. - Let γ' be a $\mathfrak{gl}_{\infty/2}$ -weight and let λ be a tailless dominant $\mathfrak{osp}(2m+1, 2n)$ -weight with $\gamma(\lambda) = \gamma$. Corollary 1 in [11] implies that

$$(4) \quad \Gamma_i(G/B, (C_\lambda \otimes E)_{\Phi^{-1}(\gamma')}) = (\Gamma_i(G/B, C_\lambda) \otimes E)_{\gamma'},$$

where $\Phi^{-1}(\gamma')$ is the set of all weights μ such that $\gamma(\mu) = \gamma'$.

The \mathfrak{b} -module $L_\lambda(\mathfrak{b}) \otimes E$ has a filtration by all the C_μ with $\mu = \lambda \pm \varepsilon_j, \lambda \pm \delta_k, \lambda$. Since the Euler characteristic of the associated graded sheaf coincides with the Euler characteristic of the original sheaf, we obtain

$$T_{a,a+1}(\mathcal{E}(\lambda)) = \sum_{\mu} \mathcal{E}(\mu),$$

where the sum is taken over all $\mu = \lambda \pm \varepsilon_j, \lambda \pm \delta_k$ such that $\gamma(\mu) = \gamma(\lambda) + \gamma_a - \gamma_{a+1}$.

By direct inspection if μ is not dominant, then $(\mu + \rho, \beta) = 0$ for some even root β and therefore by (1) $\mathcal{E}(\mu) = 0$. Hence in fact only dominant μ contribute in the summation. To find all such μ we use the weight diagram f_λ of λ , and we only have to look at the positions a and $a+1$ in f_λ . Here is a table of the different f_μ which can occur (see figure 1). Note that since only tailless weights appear, if the left position is $\frac{1}{2}$, and there is a \times then the sign before it should be (+) and we omit it for simplicity of notation.

f_λ	a	$a+1$	f_μ	a	$a+1$	
	o	o	\emptyset			
	o	<	<		o	
	o	>	\emptyset			
	o	x	<		>	
	<	o	\emptyset			
	<	<	\emptyset			
	<	>	\emptyset			
	<	x	\emptyset			
	>	o	o		>	
	>	<	o	x	\oplus	x o
	>	>	\emptyset			
	>	x	x		>	
	x	o	<		>	
	x	<	<		x	
	x	>	\emptyset			
	x	x	\emptyset			

(figure 1)

Now we conclude, noticing that

$$T_{a,a+1}(\mathcal{E}(\lambda)) = \sum_{\mu} \mathcal{E}(\mu)$$

with the weights μ we computed, and since

$$E_{a,a+1}(x_\lambda) = \sum_{\mu} x_{\mu}$$

by construction, we have proved the first identity. The proof of the second identity is similar and we leave it to the reader. \square

Remark. - The operators $E_{a,a+1}$ and $E_{a+1,a}$ satisfy the Serre relations.

5.3. The switch functor. Recall that the switch functor $sw : \mathcal{F}_\gamma \rightarrow \mathcal{F}_\gamma$ is defined by

$$sw(M) = (M \otimes E)_\gamma.$$

We also denote by sw the corresponding linear operator in $\mathcal{K}(\mathcal{F})$.

The following proposition explains how sw acts on $\mathcal{E}(\lambda)$.

Lemma 3. - *Let λ be a tailless dominant weight. If f_λ has an empty tail position, then $sw(\mathcal{E}(\lambda)) = \mathcal{E}(\lambda)$.*

If f_λ has \times at the tail position, then $sw(\mathcal{E}(\lambda)) = -\mathcal{E}(\lambda)$.

If f_λ has $>$ or $<$ at the tail position, then $sw(\mathcal{E}(\lambda)) = 0$.

Proof. - We use the same idea as in the proof of Lemma 2. As follows from Corollary 1 in [11]

$$sw(\mathcal{E}(\lambda)) = \sum_{\mu} \mathcal{E}(\mu),$$

where the summation is over all $\mu = \lambda \pm \varepsilon_j, \lambda, \lambda \pm \delta_k$ such that $\gamma(\mu) = \gamma(\lambda)$.

If f_λ has empty tail position, then the only weight μ appearing in the sum is λ .

If f_λ has $>$ at the tail position, then the weights μ appearing in sum are λ and $\lambda - \varepsilon_m$. But $\lambda - \varepsilon_m = s(\lambda + \rho) - \rho$ where s is the reflection with respect to the root ε_m . Hence by (1) $\mathcal{E}(\lambda - \varepsilon_m) = -\mathcal{E}(\lambda)$ and $sw(\mathcal{E}(\lambda)) = 0$.

Similarly, if f_λ has $<$ at the tail position,

$$sw(\mathcal{E}(\lambda)) = \mathcal{E}(\lambda) + \mathcal{E}(\lambda - \delta_n) = 0.$$

Finally, if f_λ has \times at the tail position, then

$$sw(\mathcal{E}(\lambda)) = \mathcal{E}(\lambda) + \mathcal{E}(\lambda - \varepsilon_m) + \mathcal{E}(\lambda - \delta_n) = -\mathcal{E}(\lambda).$$

\square

6. TRANSLATION FUNCTORS ACTION ON SIMPLE AND PROJECTIVE MODULES

In this section $G = \text{SOSP}(2m+1, 2n)$.

6.1. Functors $T_{a,a+1}$ and $T_{a+1,a}$.

Lemma 4. *Let $L_\lambda, L_\nu \in \mathcal{F}$, $\nu = \lambda - \alpha$ for some isotropic positive root $\alpha = \varepsilon_i + \delta_j$ such that $(\lambda + \rho, \alpha) = 0$. Then*

$$[\Gamma_0(G/B, C_\lambda) : L_\lambda] = 1, \quad [\Gamma_0(G/B, C_\lambda) : L_\nu] \geq 1.$$

Proof. - By Lemma 2 in [11] $\Gamma_0(G/B, C_\lambda)$ is the maximal finite dimensional quotient of the Verma module with highest weight λ . Hence $[\Gamma_0(G/B, C_\lambda) : L_\lambda] = 1$. Propositions 2 and 5 in [11] imply that for some parabolic subgroup P containing B we have $[\Gamma_0(G/P, L_\lambda(P)) : L_\nu] = 1$, where $L_\lambda(P)$ is a simple P -module with highest weight λ . By Lemma 2 in [11] there is a surjective homomorphism

$$\Gamma_0(G/B, C_\lambda) \rightarrow \Gamma_0(G/P, L_\lambda(P)).$$

Hence the statement. \square

Lemma 5. - Let χ and θ be two distinct central characters such that $at(\chi) \geq at(\theta)$. Denote by T the translation functor which maps \mathcal{F}_χ to \mathcal{F}_θ . Then, if some dominant weight λ (resp. λ_1, λ_2) has central character χ , then $T(L_\lambda)$ is either a simple module of \mathcal{F}_θ or zero.

Moreover, if $T(L_{\lambda_1}) = T(L_{\lambda_2}) \neq 0$, then $\lambda_1 = \lambda_2$.

Proof. - All \mathfrak{b} -singular vectors in $L_\lambda \otimes E$ have weights of the form $\lambda \pm \varepsilon_j, \lambda, \lambda \pm \delta_k$. At most one of those has the central character θ , as one can figure out writing the weight diagrams. Hence the first statement.

Assume now that $T(L_{\lambda_1}) = T(L_{\lambda_2}) = L_\mu$, then $\lambda_1 - \lambda_2$ is an isotropic root, a multiple of an even root, or zero.

Case 1 : $\lambda_1 - \lambda_2$ is a multiple of an even root say β (one has $\lambda_1 - \lambda_2 = \pm 2\varepsilon_i, \varepsilon_i - \varepsilon_j, \pm 2\delta_i$ or $\delta_i - \delta_j$) and then $\lambda_1 + \rho = s_\beta(\lambda_2 + \rho)$, where s_β is the reflection with respect to the root β . Hence λ_1 and λ_2 cannot be both dominant except if $(\lambda_1 + \rho, \beta) = 0$, in which case $\lambda_1 = \lambda_2$.

Case 2 : $\lambda_1 - \lambda_2$ is an isotropic root $\alpha = \pm(\varepsilon_i + \delta_j)$, which we can assume to be a positive root. As follows from Lemma 4

$$[\Gamma_0(G/B, C_{\lambda_1}) : L_{\lambda_1}] = 1 \quad [\Gamma_0(G/B, C_{\lambda_1}) : L_{\lambda_2}] \geq 1.$$

By Corollary 1 in [11] (see also (4))

$$T(\Gamma_0(G/B, C_{\lambda_1})) = \Gamma_0(G/B, C_\mu).$$

As $[\Gamma_0(G/B, C_\mu) : L_\mu] = 1$, either $T(L_{\lambda_1})$ or $T(L_{\lambda_2})$ is zero. \square

Lemma 6. - Let χ and θ be two distinct central characters such that $at(\chi) \leq at(\theta)$. Then the translation functor T which maps \mathcal{F}_χ to \mathcal{F}_θ maps projective indecomposable modules (PIMs for short) to PIMs or to zero.

Proof. - Let λ be a dominant weight with central character χ . Since $T(P_\lambda)$ is a projective module in \mathcal{F}_θ , it is sufficient to show that

$$Hom_{\mathfrak{g}}(T(P_\lambda), L_\mu) = \mathbb{C}$$

for at most one dominant weight μ , and is zero otherwise.

Let us denote by T^* the adjoint functor of T , which is the translation functor mapping \mathcal{F}_θ to \mathcal{F}_χ ; one has

$$Hom_{\mathfrak{g}}(T(P_\lambda), L_\mu) = Hom_{\mathfrak{g}}(P_\lambda, T^*(L_\mu))$$

and by the Lemma 5, the statement follows. \square

6.2. The action of the switch functor on simple modules and PIMs.

Lemma 7. - *Let λ be an integral dominant weight.*

If f_λ has the empty tail position, then $sw(L_\lambda) = L_\lambda$.

If f_λ has $>$ or $<$ at the tail position, then $sw(L_\lambda) = 0$.

If f_λ has several \times at the tail position and no core symbols, then $sw(L_\lambda) = L_\mu$, where f_μ is obtained from f_λ by change of sign.

The action of the switch functor on PIMs is given by the same rule.

Proof. - The first assertion easily follows from

$$sw(\Gamma_0(G/B, C_\lambda)) = \Gamma_0(G/B, C_\lambda).$$

To prove the second assertion choose the parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ with semisimple part $\mathfrak{osp}(2k+3, 2k)$ if f_λ has $k \times$ and $>$ at the tail position and with semisimple part $\mathfrak{osp}(2k+1, 2k+2)$ if f_λ has $k \times$ and $<$ at the tail position. Then C_λ has a natural \mathfrak{q} -module structure and L_λ is a quotient of the parabolically induced module $S_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} C_\lambda$. It is easy to check that

$$sw(S_\lambda) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} (C_\lambda \otimes E))_{\gamma(\lambda)} = 0.$$

Hence $sw(L_\lambda) = 0$.

Now let f_λ have $k \times$ -s at the tail position and no core symbols. Let \mathfrak{q} be the parabolic subalgebra with semisimple part $\mathfrak{osp}(2k+1, 2k)$. If the sign of f_λ is $-$ then L_λ is a quotient of S_λ . On the other hand,

$$S'_\lambda := sw(S_\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} (C_\lambda \otimes E'),$$

where E' is the standard module over semisimple part of \mathfrak{q} . The unique simple quotient of S'_λ is isomorphic to L_μ , where f_μ is obtained from f_λ by change of sign. Since the application of the switch functor to any other simple subquotient of S_λ can not produce L_μ we have $sw(L_\lambda) = L_\mu$. If the sign of f_λ is $+$, the assertion follows similarly from the fact that $sw(S'_\lambda) = S_\lambda$.

Finally, the statement about PIMs follows by duality. \square

6.3. Elementary changes. We define elementary changes on the weight diagram by the list.

- The change of the sign $(+)$ and $(-)$ in front of the diagram.
- An elementary change which involves positions a and $a+1$ with $a > \frac{1}{2}$:
 - a) $\dots \times \circ \dots \rightarrow \dots > < \dots$ (decreases the degree of atypicality by 1),
 - b) $\dots < \circ \dots \leftrightarrow \dots \circ < \dots$ (doesn't change the degree of atypicality),
 - c) $\dots > \circ \dots \leftrightarrow \dots \circ > \dots$ (doesn't change the degree of atypicality),
- An elementary change which involves positions $\frac{1}{2}$ and $\frac{3}{2}$
 - at) $(+) \times^k \circ \dots \rightarrow \underset{\times^{k-1}}{>} < \dots$ (decreases the degree of atypicality by 1),
 - bt) $\underset{\times^k}{>} \circ \dots \leftrightarrow (-) \times^k > \dots$ (doesn't change the degree of atypicality),
 - ct) $\underset{\times^k}{<} \circ \dots \leftrightarrow (-) \times^k < \dots$ (doesn't change the degree of atypicality),

The \times^k sign indicates that there are $k \times$ at the tail position and \leftrightarrow means that we can go in either direction.

Lemma 8. - *If (f_λ, f_μ) is a pair of weight diagrams of the list a), ..., ct), then for a suitable choice of a translation functor T we have $T(L_\lambda) = L_\mu$ and $T^*(P_\mu) = P_\lambda$.*

Proof. - In the cases b),c),bt) and ct), we already know ([11] Section 6) that T is an equivalence of the corresponding blocks, thus we have nothing to prove.

It remains to prove the statement for a pair (f_λ, f_μ) in the cases a) and at).

We have $\mu = \lambda + \delta_{a+1}$. It is easy to see that among the weights $\lambda \pm \varepsilon_i, \lambda \pm \delta_j$ only μ has weight $\gamma(\mu)$. Therefore by Corollary 1 in [11] we have

$$(5) \quad T(\Gamma_0(G/B, C_\lambda)) = \Gamma_0(G/B, C_\mu).$$

The multiplicity of L_λ in $\Gamma_0(G/B, C_\lambda)$ is 1 as well as the multiplicity of L_μ in $\Gamma_0(G/B, C_\mu)$. If L_ν is a simple subquotient of $\Gamma_0(G/B, C_\lambda)$ and $\nu \neq \mu$, then $\nu < \mu$ and by Lemma 5 $T(L_\nu) \neq L_\mu$. Therefore (5) implies $T(L_\lambda) = L_\mu$. \square

Lemma 9. - *Let λ be any dominant weight with atypicality degree $k > 0$. Then one can find a dominant typical weight μ such that there exist a sequence of weights $\mu = \mu_1, \mu_2, \dots, \mu_r = \lambda$ and translation functors T_1, \dots, T_{r-1} such that $P_{\mu_i} = T_{i-1}(P_{\mu_{i-1}})$, and $at(\mu_i) \leq at(\mu_{i+1})$.*

Proof. - Due to the previous Lemma we have to check that f_λ can be transformed to a typical f_μ by elementary changes.

We prove the statement by induction on degree of atypicality of λ . Let t be the position of the rightmost \times in the weight diagram f_λ . Assume first that $t \neq \frac{1}{2}$. If the position $t+1$ is empty we can use elementary change a) to decrease the atypicality degree of λ . If the position $t+1$ is occupied by a core symbol, we can use elementary changes of type b) and c) to move all core symbols in positions $t+1, t+2, \dots$ to the right. The diagram obtained in this way will have the position $t+1$ empty and now we can decrease the degree of atypicality using elementary change a).

Now let $t = \frac{1}{2}$. That means f_λ has only core symbols outside the tail position. Using elementary changes of type b) and c) we can transform f_λ to the diagram that has an empty position $\frac{3}{2}$. Hence without loss of generality we may assume that the position $\frac{3}{2}$ in f_λ is empty. Now we are going consider 3 different cases.

If all symbols at the tail positions are \times and the sign is $+$, we apply elementary change at) to decrease the degree of atypicality.

If all symbols at the tail positions are \times and the sign is $-$, we apply the switch functor to f_λ and reduce the situation to the previous case.

If the tail position contains a core symbol, we apply elementary change bt) or ct) and reduce the situation to the previous case.

Thus, the statement follows by induction. \square

7. THE ACTION OF TRANSLATION FUNCTORS IN THE CASE $SOSP(2m, 2n)$

In this section $G = SOSP(2m, 2n)$. There is an involutive automorphism σ preserving the maximal torus H which acts on Λ by the formula

$$\sigma(\varepsilon_m) = -\varepsilon_m, \quad \sigma(\varepsilon_i) = -\varepsilon_i \text{ if } i \neq m, \quad \sigma(\delta_j) = \delta_j.$$

The induced action of σ on the weight diagrams is the change of the sign if the diagram has a sign, otherwise σ preserves the weight diagram.

The action of σ on \mathcal{F} preserves blocks except the case when χ is typical and the diagram has an empty tail position. In the latter case σ permutes two blocks. If $M \in \mathcal{F}$ we denote by M^σ the twist of M by σ .

The induced action of σ on the Grothendieck ring $\mathcal{K}(\mathcal{F})$ has two eigenspaces $\mathcal{K}(\mathcal{F})^\pm$ with eigenvalues ± 1 . So we have the following decompositions

$$\mathcal{K}(\mathcal{F}) = \mathcal{K}(\mathcal{F})^+ \oplus \mathcal{K}(\mathcal{F})^-$$

and

$$\mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E})^+ \oplus \mathcal{K}(\mathcal{E})^-, \quad \mathcal{K}(\mathcal{P}) = \mathcal{K}(\mathcal{P})^+ \oplus \mathcal{K}(\mathcal{P})^-.$$

Note that the translation functors preserve those splittings.

7.1. Categorification of \mathfrak{gl}_∞ . As in Section 5 we realize translation functors by certain linear operators in a representation of \mathfrak{gl}_∞ . By \mathfrak{gl}_∞ we understand the Lie algebra with Dynkin diagram

$$\dots - \circ - \circ - \circ - \circ \dots,$$

and by U its standard representation with basis u_i with $i \in \mathbb{Z}$. Let $E_{i,j}$ be the element of \mathfrak{gl}_∞ which acts on V the following way: $E_{i,j}(v_k) = \delta_{jk}v_i$. Denote the weight of v_a by γ_a for $a > 0$ and by $-\gamma_{-a}$ for $a < 0$. Set weight of v_0 to be zero. Let $U^{\leq 0}$ be the span of u_i for all $i \leq 0$, U^+ and U^- be the span of u_i for $i > 0$ and $i < 0$ respectively. Set

$$X^+ = \Lambda^m(U^{\leq 0}) \otimes \Lambda^n(U^+),$$

$$X^- = \Lambda^m(U^-) \otimes \Lambda^n(U^+).$$

Finally set

$$F_{i,j} = (-1)^{i+j+1}(E_{i,j} + E_{-j,-i}), \text{ for } i, j > 0; \quad F_{i,0} = 2E_{0,-i}, \quad F_{0,i} = E_{-i,0}.$$

Those elements generate the Lie algebra $\mathfrak{sl}_{\infty/2} \oplus \mathfrak{sl}_{\infty/2}$ inside \mathfrak{gl}_∞ .

Let $\lambda + \rho \in \Lambda^+$ be such that $\lambda + \rho = \sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j$ with $a_i \in \mathbb{Z}_{\geq 0}, b_j \in \mathbb{Z}_{> 0}$, we associate to λ the vector

$$x_\lambda := u_{-a_m} \wedge \dots \wedge u_{-a_1} \otimes u_{b_1} \wedge \dots \wedge u_{b_n}.$$

The weight of x_λ and the translation functors $T_{a,a+1}$ and $T_{a+1,a}$ (for $a \in \mathbb{Z}_{> 0}$) are defined as in Section 5. In addition we define the translation functors $T_{0,1}, T_{1,0} : \mathcal{F} \rightarrow \mathcal{F}$ by the formulae:

$$T_{0,1}(M) = (M \otimes E)_{\gamma - \gamma_1} \text{ for } M \in \mathcal{F}_\gamma$$

$$T_{1,0}(M) = (M \otimes E)_{\gamma + \gamma_1} \text{ for } M \in \mathcal{F}_\gamma$$

Next we define isomorphisms of \mathbb{Z} -modules $\psi^\pm : \mathcal{K}(\mathcal{E})^\pm \rightarrow X^\pm$ by setting

$$\psi^+(\mathcal{E}(\lambda)) = x_\lambda, \quad \psi^-(\mathcal{E}(\lambda)) = 0$$

if $a_m = 0$,

$$\psi^\pm(\mathcal{E}(\lambda) \pm \mathcal{E}(\lambda)^\sigma) = x_\lambda,$$

if $a_m > 0$.

Lemma 10. - For any $a > 0$ one has

$$\psi^\pm \circ T_{a+1,a} = F_{a+1,a} \circ \psi^\pm$$

and

$$\psi^\pm \circ T_{a,a+1} = F_{a,a+1} \circ \psi^\pm.$$

If $a = 0$, then

$$\psi^+ \circ T_{1,0} = F_{1,0} \circ \psi^+$$

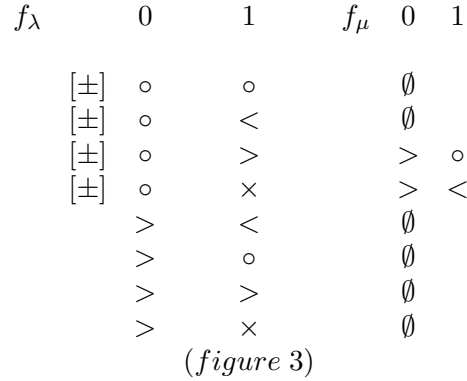
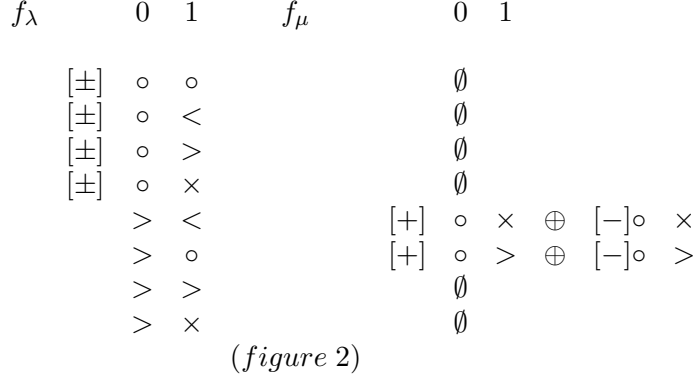
and

$$\psi^+ \circ T_{0,1} = F_{0,1} \circ \psi^+.$$

Proof. This lemma can be proven exactly as Lemma 2 by direct comparison.

The action of the functors $T_{a,a+1}$ for $a \geq 1$ in terms of weight diagrams is given in figure 1.

Below in figure 2 and figure 3 we give the action of $T_{0,1}$ and $T_{1,0}$ respectively. □



7.2. Action of translation functors on simple modules and PIMs. The following statement is analogous to Lemma 5.

Lemma 11. - (a) Let χ and θ be two distinct central characters such that $at(\chi) \geq at(\theta)$. Assume that $a \geq 1$. Let $T = T_{a,a+1}$ or $T_{a+1,a}$ be a translation functor which maps \mathcal{F}_χ to \mathcal{F}_θ . Then, if some dominant weight λ (resp. λ_1, λ_2) has central character χ , then $T(L_\lambda)$ is either a simple module of \mathcal{F}_θ or zero.

Moreover, if $T(L_{\lambda_1}) = T(L_{\lambda_2}) \neq 0$, then $\lambda_1 = \lambda_2$.

(b) If $at(\chi) \leq at(\theta)$ and $a \geq 1$, then $T(P_\lambda)$ is either PIM or zero and if $T(P_{\lambda_1}) = T(P_{\lambda_2}) \neq 0$, then $\lambda_1 = \lambda_2$.

We define non-tail elementary changes as a), b), c) in subsection 6.3 with the convention that a non-tail elementary change does not change the sign of the diagram.

Lemma 12. - Let λ be a dominant weight. Assume that the degree of atypicality of λ is not less than the degree of atypicality of $\gamma(\lambda) - \gamma_1$. If f_λ does not have > at the tail position, then $T_{0,1}(L_\lambda) = 0$. If f_λ has > at the tail position, then the action of $T_{0,1}$ is given by the following change in the diagram f_λ .

- 1) $\overset{>}{\times^k} \circ \dots \rightarrow \times^k > \dots;$
- 2) $\overset{>}{\times^{k-1}} \times \dots \rightarrow \times^k > \dots;$
- 3) $> \circ \dots \rightarrow [+]\circ > \dots \oplus [-] \circ > \dots$

Assume that the degree of atypicality of λ is not less than the degree of atypicality of $\gamma(\lambda) + \gamma_1$. If $T_{1,0}(L_\lambda) \neq 0$, then f_λ has no $>$ at the tail position and $f_\lambda(1) = \circ$ or $>$. The action of $T_{1,0}$ is given by the following change in f_λ .

- 4) $\times^k \circ \dots \rightarrow \overset{>}{\times^{k-1}} < \dots;$
- 5) $\times^k > \dots \rightarrow \overset{>}{\times^k} \circ \dots;$
- 6) $[\pm]\circ > \dots \rightarrow > \circ \dots$

Proof. The statement is an immediate consequence of Theorem 3.2 (iii) in [22]. \square

Corollary 4. - Let λ be dominant and with degree of atypicality not greater than the degree of atypicality of $\gamma(\lambda) + \gamma_1$. Then $T_{1,0}(P_\lambda) = 0$ if f_λ contains $>$ at the tail position. If f_λ does not have $>$ at the tail position the action of $T_{1,0}$ can be described by the following diagrams.

- a') $\times^k > \dots \rightarrow \overset{>}{\times^k} \dots \oplus \overset{>}{\times^{k-1}} \times \dots;$
- b') $[\pm]\circ > \dots \rightarrow > \circ \dots$

Now assume that the degree of atypicality of λ is not greater than the degree of atypicality of $\gamma(\lambda) - \gamma_1$. If $T_{0,1}(P_\lambda) \neq 0$ then f_λ contains $>$ at the tail position. The action of $T_{0,1}$ on P_λ is given by one of the following changes:

- c') $\overset{>}{\times^{k-1}} < \dots \rightarrow \times^k \circ \dots;$
- d') $\overset{>}{\times^k} \circ \dots \rightarrow \times^k > \dots;$
- e') $> \circ \dots \rightarrow [+]\circ > \dots \oplus [-] \circ > \dots$

We call a')-e') the elementary tail changes. Note that a translation functor corresponding to an elementary tail change does not always map a PIM to a PIM, a') and e') map sometimes a PIM to the direct sum of two PIMs. Thus, a straightforward analogue of Lemma 9 does not hold.

Lemma 13. - The Grothendieck group $\mathcal{K}(P)$ is generated by $[P_\lambda]$ for typical λ and $[T_m \circ \dots \circ T_1(P_\lambda)]$ (λ typical) where T_i does not decrease the degree of atypicality of $T_{i-1} \circ \dots \circ T_1(P_\lambda)$.

Proof. - We prove the statement by induction on the degree of atypicality. Let S denote the span of $[T_m \circ \dots \circ T_1(P_\lambda)]$. It is clear that $[P_\lambda] \in S$ for typical λ . Assume that $[P_\mu] \in S$ if $at(\mu) = k - 1$. Suppose that $at(\lambda) = k$. If f_λ has at least one \times which is not at the tail position, then we can obtain $[P_\lambda]$ from some $[P_\nu]$ with $at(\nu) = k - 1$ in the same way as in the proof of Lemma 9.

Now assume that all the \times of f_λ are at the tail position. We check all the possible cases for f_λ .

If $f_\lambda = \times^k \circ \dots$, then $P_\lambda = T_{0,1}(P_\nu)$ with $f_\nu = \overset{>}{\times^{k-1}} < \dots$ (Corollary 4 c')). By induction assumption $[P_\nu] \in S$. Hence $[P_\lambda] \in S$.

If $f_\lambda = \times^k > \dots$ or $f_\lambda = \times^k < \dots$, we use non-tail elementary changes to move the non-tail symbols to the right, then apply the translation functor as in the previous case and then move the non-tail symbols back. For instance, if $f_\lambda = \times^k > \circ$, we use

$$\overset{>}{\times^{k-1}} < \circ \rightarrow \times^k \circ > \rightarrow \times^k > \circ.$$

Now let $f_\lambda = \underset{\times^k}{>} \dots$. Using non-tail elementary changes we can reduce to the case $f_\lambda = \underset{\times^k}{>} \circ \dots$ in the same way as above. By Corollary 4 a')

$$T_{1,0}(P_\nu) = P_\lambda \oplus P_\mu,$$

where $f_\nu = \times^k > \dots$ and $f_\mu = \underset{\times^{k-1}}{>} \times \dots$. But f_μ has \times at non-tail position. We have proved above that $[P_\mu] \in S$. We also have checked above that $[P_\nu] \in S$. This implies $[P_\lambda] \in S$. \square

8. PIM AS A LINEAR COMBINATION OF EULER CHARACTERISTICS

8.1. The case of a tailless weight. In this subsection $G = SOSP(2m, 2n)$ or $SOSP(2m+1, 2n)$.

Theorem 2. - Let λ be a tailless dominant weight. One has:

$$[P(\lambda)] = \sum_{\mu \in \mathcal{P}(\lambda)} \mathcal{E}(\mu).$$

Proof. - This lemma has the same proof as the corresponding statement, due to Jonathan Brundan, in the case $\mathfrak{gl}(m, n)$. However, we write down the argument in our setting.

Due to Lemma 8 and Lemma 13 it is sufficient to check that if the statement holds for P_κ , then it holds for $P_\lambda = T(P_\kappa)$, where T is a translation functor corresponding to some elementary change. If the elementary change is of type b) or c) consisting of moving a core symbol from position $t+1$ to t , then clearly the weight diagrams of $\mathcal{P}(\lambda)$ are obtained from those of $\mathcal{P}(\kappa)$ by exchanging symbols in position $t+1$ and t . If the elementary change is of type a)

$$> < \rightarrow \times \circ,$$

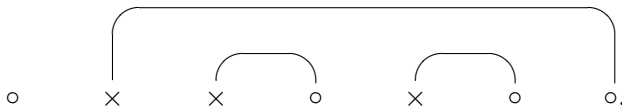
then the cap diagram of f_λ has exactly one new cap joining \times and \circ in f_λ . All other caps remain the same. Hence the statement holds in this case as well. \square

Examples. Let $G = SOSP(7, 6)$.

$$[P(\circ \times > < \times)] = \mathcal{E}(\circ \times > < \times) + \mathcal{E}(\circ \times > < \circ \times) + \mathcal{E}(\circ \circ > < \times \circ \times) + \mathcal{E}(\circ \circ > < \circ \times \times),$$

$$[P(\circ \times \times \circ \times)] = \mathcal{E}(\circ \times \times \circ \times) + \mathcal{E}(\circ \times \circ \times \times) + \mathcal{E}(\circ \times \times \circ \circ \times) + \mathcal{E}(\circ \times \circ \times \circ \times) + \mathcal{E}(\circ \circ \times \circ \times \circ \times) + \mathcal{E}(\circ \circ \times \circ \times \times) + \mathcal{E}(\circ \circ \times \circ \times \times).$$

and the caps are the following:



8.2. The general case $G = SO(2m + 1, 2n)$. Now let λ have a tail. Color all \times at the tail position emerald. We define the tailless weight $\bar{\lambda}$ as follows. Ignore for a moment the tail position and consider the cap diagram associated to the weight diagram for the remaining positions. We call a position free if it is empty and is not an end of a cap. Now, we move \times -s from the tail position to free positions according to the rule below:

- If f_λ has a core symbol at the tail, move all \times from the tail position to the free positions number 1,3,... counting from the left.
- If f_λ does not have a core symbol at the tail, move all but one \times from the tail to the free positions number 2,4,... counting from the left.

Example. If $f_\lambda = (\times^3 \times \circ \dots)$, then $f_{\bar{\lambda}} = (\times \times \circ \circ \times \circ \times \circ \dots)$.

Let

$$[P(\mu)] = \sum_{\nu} a(\mu, \nu) \mathcal{E}(\nu).$$

As follows from Theorem 2, if μ is tailless $a(\mu, \nu) = 0$ or 1.

Theorem 3. - Let $G = SO(2m + 1, 2n)$. If the tail position of the diagram f_λ contains $<$, $>$ or $(-)$ sign, then

$$a(\lambda, \nu) = (-1)^{c(\lambda, \nu)} a(\bar{\lambda}, \nu)$$

where $c(\lambda, \nu) = x + y$ where x is the total number of emerald \times in $f_{\bar{\lambda}}$, and y is the number of emerald \times in $f_{\bar{\lambda}}$ moved along the caps in order to get f_ν from $f_{\bar{\lambda}}$.

If the tail position of f_λ has a $(+)$, we change the sign of $a(\lambda, \nu)$ for all ν such that f_ν has a \times at the tail position.

Proof. - As in the proof of Theorem 2 we have to check that the statement for P_μ implies the statement for $T(P_\mu)$ for a translation functor T corresponding to some elementary change. This check for elementary changes a)-c) is completely analogous to the case of tailless λ . So we leave it to the reader.

Let λ and μ be related by an elementary change b) i.e.

$$f_\mu = (-) \times^k > \dots \rightarrow f_\lambda = \underset{\times^k}{>} \circ \dots$$

Then $\bar{\lambda}$ is obtained from $\bar{\mu}$ by switching \times at the tail position with $>$ at position $\frac{3}{2}$, and the number of emerald \times in $f_{\bar{\lambda}}$ and in $f_{\bar{\mu}}$ is the same. All $\nu' \in \mathcal{P}(\bar{\lambda})$ are obtained from $\nu \in \mathcal{P}(\bar{\mu})$ by interchanging symbols at positions $\frac{1}{2}$ and $\frac{3}{2}$. Clearly $c(\lambda, \nu) = c(\mu, \nu)$. The case of elementary change ct) is similar.

Now let λ and μ be related by an elementary change a), namely

$$f_\mu = \underset{\times^k}{>} < \dots \rightarrow f_\lambda = (+) \times^{k+1} \circ \dots$$

Then $\bar{\lambda}$ is obtained from $\bar{\mu}$ by removing core symbols from $\frac{1}{2}$ and $\frac{3}{2}$ and adding \times to the tail position $\frac{1}{2}$. The cap diagram for $f_{\bar{\lambda}}$ has an additional cap joining $\frac{1}{2}$ and $\frac{3}{2}$. The number of emerald \times increases by 1. However, since sign of f_λ is $(+)$, the signs agree after applying the switch functor to P_λ . \square

Examples. $G = SO(5, 4)$

$$\begin{aligned} \left[P \left(\begin{array}{c} > \\ \times < \end{array} \right) \right] &= -\mathcal{E}(> < \times) + \mathcal{E}(> < \circ \times), \\ \left[P \left(\begin{array}{c} (+) \times \\ \times \end{array} \right) \right] &= -\mathcal{E}((+) \times \circ \times) - \mathcal{E}(\circ \times \times) + \mathcal{E}((+) \times \circ \circ \times) + \mathcal{E}(\circ \times \circ \times), \end{aligned}$$

$$\left[P \left(\begin{array}{c} (-) \quad \times \\ \times \end{array} \right) \right] = \mathcal{E}((+) \times \circ \times) - \mathcal{E}(\circ \times \times) - \mathcal{E}((+) \times \circ \circ \times) + \mathcal{E}(\circ \times \circ \times).$$

8.3. The general case $G = \text{SOSP}(2m, 2n)$. Let λ be a dominant weight for $\text{SOSP}(2m, 2n)$. Define a dominant weight λ' for $\text{SOSP}(2m+1, 2n)$ whose weight diagram satisfies the condition $f_{\lambda'}(t) = f_{\lambda}(t - \frac{1}{2})$. If f_{λ} has some \times at the tail and no $>$, then $f_{\lambda'}$ must have a sign. In this case we use the notation λ'_{\pm} depending on the sign.

Theorem 4. - Let $G = \text{SOSP}(2m, 2n)$. Then for any $\nu \in \Lambda^+ - \rho$ we have

$$a(\lambda, \nu) = a(\lambda'_{\pm}, \nu').$$

Remark. As follows from Theorem 3 $a(\lambda'_{+}, \nu') = a(\lambda'_{-}, \nu')$ since $f_{\nu'}$ can not have \times at the tail position. Thus, Theorem 4 implies that the coefficient $a(\lambda, \mu)$ for $\text{SOSP}(2m, 2n)$ can be computed using the algorithm for $\text{SOSP}(2m+1, 2n)$. If f_{λ} has $>$ at the tail the coefficients remain the same. Otherwise some coefficients disappear because $f_{\nu'}$ can not have a \times at the tail position.

Proof. Let us introduce a few notations. For any \mathbb{Z} -module A , denote ${}_{\mathbb{Q}}A = \mathbb{Q} \otimes_{\mathbb{Z}} A$. For m, n fixed, we denote by $\mathcal{K}_{ev}^+(\mathcal{F})$ the σ -invariant subgroup of the Grothendieck group for $\text{SOSP}(2m, 2n)$ and by $\mathcal{K}_{odd}(\mathcal{F})$ the Grothendieck group for $\text{SOSP}(2m, 2n)$. The definitions of $\mathcal{K}_{ev}^+(P)$, $\mathcal{K}_{ev}^+(\mathcal{E})$, $\mathcal{K}_{odd}(P)$ and $\mathcal{K}_{odd}(\mathcal{E})$ are obvious.

We define two \mathbb{Q} -linear maps $\alpha : {}_{\mathbb{Q}}\mathcal{K}_{odd}(\mathcal{E}) \rightarrow {}_{\mathbb{Q}}\mathcal{K}_{ev}^+(\mathcal{E})$ and $\beta : {}_{\mathbb{Q}}\mathcal{K}_{ev}^+(\mathcal{E}) \rightarrow {}_{\mathbb{Q}}\mathcal{K}_{odd}(\mathcal{E})$ in the following way. First we define $\alpha : U \rightarrow V \oplus V^*$ and $\beta : V \oplus V^* \rightarrow U$ by the formulae

$$\begin{aligned} \alpha(v_{1/2}) &= 0, \quad \alpha(v_i) = u_{i-1/2}, \quad i > 1, \quad \alpha(w_i) = u_{1/2-i} \\ \beta(u_i) &= w_{1/2-i}, \quad i \leq 0, \quad \beta(u_i) = v_{i-1/2}, \quad i > 0. \end{aligned}$$

Next, we extend α and β to $\Lambda^m(V^*) \otimes \Lambda^n(V)$ and X^+ in the natural way. The following diagram explains the maps α and β on the level of Grothendieck groups

$$\begin{array}{ccc} & \xrightarrow{\beta} & \\ X^+ & \xleftarrow{\alpha} & \Lambda^m(V^*) \otimes \Lambda^n(V) \\ \psi^+ \uparrow & & \uparrow \phi \\ & \xrightarrow{\beta} & \\ {}_{\mathbb{Q}}\mathcal{K}_{ev}^+(\mathcal{E}) & \xleftarrow{\alpha} & {}_{\mathbb{Q}}\mathcal{K}_{odd}(\mathcal{E}). \end{array}$$

Note that α is surjective, β is injective and $\alpha \circ \beta = \text{id}$. For a linear operator T in $\mathcal{K}_{ev}^+(\mathcal{E})$ corresponding to a translation functor we define an operator T' in $\mathcal{K}_{odd}(\mathcal{E})$ by the following rules

- if $T = T_{a, a+1}$ then $T' = T_{a+1/2, a+3/2}$,
- if $T = T_{a, a-1}$ and $a \neq 1$ then $T' = T_{a+1/2, a-1/2}$,
- if $T = T_{1, 0}$ then $T' = 2T_{3/2, 1/2}$.

Direct computation gives the following result:

Lemma 14. One has for any $T = T_{a, a \pm 1}$

$$T = \alpha \circ T' \circ \beta.$$

Next we define $\bar{\beta} : {}_{\mathbb{Q}}\mathcal{K}_{ev}^+(P) \rightarrow {}_{\mathbb{Q}}\mathcal{K}_{odd}(P)$ by setting:

$$\begin{aligned} \bar{\beta}[P_{\lambda}] &= \frac{1}{2}([P_{\lambda'_+}] \oplus [P_{\lambda'_-}]) \text{ if } f_{\lambda} = (\times^k \dots), \quad k > 0; \\ \bar{\beta}[P_{\lambda} \oplus P_{\lambda}^{\sigma}] &= [P_{\lambda'}] \text{ if } f_{\lambda} = (\circ \dots); \end{aligned}$$

$\bar{\beta}[P_\lambda] = [P_{\lambda'}]$ if $f_\lambda = (\underset{\times^k}{>} \dots)$;

and $\bar{\alpha} : \mathbb{Q}\mathcal{K}_{odd}(P) \rightarrow \mathbb{Q}\mathcal{K}_{ev}^+(P)$ as follows:

If $\nu = \lambda'_\pm$ for some dominant $SOSP(2m, 2n)$ weight λ we set $\bar{\alpha}[P_\nu] = [P_\lambda]$ or $[P_\lambda \oplus P_\lambda^\sigma]$ (in the case $P_\lambda \neq P_\lambda^\sigma$). Otherwise we set $\bar{\alpha}[P_\nu] = 0$.

It follows immediately from the definitions that $\bar{\alpha}$ is surjective, $\bar{\beta}$ is injective and $\bar{\alpha} \circ \bar{\beta} = \text{id}$.

Lemma 15. *Let $T = T_{a, a \pm 1}$, $P = P_\lambda$ or $P = P_\lambda \oplus P_\lambda^\sigma$. If T does not decrease the degree of atypicality of λ , then*

$$T([P]) = \bar{\alpha} \circ T' \circ \bar{\beta}([P]).$$

Proof. If $T \neq T_{0,1}$ or $T_{1,0}$ the statement follows from the fact that the elementary changes listed in subsection 6.3 are the same in the even and odd cases.

If $T \neq T_{0,1}$ or $T_{1,0}$ one can just compare the actions of T and T' in all possible cases. We show how it works in the most interesting cases and leave to the reader the remaining cases.

$$\begin{array}{ccc} \times^k < & \xrightarrow{\bar{\beta}} & \frac{1}{2}((+) \times^k < \oplus (-) \times^k <) \\ T_{0,1} \downarrow & & \downarrow T_{1/2, 3/2} \\ 0 & \xleftarrow{\bar{\alpha}} & \frac{1}{2}(\underset{\times^k}{<})^\circ \end{array}$$

$$\begin{array}{ccc} \underset{\times^{k-1}}{>} < & \xrightarrow{\bar{\beta}} & \underset{\times^{k-1}}{>} < \\ T_{0,1} \downarrow & & \downarrow T_{1/2, 3/2} \\ \times^k \circ & \xleftarrow{\bar{\alpha}} & (+) \times^k \circ \end{array}$$

$$\begin{array}{ccc} \times^k > & \xrightarrow{\bar{\beta}} & \frac{1}{2}((+) x^k > \oplus (-) x^k >) \\ T_{1,0} \downarrow & & \downarrow 2T_{3/2, 3/2} \\ \underset{\times^k}{>} \circ \oplus \underset{\times^{k-1}}{>} \times & \xleftarrow{\bar{\alpha}} & \underset{\times^k}{>} \circ \oplus \underset{\times^{k-1}}{>} \times \end{array}$$

$$\begin{array}{ccc} [+]\circ > \oplus [-]\circ > & \xrightarrow{\bar{\beta}} & \circ > \\ \downarrow T_{1,0} & & \downarrow 2T_{3/2, 1/2} \\ 2 > \circ & \xleftarrow{\bar{\alpha}} & 2 > \circ \end{array}$$

□

Lemma 16. *One has*

$$\beta \circ \alpha|_{\mathbb{Q}\mathcal{K}_{odd}(P)} = \bar{\beta} \circ \bar{\alpha}.$$

Proof. Both $\beta \circ \alpha$ and $\bar{\beta} \circ \bar{\alpha}$ are projectors. Observe that $\text{Ker } \beta \circ \alpha$ is generated by $\mathcal{E}(\lambda)$ for all $f_\lambda = (< \dots)$ or $(\times \dots)$. $\text{Ker } \bar{\beta} \circ \bar{\alpha}$ is generated by $[P_\lambda]$ with $f_\lambda = (\underset{\times^k}{<} \dots)$ and $[P_{\lambda_+}] - [P_{\lambda_-}]$ with $f_\lambda = ((\pm) \times^k \dots)$. By Proposition 3 if $f_\lambda = (\underset{\times^k}{<} \dots)$, then $[P_\lambda] = \sum a(\lambda, \mu) \mathcal{E}(\mu)$ with $f_\mu = (< \dots)$ and if $f_\lambda = ((\pm) \times^k \dots)$, then $[P_{\lambda_+}] - [P_{\lambda_-}] = \sum a(\lambda, \mu) \mathcal{E}(\mu)$ with $f_\mu = (\times \dots)$. That implies

$$\text{Ker } \bar{\beta} \circ \bar{\alpha} = \text{Ker } \beta \circ \alpha \cap \mathbb{Q}\mathcal{K}_{odd}(P).$$

Similarly

$$\text{Im } \bar{\beta} \circ \bar{\alpha} = \text{Im } \beta \circ \alpha \cap_{\mathbb{Q}} \mathcal{K}_{\text{odd}}(P).$$

The statement follows. \square

Lemma 17. *One has*

$$\beta|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)} = \bar{\beta}, \quad \alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)} = \bar{\alpha}.$$

Proof. By definition $\beta([P]) = \bar{\beta}([P])$ if P is typical or a direct sum of two typical PIMs. By Lemma 13 it is sufficient to check that $\beta([P]) = \bar{\beta}([P])$ implies $\beta(T[P]) = \bar{\beta}(T[P])$ if T does not decrease the degree of atypicality of P . Indeed we have

$$\bar{\beta}(T([P])) = \bar{\beta} \circ \bar{\alpha}(T'(\bar{\beta}([P]))) = \bar{\beta} \circ \bar{\alpha}(T'(\beta([P])))$$

and similarly

$$\beta(T([P])) = \beta \circ \alpha(T'(\beta([P]))).$$

Thus the statement about β follows from the previous lemma.

The statement about α follows immediately, since

$$\bar{\beta} \circ \bar{\alpha} = \beta \circ \alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)}$$

implies

$$\alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)} \circ \bar{\beta} \circ \bar{\alpha} = \alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)} \circ \beta \circ \alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)},$$

hence $\alpha|_{\mathbb{Q}\mathcal{K}_{\text{odd}}(P)} = \bar{\alpha}$. \square

Now we are ready to prove Theorem 4. The case of a tailless λ was covered in Theorem 2. Write

$$[P_\lambda] = \alpha(\beta([P_\lambda])).$$

Use Theorem 3. If f_λ has $>$ at the tail we have

$$\beta([P_\lambda]) = [P_{\lambda'}] = \sum a(\lambda', \nu') \mathcal{E}(\nu'),$$

and since $\alpha(\mathcal{E}(\nu')) = \mathcal{E}(\nu)$ we obtain

$$[P_\lambda] = \sum a(\lambda', \nu') \mathcal{E}(\nu).$$

If f_λ does not have $>$ at the tail position, then

$$\beta([P_\lambda]) = \frac{1}{2}[P_{\lambda'_+} \oplus P_{\lambda'_-}] = \sum a(\lambda', \nu') \mathcal{E}(\nu').$$

Here all $f_{\nu'}$ have \circ at the tail position. Hence again

$$[P_\lambda] = \sum a(\lambda', \nu') \mathcal{E}(\nu).$$

\square

Examples. $G = \text{SOSP}(4, 2)$.

$$\begin{aligned} [P(>><)] &= \mathcal{E}(>><), \\ [P(> \times)] &= \mathcal{E}(> \times \circ) + \mathcal{E}(> \circ \times), \\ [P(><>)] &= \mathcal{E}(><>), \\ [P(\times \circ >)] &= \mathcal{E}(+ \circ \times >) + \mathcal{E}(- \circ \times >), \\ [P(\times > \circ)] &= \mathcal{E}(+ \circ > \times) + \mathcal{E}(- \circ > \times), \\ [P(\overset{\sim}{\times} \circ \circ) + P(> \times \circ)] &= 2\mathcal{E}(> \circ \times), \\ [P(\overset{\sim}{\times} \circ \circ)] &= -\mathcal{E}(> \times \circ) + \mathcal{E}(> \circ \times). \end{aligned}$$

$$G = \text{SOSP}(4, 4)$$

$$\left[P \begin{pmatrix} \times \\ \times \\ \times \end{pmatrix} \right] = -\mathcal{E}((+) \circ \times \times) - \mathcal{E}((-) \circ \times \times) + \mathcal{E}((+) \circ \times \circ \times) + \mathcal{E}((-) \circ \times \circ \times).$$

REFERENCES

- [1] J. Bernstein, I. Gel'fand, S. Gel'fand, Category of \mathfrak{g} -modules. *Func. Anal. Appl.*, 10 (1976), 87–92.
- [2] R. Bott, Homogeneous vector bundles. *Annals of Mathematics*, 66 (1957), no. 2.
- [3] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $gl(m|n)$. *J. Amer. Math. Soc.* 16 (2003), no. 1, 185–231.
- [4] R. Brauer, On modular and p -adic representations of algebras. *Proc. Nat. Acad. Sci. USA*, 25. (1939), 252–258.
- [5] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra I: cellularity. Preprint, 2008.
- [6] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra II: Koszulity. *Transform. Groups* 15 (2010), 1–45.
- [7] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra III: Category \mathcal{O} . To appear in *Represent. Theory* (2011).
- [8] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup. To appear in *J. Eur. Math. Soc.* (2011),
- [9] S.-J. Chang, N. Lam, W. Wang, Super duality and irreducible characters of ortho-symplectic Lie superalgebras. *Inventiones Mathematicae*, to appear (2011) (<http://www.springerlink.com/content/y4633362461h8365>).
- [10] E. Cline, B. Parshall, L. Scott, Finite-dimensional algebras and highest weight categories. *J. Reine Agnew. Math.*, 391 (1988), 85–99.
- [11] C. Gruson, V. Serganova, Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras. *Proc. of the LMS*, 101 (2010), no.3, 852–892.
- [12] J. Humphreys, Modular representation of classical Lie algebras and semisimple groups. *J. Algebra*, 19 (1971), 51–79.
- [13] R. Irving, BGG algebra and BGG reciprocity principle. *J. Algebra*, 135 (1990), 363–380.
- [14] J. Jantzen, Representations of algebraic groups. Second edition. *Mathematical Surveys and Monographs*, 107. American Mathematical Society, Providence, RI, 2003.
- [15] V. Kac, Lie superalgebras. *Advances in Math.* 26 (1977), no. 1, 8–96.
- [16] V. Kac, Characters of typical representations of classical Lie superalgebras. *Comm. Algebra* 5 (1977), no. 8, 889–897.
- [17] Yu. Manin, Gauge field theory and complex geometry. Translated from the 1984 Russian original by N. Koblitz and J. R. King. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 289. Springer-Verlag, Berlin, 1997.
- [18] Yu. Manin, I. Penkov, A. Voronov, Elements of supergeometry. (Russian) Translated in *J. Soviet Math.* 51 (1990), no. 1, 2069–2083.
- [19] I. Musson, V. Serganova, Combinatorics of character formulas for the Lie superalgebra $gl(m, n)$. *Transformation groups*, 16 (2011), no. 2, 555–578.
- [20] I. Penkov, Borel-Weil-Bott theory for classical Lie supergroups. (Russian) Translated in *J. Soviet Math.* 51 (1990), no. 1, 2108–2140.
- [21] V. Serganova, Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra $gl(m|n)$. *Selecta Math. (N.S.)* 2 (1996), no. 4, 607–651.
- [22] V. Serganova, On the superdimension of a finite-dimensional representation of a basic classical Lie superalgebra. *Supersymmetry in mathematics and physics. LNM*. To appear.
- [23] V. Serganova, Quasireductive supergroups. *New Developments in Lie theory and its applications, Contemporary Mathematics*, AMS, 544, 2011, 141–159.
- [24] V. Serganova, Structure and representation theory of Kac-Moody superalgebras. *Highlights in Lie algebraic methods*, Birkhauser, 2011, to appear.
- [25] A. Sergeev, A. Veselov. Grothendieck ring of basic classical Lie superalgebras. *Annals of Mathematics*, 173 (2011), no.2.
- [26] Y. M. Zou, Categories of finite-dimensional weight modules over type I classical Lie superalgebras. *J. of Algebra*, 180 (1996), 459–482.