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A Hodge Theorem for Noncompact Manifolds

Theorem *If M is a riemannian manifold, then the inclusion of the complex of coclosed harmonic forms into the de Rham complex induces a linear isomorphism in cohomology. If M has at most countably many connected components, this linear isomorphism is a Fréchet isomorphism.*

The simplest example is that of the real line with its standard metric. In degree zero the complex of coclosed harmonic forms is $\mathbb{C} \oplus \mathbb{C}x$, and in degree one it is $\mathbb{C}dx$, which gives the right cohomology.

[Manifolds are assumed to be C^∞ and Hausdorff.]

Proof. Theorem 5 in Section I.9.10 of Bourbaki [2] implying that M is paracompact, we can assume that it is connected, and also that it is non-compact (the result being classical in the compact case). Then the claim follows easily (using the Open Mapping Theorem and the fact that the de Rham cohomology is a Fréchet space) from the surjectivity of the laplacian on the de Rham complex (see *Algebra Background* below). Let us check this surjectivity. In [4, p. 158] de Rham proves (using results of Aronszajn, Krzywicki and Szarski [1]) that a harmonic form which has a zero of infinite order vanishes identically; this implies in particular that the laplacian satisfies Property (A) in Definition 5 of Malgrange [3, p. 333]; it is well known that the laplacian satisfies also Condition (P) — called **ellipticity** nowadays — in Definition 6 of [3, p. 338]; in view of Theorem 5 in [3, p. 341] this implies the desired surjectivity. QED

Algebra Background. Let A be a module over some unnamed ring, and let d, δ be two endomorphisms of A satisfying $d^2 = 0 = \delta^2$. Put $\Delta := d\delta + \delta d$. Assume $A = \Delta A + A_{d,\delta}$ where $A_{d,\delta}$ stands for $\ker d \cap \ker \delta$. Write $A_{\delta,\Delta}$ for $\ker \Delta \cap \ker \delta$. Note $dA_{\delta,\Delta} \subset A_{\delta,\Delta}$.

We claim that the natural map

$$H(A_{\delta,\Delta}, d) \to H(A, d)$$

between homology modules is bijective.

Injectivity. Assume $\delta da = 0$ form some a in A. We must find an x in $A_{\delta,\Delta}$ such that $dx = da$. We have $a = \Delta b + c$ for some $b \in A$ and some $c \in A_{d,\delta}$. One easily checks that $x := \delta db + c$ does the trick.

Surjectivity. Let a be in $\ker d$. We must find $x \in A$, $y \in A_{d,\delta}$ such that $a = dx + y$. We have $a = \Delta b + c$ for some $b \in A$ and some $c \in A_{d,\delta}$. One easily checks that $x := \delta b$, $y := \delta db + c$ works.

References

[1] Aronszajn N., Krzywicki A., Szarski J., A unique continuation theorem for exterior differential forms on Riemannian manifolds, *Ark. Mat.*, Volume 4, Number 5 (1962) 417-453. Available at springerlink.com.

[2] Bourbaki, N., **Topologie générale**, Vol. 1, Chapitres 1 à 4, Hermann, 1971.

[3] Malgrange B., Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier*, Grenoble **6** (1955-56) 271-354. Available at numdam.org.

[4] de Rham G., **Differentiable manifolds**, Springer-Verlag, 1984.

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