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Curriculum Vitae

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Positions held:

1994-1999: École Normale Supérieure de Cachan
1999-2002: Assistant Moniteur Normalien, Université de Cergy-Pontoise
2002-2003: Assistant, ETH Zürich
2003-2009: Maître de Conférences (Associate Professor), Université de Nice-Sophia Antipolis.
2009- : Professeur des Universités (Full Professor), Université Nancy 1

Ph.d. thesis:

Title: "Études asymptotiques d'équations elliptiques issues de la géométrie".
Advisor: Emmanuel Hebey. Defended: November 2001.
Ph.D. Committee: Henri Berestycki (Paris), Emmanuel Hebey (Cergy-Pontoise), Frédéric Helein (ENS Cachan), Patrice le Calvez (Paris), Lambertus Peletier (Leiden), Eric Séré (Paris), Michel Vaugon (Paris).
Mention: très honorable avec les félicitations du jury.

"Habilitation à diriger des recherches":

Title: "Analyse variationnelle et phénomènes non-linéaires pour des équations elliptiques critiques". Defended: June 28th 2007 at Nice.
Committee: Philippe Delanoë (Nice), Nassif Ghoussoub (Vancouver), Emmanuel Hebey (Cergy-Pontoise), Gilles Lebeau (Nice), Frank Pacard (Paris 12), Tristan Rivière (Zürich), Michaël Struwe (Zürich), Gabriella Tarantello (Rome 2).

A few activities in the mathematics department:

2004-2008: Member of the Hiring Committee, University of Nice.
2004-2009: Member of the Department Committee, University of Nice.
2006-2009: Co-organization of the "Geometry and Analysis" seminar, University of Nice.
2010- : In charge of the Master of Mathematics, University Nancy 1.

Editorial activities:
Associate editor of "International Journal of Mathematics and Mathematical Sciences" .

Participation to research grants:
2009-2012: Member of the French ANR Grant "Phénomènes de Concentration en Analyse Géométrique" (Concentration phenomena in geometric analysis).
2010: grant from the "Region Lorraine", France.
2011: joint "Hubert Curien" travel grant with Juncheng Wei, Hongkong.

Organization of conferences:
11/2005: Co-organization of the conference "Week-end niçois d'analyse géométrique conforme" in Nice (France).
08/2006: Co-organization of the conference "Geometric and Nonlinear Analysis" in Banff (Canada).
06/2007: Co-organization of the conference "Geometric Analysis in Nice" in Nice (France).
03/2008: Co-organization of the meeting "Nonlinear analysis and critical phenomena" in Cergy-Pontoise (France).
06/2011: Co-organization of the meeting "Geometric and Nonlinear Analysis: meeting in Lorraine" in Nancy (France).

Invited talks:
12/2000: Series of talks at the University of Rome III (Italy).
01/2001: PDE seminar, University of Cergy-Pontoise (France).
03/2001: Analysis seminar, University of Amiens (France).
01/2002: Geometry seminar, University of Nancy (France).
01/2002: Analysis seminar, University of de Tours (France).
02/2002: Mathematics colloquium, Tata Institute, Bangalore (India).
03/2002: Series of talks at the University of Rome III (Italy).
10/2002: Analysis seminar, ETH Zürich (Switzerland).
01/2003: Colloquium, University of Fribourg (Switzerland).
02/2003: PDEs and geometry seminar, Notre Dame University (USA).
02/2003: Geometric analysis seminar, Princeton University (USA).
03/2003: Analysis and geometry seminar, University of Nice (France).
04/2003: Complex Geometry seminar, University of Marseille I (France).
06/2003: Department seminar, University of la Rochelle (France).
08/2003: Conference "Partial Differential Equations", Oberwolfach (Germany).
09/2003: Conference "Affine Differential Geometry", Bedlewo (Poland).
01/2004: Analysis seminar, University of Amiens (France).
01/2004: Analysis seminar, ETH Zürich (Switzerland).
03/2004: PDEs and applications seminar, ENS Lyon (France).
11/2004: Differential Geometry and PDEs seminar, University of British Columbia, Vancouver (Canada).
03/2005: Conference "Recent Advances in Calculus of Variations and PDEs", Pisa (Italy).
05/2005: Conference "Analytic Aspects of Problems in Riemannian Geometry", Brest (France).
09/2005: Conference "Interactions between Complex Geometry and Real Analysis",

Hanover (Germany).
11/2005: Complex Geometry seminar, University of Marseille I (France).
02/2006: Analysis seminar, University of Reims (France).
02/2006: Differential Geometry and PDEs seminar, University of British Columbia, Vancouver (Canada).
03/2006: Series of lectures at the University of Wisconsin, Madison (USA).
03/2006: Geometric Analysis seminar, University of Wisconsin, Madison (USA).
04/2006: Analysis Seminar, ETH Zürich (Switzerland).
05/2006: Differential equations seminar, University of Rome II (Italy).
05/2006: Seminar, University of Rome III (Italy).
06/2006: Analysis Seminar, University of Warwick (England).
06/2006: AIMS Conference, Poitiers (France).
12/2006: Algebra and Geometry seminar, University of Brest (France).
01/2007: Differential geometry and analysis Seminar, University of Magdeburg (Germany).
01/2007: Analysis seminar, University of Amiens (France).
02/2007: Lecture series in PDE, Chinese University of Hong-Kong (Hong-Kong).
02/2007: Colloquium, Chinese University of Hong-Kong (Hong-Kong).
08/2007: Conference "Loss of compactness in nonlinear PDE: Recent trends", Banff (Canada).
11/2007: Colloquium, Australian National University, Canberra (Australia).
01/2008: MIP seminar, University of Toulouse III (France).
02/2008: Analysis seminar, SISSA, Trieste (Italie).
06/2008: Second Canada-France Mathematical Congress 2008, Montréal (Canada).
06/2008: Conference "Conformal Geometry: invariant theory and the variational method", Roscoff (France).
10/2008: Complex geometry seminar, University of Marseille I (France).
11-12/2008: Series of lectures "Elie Cartan", Technic University of Berlin (Germany)
01/2009: "Journées Nancéiennes de Géométrie", University of Nancy I (France).
02/2009: Geometric Analysis seminar, Albert-Einstein-Institut, Potsdam, (Germany).
02/2009: Geometry seminar, Technic University of Berlin (Germany).
03/2009: Analysis seminar, Free University of Berlin (Germany).
03/2009: Geometry seminar, Humboldt University, Berlin (Germany).
04/2009: Analysis seminar, ETH Zürich (Switzerland).
05/2009: Conference "Nonlinear analysis and related problems", University of Cergy-Pontoise (France).
07/2009: Conference "Asymptotic analysis in the calculus of variations and PDEs", University of British Columbia, Vancouver (Canada).
09/2009: Conference "Workshop in Nonlinear Elliptic PDEs" for Jean-Pierre Gossez's 65th birthday, Free University of Bruxelles (Belgium).
10/2009: Conference "Problèmes non linéaires avec perte de compacité: perspectives et applications", Luminy (France).
10/2009: Geometry and Analysis seminar, University of Nice (France).
10/2009: Interregional Colloquium, University of Saarbrücken (Germany).
11/2009: Analysis seminair, Chinese University of Hong-Kong (Hong-Kong).
12/2009: Analysis, Geometry and Algebra seminar, University of Metz (France).

04/2010: Series of lectures, Tata Institute, Bangalore (India).
06/2010: Geometry and Analysis seminar, University of Nice (France).
07/2010: Summer School "Singulatities in Partial DIfferential Equations", IHES (France).
11/2010: Series of talks at the PDE semianr, University of Sydney (Australia).
11/2010: Analysis and Pde seminar, Australian National University, Canberra (Australie).

Invitations by foreign institutions:

12/2000: University of Rome III (Italy).
02/2002: Tata Institute, Bangalore (India).
03/2002: University of Rome III (Italy).
02/2003: Notre Dame University (USA).
02/2003: Princeton University (USA).
01/2004: ETH Zürich (Switzerland).
11/2004: University of British Columbia, Vancouver (Canada).
07/2005: University of British Columbia, Vancouver (Canada).
02/2006: University of British Columbia, Vancouver (Canada).
03/2006: University of Wisconsin, Madison (USA).
04/2006: ETH Zürich (Switzerland).
05/2006: University of Rome II (Italy).
01/2007: University of Magdeburg (Germany).
02/2007: Chinese University of Hong-Kong (Hong-Kong).
05/2007: University of Perugia (Italy).
08/2007: University of British Columbia, Vancouver (Canada).
11/2007: Australian National University, Canberra (Australia).
02/2008: Chinese University of Hong-Kong (Hong-Kong).
06/2008: McGill University, Montréal (Canada).
11/2008: University of Rome II (Italy).
02/2009: Technic University of Berlin (Germany).
02/2009: University of Magdeburg (Germany).
04/2009: ETH Zürich (Switzerland).
07/2009: University of Washington, Seattle (USA).
07/2009: University of British Columbia, Vancouver (Canada).
11/2009: Chinese University of Hong-Kong (Hong-Kong).
04/2010: Tata Institute, Bangalore (India).
04/2010: Chinese University of Hong-Kong (Hong-Kong).
10/2010: University of Sydney (Australia).

Main publications:

[1] Asymptotic behaviour of a nonlinear elliptic equation with critical Sobolev exponent. The radial case. *Advances in Differential Equations*, **6**, (2001), 821-846.

[2] Asymptotic profile for the sub-extremals of the sharp Sobolev inequality on the sphere, with O.Druet.
Communications in Partial Differential Equations, **26**, (2001), 743-778.

[3] Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients, with E.Hebey.
Calculus of Variations and Partial Differential Equations, **13**, (2001), 491-517.

[4] Asymptotic behaviour of a nonlinear elliptic equation with critical Sobolev exponent. The radial case II.
Nonlinear Differential Equations and Applications **9**, (2002), 361-384.

[5] Mountain pass critical points for Paneitz-Branson operators, with P.Esposito.
Calculus of Variations and Partial Differential Equations, **15**, (2002), 493-517.

[6] Positive solutions for a fourth order equation invariant under isometries.
Proceedings of the AMS, **131**, (2003), 1423-1431.

[7] Blow-up theory for elliptic PDEs in Riemannian geometry, with O.Druet and E.Hebey. *Mathematical Notes, Princeton University Press*, Volume 45.

[8] Sharp solvability conditions for a fourth order equation with perturbation, with K.Sandeep. *Differential and Integral Equations*, **16**, (2003), 1181-1214.

[9] Asymptotic profile for a fourth order PDE with critical exponential growth in dimension four, with M.Struwe. *Advanced Nonlinear Studies* **4**, (2004), 397-415.

[10] Compactness and global estimates for the geometric Paneitz equation in high dimensions, with E.Hebey.
Electronic Research Announcements of the AMS, **10**, (2004), 135-141.

[11] Critical functions and optimal Sobolev inequalities.
Mathematische Zeitschrift, **249**, (2005), 485-492.

[12] Fourth order equations of critical Sobolev growth. Energy function and solutions of bounded energy in the conformally flat case, with V. Felli and E. Hebey.
Nonlinear Differential Equations and Applications, **12**, (2005), 171-213.

[13] Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth, with O.Druet. *Proceedings of the AMS*, **134**, (2006), 897-908.

[14] Compactness and global estimates for a fourth order equation of critical Sobolev growth arising from conformal geometry, with E.Hebey and Y.Wen.
Commununications in Contemporary Mathematics, **8**, (2006), 9-65.

[15] Concentration estimates for Emden-Fowler equations with boundary singularities and critical growth, with N.Ghoussoub. *International Mathematics Research Papers (IMRP)*, Volume 2006, Article ID 21867, 1-85.

[16] Concentration phenomena for Liouville's equation in dimension four, with Adimurthi and M.Struwe.
Journal of the European Mathematical Society, **8**, (2006), 171-180.

[17] The effect of curvature on the best constant in Hardy-Sobolev inequalities, with N.Ghoussoub. *Geometric And Functional Analysis (GAFA)*, **16**, (2006), 1201-1245.

[18] Concentration phenomena for a fourth order equation with exponential growth: the radial case. *Journal of Differential Equations*, **231**, (2006), 135-164.

[19] Quantization effects for a fourth order equation of exponential growth in dimension four. *The Royal Society of Edinburgh Proceedings A*, **137A**, (2007), 531-553.

[20] On the local Nirenberg problem for the Q-curvatures, with Ph.Delanoë.
Pacific Journal of Mathematics, **231**, (2007), 293-304.

[21] On the influence of the Kernel of the bi-harmonic operator on fourth order equations with exponential growth. Sixth international conference on "Dynamical Systems and Differential Equations", Poitiers, June 2006.
Discrete and continuous dynamical systems, Supplement 2007, 875-882.

[22] Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition, with J.Wei.
Indiana University Mathematics Journal, **57**, (2008), 2039-2060.

[23] Elliptic equations with critical growth and a large set of boundary singularities, with N.Ghoussoub. *Transactions of the AMS*, **361**, (2009), no. 9, 4843-4870.

[24] On a p-Laplace equation with multiple critical nonlinearities, with R.Filippucci and P.Pucci. *Journal des Mathématiques Pures et Appliquées*, **91**, (2009), 156-177.

[25] Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions, with H.-C.Grunau.
Archive for Rational Mechanics and Analysis, **195**, (2010) 865-898.

[26] The heat flow with a critical exponential nonlinearity, with Tobias Lamm and Michael Struwe. *Journal of Functional Analysis*, **257**, (2009), 2951-2998.

[27] Admissible $Q-$curvatures invariant under isometries for the general GJMS operators. *Contemporay Mathematics*, Volume in the honor of Jean-Pierre Gossez, to appear.

[28] The Lin-Ni's conjecture for mean convex domains, with O.Druet and J.Wei. *Preprint.*

[29] Asymptotic analysis for fourth order Paneitz equations with critical growth, avec E.Hebey. *Advances in the Calculus of Variations*, to appear.

[30] Optimal estimates from below for biharmonic Green functions, avec H.-Ch.Grunau et G.Sweers. *Proceedings of the AMS*, to appear.

Other publications:

[31] Étude asymptotique d'une équation non-linéaire à croissance de Sobolev critique. *Preprint (not submitted)*, (1999), 40 pages.

[32] On the equivalence of the Kazdan-Warner and the Pohozaev identities, with O.Druet. *Preprint (not submitted)*, (1999), 2 pages.

[33] Asymptotic profile and blow-up estimates on compact Riemannian manifolds, with O.Druet. *Preprint*, (2000), 21 pages. Reproduced in "The *AB* program in Geometric Analysis. Sharp Sobolev inequalities and related problems", by O.Druet and E.Hebey, *Memoirs of the AMS*, **160**, (2002).

[34] Struwe's compactness for free functionals involving the bi-harmonic operator. *Preprint (not submitted)*, (2000), 22 pages.

[35] A C^0-theory for the blow-up of second order elliptic equations of critical Sobolev growth, with O.Druet and E.Hebey.
Electronic Research Announcements of the AMS, **9**, (2003), 19-25.

[36] Sobolev spaces on manifolds. *Handbook of global analysis*, 375-415, 1213, *Elsevier Sci. B. V., Amsterdam*, 2008.

[37] Extremals for Hardy-Sobolev type inequalities: the influence of the curvature. 15 pages. In *"Aspects analytiques de la géométrie riemannienne"*, to appear in the series "Séminaires et Congrès", SMF.

[38] Boundedness of the negative part of biharmonic Green's functions under Dirichlet boundary conditions in general domains, with H.-Ch.Grunau.
C. R. Acad. Sci. Paris, Ser. I, **347**, (2009), no. 3-4, 163-166.

[39] Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d'ordre deux. *Available on http://www.iecn.u-nancy.fr/~frobert.*

[40] Fourth order equations with critical growth in Riemannian geometry. Notes from lectures given in Madison and in Berlin. *Available on http://www.iecn.u-nancy.fr/˜frobert.*

[41] Construction and asymptotics for the Green's function with Neumann boundary condition. *Available sur http://www.iecn.u-nancy.fr/˜frobert.*

Teaching experience:

1997-1998: Ecole d'Application du Train de Tours.
1998-2002: University of Cergy-Pontoise.
2002-2003: ETH Zürich.
2003-2009: University of Nice-Sophia-Antipolis.
2009- : University of Nancy 1.
06/2006 : Posdoctoral lecture at the University of Wisconsin at Madison (USA).

1. Short description of research

The questions I am interested in are located at the intersection of analysis and geometry: they arise naturally from geometric analysis and the variational analysis of partial differential equations. The context can be either an open subset of the Euclidean space $\mathbb{R}^n$ or the more geometric framework of a Riemannian manifold. The investigated equations will be either elliptic or parabolic, linear or nonlinear. In this latest situation, the nonlinearity induces an invariance of the equation under a suitable rescaling: this leads us to developing a renormalization theory. The criticality of the equations is one of the fundamental difficulties of the theory.

The equations considered here are quite general and can be connected to various contexts. For instance, the classical simplification of the Gierer-Meinhardt cell system leads to an nonlinear equation of the type studied in the first part. In astrophysics, some theories developed models based on Emden-Fowler equations with singularities similar to the ones we will consider: by the way, many other equations here arise in physics and some of then can be interpretated via statistic mechanics. It is also in the conformal context that these contributions have a natural place: indeed, the rescaling and the nonlinearity are direct consequences of the conformal invariance of natural operators.

We refer to the "Habilitation" memoir (in French) of the author for more details on the themes considered in this short description. The references correspond to the numbering of the articles above. Most of these articles are available on the author's webpage.

1. Structure and stability of some elliptic equations with polynomial growth.

We consider functions $u \in C^\infty(D)$ solutions to the pde

$$L_\alpha u = f_\alpha(x, u) \text{ in } D. \tag{1}$$

Her, $\alpha \in \mathbb{N}$ is a parameter and $(L_\alpha)_\alpha$ is a family of elliptic operators of the type $L_\alpha = (-\Delta)^k + ...$, with $\Delta = \text{div}(\nabla)$ and $k \geq 1$ is a positive integer. The function u is defined on D, an $n-$dimensional set, with $n > 2k$ (the critical cas $n = 2k$ will be considered in Part 2): as mentioned above, D will be a domain of $\mathbb{R}^n$ or a Rimannian manifold with Dirichlet type conditions if necessary. Finally, the family of functions $(f_\alpha)_\alpha : D \times \mathbb{R} \to \mathbb{R}$ is nonlinear with polynomial growth wrt the second variable and can enjoy singularities wrt the first variable.

The structure of the propblems we analyse implies that for any α, each solution u to (1) is smooth. This static approach hides a much more complex reality: indeed, the coupling of the elliptic operator and the nonlinearity generates an intrinsic invariance of the equation under the action of a transformation group, and then a highly instable dynamics of its solutions. For instance, the equation $(-\Delta)^k u = u^{\frac{n+2k}{n-2k}}$ on a domain of $\mathbb{R}^n$ is invariant under the change of functions

$$\tilde{u}_\alpha := \mu_\alpha^{\frac{n-2k}{2}} u(x_\alpha + \mu_\alpha \cdot), \tag{2}$$

where $(x_\alpha)_\alpha \in \mathbb{R}^n$ and $(\mu_\alpha)_\alpha \in (0, +\infty)$ are arbitrary. This invariance induces very rich asymptotic behaviors. This is the reason why we adopt here a dynamical approach and we will consider families $(u_\alpha)_\alpha$ of solutions to (1): the invariance under transformation induces a blowing-up behavior of u_α when $\alpha \to +\infty$. Coming back to the example above, taking $\mu_\alpha \to +\infty$ when $\alpha \to +\infty$ yields the blowing-up of $\tilde{u}_\alpha$ at 0.

After the early descriptions of Wente, Sacks-Uhlenbeck and Lions, Struwe described for $k = 1$ this blowing-up behavior via the arising of singularities refered to as bubbles. In general, these bubbles are explicit and they are modeled on the ground states of (1) (We will come again to thes ground states in Part 3): they depend also on two parameters that are the location and the height. Moreover, the bubbles are approximate solutions to (1) and they blow-up, that is their height goes to infinity when the parameter $\alpha \to +\infty$. Struwe proved that for $k = 1$, one can write in general that

$$u_\alpha = u_\infty + \sum_i B_{i,\alpha} + R_\alpha$$

where u_∞ is a solution to the limit equation, the $B_{i,\alpha}$'s are bubbles and R_α goes to zero in the natural Sobolev space. This is an integral description.

In the constributions described in this part, we give pointwise (or C^0) descriptions of the u_α's, that is we prove the validity of the equation

$$u_\alpha \simeq u_\infty + \sum_i B_{i,\alpha}. \tag{3}$$

The gain of informations and of control is clear: since the rest (R_α) enjoys a pointwise control, we decribe precisely the behavior of $u_\alpha(z_\alpha)$ when $\alpha \to +\infty$ for all the families of points $(z_\alpha)_\alpha$. Apart from its theoretical interest, this type of results allows a fine analysis of the bubbles and of the singularities and gives important informations on their location, which allows to give conditions for stability.

Here are now some more explicit situations.

We first describe the context of a Riemannian manifold without boundary (M, g) of dimension $n \geq 3$. We consider a sequence $(h_\alpha)_\alpha \in C^0(M)$ bounded in C^0. Then, we consider an arbitrary family $(u_\alpha)_\alpha \in C^2(M)$ of solutions to

$$-\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{\frac{n+2}{n-2}} \text{ in } M, \text{ with } u_\alpha > 0. \tag{4}$$

Here $\Delta_g = \text{div}_g(\nabla)$ and we recover equation (1) with $k = 1$. The invariance appearing naturally comes from the conformal invariance: if u_α is a solution to (4), then, after a transformation of the type of (2), we recover an equation close to (4). Assuming a bound on the Dirichlet energy, we show with Olivier Druet and Emmanuel Hebey in [7] that the control (3) holds pointwisely. We describe completely the pointwise asymptotics for sequences of solutions to (4). This optimal description is the first C^0 theory in the field. To be more precise, the bubbles are of the type

$$B_\alpha := \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha,\cdot)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

with $(x_\alpha)_\alpha \in M$ and $(\mu_\alpha)_\alpha \in (0,+\infty)$ converges to 0.

The same questions are pertinent for fourth-order equations. Given (M,g) a compact Riemannian manifold of dimension $n \geq 5$, we consider families $(u_\alpha)_\alpha \in C^4(M)$ of solutions to

$$\Delta_g^2 u_\alpha - \text{div}_g(A_\alpha du_\alpha) + a_\alpha u_\alpha = u_\alpha^{\frac{n+4}{n-4}},\ u_\alpha > 0 \text{ in } M. \tag{5}$$

Here, (A_α) is a family of $(2,0)-$tensors and (a_α) is a family of functions. Here again, we recover an invariance under renormalization. However, a specificity of fourth-order problems is the lack of a general comparison principle which is a fundamental tool in the proof of estimates of the type (3) (we will go back to this point in Part 3). Despite this difficulty, in a joint work with Emmanuel Hebey and Yuliang Wen [14], we are succesful in proving precise asymptotics for families of solutions to (5): a consequence is the localization of singularities at points on M where the limit operator "touches" the conformally invariant Paneitz operator. Therefore, for an operator far from the conformal one, we recover the stability of the solutions to the equation (5). Conversely, when the operator is exactly the Paneitz operator, we prove with Emmanuel Hebey in [10] that stability is valid when the manifold is essentially not the standard sphere, which is a result in the spirit of Schoen.

When one eliminates singularities, and then when one gets stability, we show existence and multiplicity results for the limit equations. In the context of GJMS conformally invariant operators, we are led natirally to results about the prescription of Branson's Q-curvature. For instance, in a joint work with Philippe Delanoë [20], we show that under invariance under isometries, the singularities are ruled out and we give a large class of admissible Q-curvatures on the conformal sphere. Note that it follows from the conformal invariance that the invariance under isometries we impose is necessary.

In the context of a flat space, we studied Hardy-Sobolev type inequalities with Nassif Ghoussoub. Thes inequalities lead naturally to consider solutions $(u_\alpha)_\alpha \in C^2$ on a domain Ω of the equation

$$-\Delta u_\alpha + a_\alpha u_\alpha = \frac{u_\alpha^{\frac{n+2-2s}{n-2}-\epsilon_\alpha}}{|x|^s} \text{ in } \Omega \text{ eand } u_{\alpha|\partial\Omega} = 0 \tag{6}$$

where $s \in (0,2)$, (a_α) is bounded C^1, $\epsilon_\alpha \geq 0$ and $\lim_{\alpha\to+\infty} \epsilon_\alpha = 0$. In this boundary problem, we recover again an invariance under the transformation (2) (with $x_\alpha = 0$). In the case when 0 is on the boundary of the domain, some singularities might appear and they are located at 0: in [15,17], we describe them precisely and we show (3) with bubbles adapted to our context. Pushing further our analysis, we prove that when a singularity arises, then the curvature at 0 is nonnegative. Conversely, when the curvature is negative (which is a weak concavity condition at 0), then no singularity appears and (6) is stable: as above, we can use topological methods of mountain-pass type to obtain existence of extremals fo the hardy-Sobolev inequalities and multiplicity for solutions of (6).

Still in the Euclidean context, Olivier Druet, Juncheng Wei and I solved the Lin-Ni conjecture. Let us consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ and a positive solution $u \in C^2(\overline{\Omega})$ to

$$-\Delta u + \epsilon u = u^{\frac{n+2}{n-2}} \text{ with } \partial_\nu u = 0 \text{ on } \partial\Omega. \tag{7}$$

This Neuman-type problem has its origins in a mathematical problem sugegsted by Gierer and Meinhardt to describe the formation of a head on microscopic cellular beings. This model is a complex evolution system very difficult to tackel: via some natural restrictions and simplifications inspired by biologists, we recover (7). In particular, the Neuman condition is natural for this type of model.
When $n \geq 7$, Lin and Ni conjectures taht for $\epsilon > 0$ small enough, (7) has only one solution (the constant one). The difficulty of this conjecture comes here again from the invariance under transformation of the type (2). Following the dynamical approach, we prove in [28] that the families $(u_\epsilon)_\epsilon$ of solutions to (7) may develop bubbles and we prove a control of the type (3). Thanks to a more refined asymptotic analysis, we prove that these singularities are located on the boundary at points of negative mean curvature. An immediate consequence is that the Lin-Ni conjecture holds for positive curvature and bounded energy. These two considtions might seem technical, but they are not: since recent works from Wang, Wei and Yan, we have known that these two conditions are necessary. Finally, our result exhibits the only context in which the Lin-Ni conjecture is true.

2. The case of critical dimensions.
Problems of the same nature as the ones described above arise in the case of the critical dimension, that is when $n = 2k$ in (1). But here, the invariance is generated by exponential nonlinearituies and we will also deal with parabolic evolution problems. We consider two distinct exponential nonlinearities: one is related to conformal geometry and the mean-field equation, and the other is related to the Moser-Trudinger or Adams inequalities. Despite the evident analogies with the problems when $n > 2k$, the questions related to the critical dimension $n = 2k$ give rise to huge differences: in particular, getting a Struwe-type decomposition in SObolev spaces is very delicate, specially in the case of nonlinearities of the type of Moser-Trudinger or Adams.

In dimension two, and therefore for second-order problems, we tackle the second type of nonlinearities. In a joint work with Tobias Lamm and Michaël Struwe [26], we study a highly nonlinear parabolic problem of the type of Moser-Trudinger. After elliptic problems, the analysis of parabolic problems might seem completely different at first glance. Indeed, the appearance of singularities along the flow (in finite or infinite time) is a frequent phenomenon: it is related to the notion of bubbles and singularities described in the first part. Therefore, the vision adopted for the study of elliptic problems will be applied for the best to parabolic problems. More precisely, on a smooth bounde domain $\Omega \subset \mathbb{R}^2$, we consider a solution $u : [0,T) \times \overline{\Omega} \to \mathbb{R}$ to the system

$$\left\{ \begin{array}{ll} e^{u^2}\partial_t u - \Delta_x u = \lambda(t) u e^{u^2} & \text{in } (0,T)\times\Omega \\ u(t,x) = 0 & \text{for all } t \geq 0 \text{ and } x \in \partial\Omega \\ u(0,\cdot) = u_0 & \end{array} \right\} \tag{8}$$

where $t \mapsto \lambda(t)$ is chosen such that $\int_\Omega e^{u^2}\,dx$ is preservd along the flow. This flow is eternal, but it can develop singularities. We describe these singularities precisely and show an asymptotic quantization of the energy: indeed, we show that the defect of energy convergence of the flow to a stationnary solution is a multiple of the energy of the ground state, that is 4π.

In the case of fourth-order operators (and therefore in dimension four), some very interesting phenomena arise. Indeed, the lack of comparison principle and the existence of a very rich Kernel for the bi-Laplacian induce the formation of a new type of singularities that do not existe for second-order problems. In particular, there is no quantization of the singularities: this is specific to the critical dimension $n = 2k \geq 4$, it does not appear in dimension $n > 2k$.

With Adimurthi and Michaël Struwe, we give in [16] a general and optimal description of the asymptotic behavior of sequences of solutions for the first type of nonlinearities, that is for the conformal problem. More precisely, given $\Omega \subset \mathbb{R}^4$ a domain, we consider a family $(u_\alpha)_\alpha \in C^4(\Omega)$ of solutions to equations of the type

$$\Delta^2 u_\alpha = V_\alpha e^{4u_\alpha} \text{ in } \Omega, \tag{9}$$

wher $(V_\alpha)_\alpha$ is a family of potentials converging uniformly. This type of equation arises in many areas in mathematics: for instance, it is a particular case of the prescription problem for the Q-curvature in dimension four, but it also models the repartition density of particules on a sphere in statistical mechanics. The invariance of (9) has its roots in these origins: of course, the concentration phenomena are natural consequences of this. The study we perform in [16] permits to understand the quadratic concentration phenomenon which is at the origin of the lack of quantization of the energy of (9). In the sequel, we get more refined descriptions in more precise contexts that are still fairly general like in the case of a trivial kernel ([13], with Olivier Druet), the radial case ([18]) and generic situations of quantization ([19]). These contributions have applications in the study of the four-dimesnional mean-field equation with Dirichlet boundary condition in collaboration with Jun-Cheng Wei [22].
Still in dimension four, we have dealt with the second type of nonlinearity (of Adams type) in [9] in collaboration with Michaël Struwe: we prove pointwise estimates and a quantization phenomenon due to the Dirichlet boundary condition.

3. Ground states and fundamental solutiosn.
The analysis of the singularities above requires a precise understanding of limit solutions to(1). Depending on the location, thes limit solutions can be of two type. Around a singualr point, the rescalinf captures the essence of the operator L_α, that is its principal part, and of the nonlinearity f_α. We show that solutions to (1) behave locally like solutions U to

$$(-\Delta)^k U = f_\infty(\cdot, U) \text{ in } \mathbb{R}^n \text{ or the half space } \mathbb{R}^n_-. \tag{10}$$

Outside the singularities, the linear part dminates and the nonlinearity is neglictible. We recover a behavior ruled by the solutions U to

$$L_\infty U = 0 \text{ in } D \setminus S, \tag{11}$$

wher S is the (finite) set of singularities and L_∞ is the limit of L_α.
Studying independantly the solutions to (10) and (11) is then not only natural but also crucial to develop the analysis described in Parts 1 and 2 above. The

qualitative properties of these solutions are very precisely known (sometimes, they are even explicit).

The solutions to (10) are the ground states of (1). In the case of (4), we recover the solutions to $-\Delta U = U^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$ which are well known since the work of Caffarelli-Gidas-Spruck. For the problem (6), the ground states equation is

$$-\Delta U = \frac{U^{\frac{n+2-2s}{n-2}}}{|x|^s} \text{ in the half-space } \mathbb{R}^n_-.$$

Despite these solutions are not explicit, we show with Nassif Ghoussoub in [17] that they enjoy the best possible symmetry and we give their eaxct asymptotic profile. To prove this, we use arguments of moving-plane type of Gidas-Ni-Nirenberg.

Equation (11) is satisfied by the fundamental solutions of the operator L_∞. These Green kernels exist and in Druet-Hebey-Robert [7], Ghoussoub-Robert [17] and Robert [39], we give their exact profile in the case $k = 1$, that is for second-order elliptic operators. The properties of the Green kernels are used many times for second-order problems, in particular via the comparison principle (or maximum principle). But for orders four and beyond ($k \geq 2$), there is no comparison principle in general, which makes the analysis of the singularities slightly more delicate: indeed, we know very few domains and operators that satisfy it. The questions of positivity and comparison principle are deeply related to the qualitative properties of the Green's function (satisfying the comparison principle is equivalent to the positivity of the Green's kernel). The most striking example is the Green's function of the bi-Laplacian with Dirichlet boundary condition, that is for Ω a domain of $\mathbb{R}^n$, the function $G : D \times D \to \mathbb{R}$ such that

$$\left\{ \begin{array}{ll} \Delta^2 G(x, \cdot) = \delta_x & \text{in } \mathcal{D}'(D) \\ G(x, \cdot) = \partial_\nu G(x, \cdot) = 0 & \text{on } \partial D. \end{array} \right. \tag{12}$$

The Green's function is naturally related to the mechanical problem of the clamped plates. Despite this model is natural and (apparently) very simple, the question of the positivity of the Green's function (and then of the comparison principle) remains widely open. This delicate issue goes back to Hadamard: surprisingly, positivity is known in very few cases. In a joint work with Hans-Christoph Grunau [25], we show that the positivity of the Green's function is stable after perturbations of the domain, which allows to exhibit new domains with positive Green's kernel. A consequence of this analysis is that, for an arbitrary domain, we get important informations on the negative part of the Green's function, especially on the boundary of the domain, and therefore on the lack of comparison principle. In particular, we show in [25] that this lack of comparison principle is very well controled.