Multifractional Rosenblatt process

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Outline

1. Preliminaries
   - Double stochastic integrals
   - Local times

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2. Hermite and Rosenblatt processes
   - Properties of Rosenblatt process

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Take $f \in \widehat{L}^2(\mathbb{R}^2)$ of the form

$$f(x,y) = \sum_{k,j=1}^{n} a_{kj}\mathbb{I}_{A_k}(x)\mathbb{I}_{A_j}(y), \tag{1}$$

where $A_1, \ldots, A_n$ are disjoint subsets of $\mathbb{R}$ of finite measure, $a_{kj} = a_{jk}$, $a_{kk} = 0$.

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$$I_2(f) = \iint_{\mathbb{R}^2} f(x,y) W(dx)\, W(dy) = \sum_{k,j=1}^{n} a_{kj} W(A_k) W(A_j).$$

$I_2/2$ is a linear isometry to $L^2(\Omega)$ which can be extended to $\widehat{L}^2(\mathbb{R}^2)$.

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Let $\lambda_{k,f}$ be its eigenvalues ordered by absolute value decrease, $\varphi_{k,f}$ be correspondent (orthonormal) eigenfunction. Then $f(x,y) = \sum_{k\geq 1} \lambda_{k,f} \varphi_{k,f} \otimes \varphi_{k,f}$, whence $I_2(f) = \sum_{k\geq 1} \lambda_{k,f}(\zeta_k^2 - 1)$, where $\zeta_k = I_1(\varphi_{k,f})$ are independent standard Gaussians and

$$\mathrm{E}\left[e^{i\alpha I_2(f)} \right] = \prod_{k\geq 1} \frac{e^{-i\alpha\lambda_{k,f}}}{\sqrt{1 - 2i\alpha\lambda_{k,f}}}.$$

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This implies

$$\left|\mathsf{E}\left[e^{i\alpha I_2(f)}\right]\right| \leq \left(1 + 4\alpha^2 \sum_{k\geq 1} \lambda_{k,f}^2 + 16\alpha^4 \sum_{j<k} \lambda_{j,f}^2\lambda_{k,f}^2 + 64\alpha^6 \sum_{j<k<l} \lambda_{j,f}^2\lambda_{k,f}^2\lambda_{l,f}^2\right)^{-1/4}$$

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and

$$\left|\mathsf{E}\left[(I_2(f))^n\right]\right| \leq C(4\|f\|)^n n!.$$

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   - Properties of mRp
   - Local time of mRp

$\{Z_t, t \geq 0\}$ separable random process.

Occupation measure for Z: for For Borel $A \subset [0, \infty)$, $B \subset \mathbb{R}$

$$\mu(A, B) = \lambda(\{s \in A, Z_s \in B\}),$$

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Analytic method: for a given path, write $L(t, x)$ as the inverse Fourier transform

$$L(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivx} \int_{\mathbb{R}} e^{iv} L(t, y) dy \, dv = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{iv(Z_s - x)} ds \, dv.$$

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Hence the following criterion of local time existence and square integrability:

Proposition ([Berman (1969)])

A process $\{Z_t, t \geq 0\}$ *has a local time* $L(A, x) \in L^2(\mathbb{R} \times \Omega)$ *iff*

$$\int_{\mathbb{R}}\int_A\int_A \mathrm{E}\left[e^{iv(Z_t - Z_s)}\right] ds\,dt\,dv < \infty.$$

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$$n^{-H}\sum_{i=1}^{[nt]} g(\xi_i) \to c(H,k)\int\cdots\int_{\mathbb{R}^k}\int_0^t \prod_{j=1}^{k}(s-y_i)_+^{-\frac{1}{2}-\frac{1-H}{k}}\,ds\,W(dy_1)\dots W(dy_k),$$

the Hermite process of rank k (with Hurst parameter H).

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$k = 2$: Rosenblatt process:

$$Z_t^H = c(H)\iint_{\mathbb{R}^2}\int_0^t (s-x)_+^{H/2-1}(s-y)_+^{H/2-1}ds\,W(dx)\,W(dy).$$

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continuity Rosenblatt process paths are Hölder continuous of any order $\delta < H$.

Definition

Multifractional Rosenblatt process with the Hurst function H

$$X_t = \iint_{\mathbb{R}^2} \int_0^t (s-x)_+^{H(t)/2-1}(s-y)_+^{H(t)/2-1} ds\, W(dx)W(dy). \tag{2}$$

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H satisfies, as usually,

$$|H(t) - H(s)| \le C\, |t-s|^\gamma \tag{3}$$

with $\gamma > \sup_t H(t)$.

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Proposition

For $t, s, u \in [0, \infty)$ *close enough,* $s \leq u \leq t$

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On any interval $[a, b] \subset [0, \infty)$ *trajectories of mRp are a.s. Hölder continuous of any order* $\delta < \min_{t\in[a,b]} H(t)$.

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MRp $\{X_t, t \ge 0\}$ *is strongly* $H(t_0)$*-localisable at point* t_0 *with the local version* $\left\{Z_t^{H(t_0)}/c(H(t_0)), t \ge 0\right\}$

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References I

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