Stochastic differential equations with fractional Brownian motion

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Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 1

The (two-sided, normalized) fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties

(i) $B_0^H = 0$,

(ii) $EB_t^H = 0, t \in \mathbb{R}$,

(iii) $EB_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s, t \in \mathbb{R}$.

Remark 2

Since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

Sometimes it is convenient to consider a one-sided fBm $B^H = \{B_t^H, t \geq 0\}$ and to represent it as a functional of the form $B_t^H = \varphi(B_s, 0 \leq s \leq t)$, of some Wiener process $B = \{B_t, t \geq 0\}$. For this purpose consider the kernels

$$l_H(t,s) = C_H^{(5)} s^{-\alpha}(t-s)^{-\alpha} I_{\{0<s<t\}},$$

and

$$m_H(t,s) = C_H^{(6)}\Big(\big(\frac{t}{s}\big)^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}du\Big),$$

where

$$C_H^{(5)} = \left(\frac{\Gamma(2-2\alpha)}{2H\Gamma(1-\alpha)^3\Gamma(1+\alpha)}\right)^{\frac{1}{2}},\ C_H^{(6)} = \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)}\right)^{\frac{1}{2}},$$

and $\alpha = H - \frac{1}{2}, \quad H \in (0,1).$

If we consider the integral

$$I_t^H(l_H) = \int_0^t l_H(t,s)dB_s^H := \int_{\mathbb{R}} l_H(t,s)dB_s^H, \tag{1}$$

then it is a square integrable martingale w.r.t. its natural filtration

$$\mathcal{F}_t^H := \sigma\left\{ I_s^H(l_H), 0 \le s \le t \right\},$$

having angle bracket $\langle I_t^H(l_H) \rangle = t^{1-2\alpha}$ and $I_0^H(l_H) = 0$. By the Lévy theorem, there exists some Wiener process $B = \{B_t, t \ge 0\}$, such that

$$M_t^H := I_t^H(l_H) = \widetilde{\alpha} \int_0^t s^{-\alpha} dB_s,$$

where $\widetilde{\alpha} = (1-\alpha)^{1/2}$. The process M^H is called the Molchan martingale, or the fundamental martingale, since it was considered originally in the papers of Molchan.

Consider the following scalar stochastic differential equation

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t f(s)dB_s^H, \qquad (2)$$

where $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable function, $H \in (0, 1)$, $x \in \mathbb{R}$.

Definition 3

By *a weak solution* to equation (2) we mean a couple of adapted continuous processes $(\tilde{B}^H, X)$ on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$ such that

(a) $\tilde{B}^H$ is an $\mathcal{F}_t$ - fractional Brownian motion;

(b) X and $\tilde{B}^H$ satisfy (2).

The general approach to existence of weak solution of (2) is the following. Let a function f be nonzero on $\mathbb{R}$ so that $g(s) := \frac{1}{f(s)}$ is well defined on $\mathbb{R}$. Consider the process $\tilde{B}_t^H := B_t^H - \int_0^t g(s)b(s, x + I_s(f))ds$, where $I_t(f) = \int_0^t f(s)dB_s^H$.

From Girsanov theorem for fBm, under the following condition

$$E \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\} = 1, \quad t \in [0, T] \tag{3}$$

where $L_t = \int_0^t s^\alpha \delta_s dB_s$, B is Wiener process, $B_t = \int_0^t s^\alpha dM_s^H$, $M_t^H = \int_0^t l_H(t,s)dB_s^H$ and

$$\int_0^t l_H(t,s)g(s)b(s, x + I_s(f))ds = \int_0^t \delta_s ds \tag{4}$$

we have that $\tilde{B}_t^H$ will be fBm w.r.t. the measure Q such that

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}$$

In this case it is very easy to check that the couple $(\tilde{B}_t^H, x + I_t(f))$ forms a weak solution of equation (2).

In turn, the equality (3), by Novikov condition, holds if

$$E \exp\left\{\frac{1}{2}\langle L\rangle_T\right\} < \infty, \tag{5}$$

where

$$\langle L\rangle_t = \int_0^t s^{2\alpha}\delta_s^2 ds. \tag{6}$$

Therefore, we must check inequality (5) under (4) and (6). Denote the stochastic process $h(s) := g(s)b(s, x + I_s(f))$.

Theorem 4

Let one of the following assumptions hold:

(i) $H \in (0, 1/2)$, *the coefficients* b *and* g *satisfy the condition: there exists* $\lambda > 0$ *such that*

$$\sup_{0 \le t \le T} E \exp\left\{ \lambda t^{2\alpha} \left(\int_0^t s^{-\alpha} (t-s)^{-\alpha-1} h(s) ds \right)^2 \right\} < \infty; \qquad (7)$$

(ii) $H \in (1/2, 1)$, *the coefficient* b *satisfies the condition:*

$$E \exp\left\{ \lambda \int_0^T \left(s^{-\alpha} |h(s)| + \right.\right.$$
$$\left.\left. + \alpha s^{\alpha} \int_0^s \frac{|s^{-\alpha} h(s) - r^{-\alpha} h(r)|}{(s-r)^{\alpha+1}} dr \right)^2 ds \right\} < \infty \qquad (8)$$

for any $\lambda > 0$.

Then the equation (2) has a weak solution.

Now we establish more convenient conditions for existence of a weak solution in terms of g and b.
Denote the function $h(s, x) := g(s)b(s, x)$.

Theorem 5

Let $|f(t)| > 0$ for any $t \in [0, T]$ and one of the following assumptions hold:

(iii) *$H \in (0, 1/2)$ and $h(t, x)$ is of linear growth:*

$$|h(t,x)| \leq C(1 + |x|), \quad (t,x) \in [0,T] \times \mathbb{R}$$

(iv) *$H \in (1/2, 1)$, f is essentially bounded on $[0, T]$ and $h(s, x)$ is Hölder continuous:*

$$|h(t,x) - h(s,y)| \leq C\left(|x-y|^{\beta} + |t-s|^{\gamma}\right),$$

where $1 \geq \beta > 1 - \frac{1}{2H}$ and $1 \geq \gamma > \alpha$.

Then the equation (2) has a weak solution.

Consider the equation (2) for the case when $f \equiv 1$, $b(s,x) = b(x)$ and $b(x)$ is Hölder continuous of order $\beta \in (1 - 1/2H, 1)$ except a finite number of points, where there is a jump discontinuity.

Theorem 6

Suppose that the function $b(x)$ is Hölder continuous of order $\beta \in (1 - 1/2H, 1)$ on a finite number of intervals $(-\infty, a_1)$, (a_1, a_2),...,(a_{N-1}, a_N), $(a_N, +\infty)$ and there are jump discontinuity at the points $a_i, 1 \le i \le N$ that is, $b(a_i-) \neq b(a_i+) = b(a_i)$. Let B_t^H be an fBm with Hurst parameter $H \in \left(\frac{1}{2}, \frac{1+\sqrt{5}}{4}\right)$. Then the equation (2) with $f \equiv 1$ has a weak solution.

Remark 7

The case $H \in (0, 1/2)$ is not specific now; for example, if b is discontinuous but bounded we have a weak solution.

The constant $\frac{1+\sqrt{5}}{4}$ appears here for the following reason. In order to prove (8) it suffices to establish the estimate

$$E \exp\left(\lambda \xi_\varepsilon^{\frac{2\alpha}{H-\varepsilon}} \int_0^T |B_s^H + x|^{\frac{-2\alpha}{H-\varepsilon}} ds\right) < \infty$$

for any $\lambda > 0$, $T > 0$, $x > 0$ and for some fixed $0 < \varepsilon < H$, where

$$\xi_\varepsilon := \left(\int_0^T \int_0^T \frac{|B_u^H - B_r^H|^{\frac{2}{\varepsilon}}}{|u-r|^{\frac{2H}{\varepsilon}}} dr du\right)^{\frac{\varepsilon}{2}}.$$

After some technical transformations, it suffices to show that

$$E \exp\left(\lambda \psi_\varepsilon^{4\alpha+\delta}\right) < \infty, \tag{9}$$

where

$$\psi_\varepsilon = \int_0^T \mathbf{1}_{\{|B_s^H + x| < 1\}} |B_s^H + x|^{-1+\varepsilon} ds.$$

Lemma 8 below states the estimate (9), provided $4\alpha H < 1$, and this leads to the condition $H < \frac{1+\sqrt{5}}{4}$.

The proof is based on the following result.

Lemma 8

Fix $\nu < 1$ *and define*

$$G = \int_0^T \mathbf{1}_{\{|B_s^H + x| < 1\}} |B_s^H + x|^{-\nu} ds.$$

Then for any $p > 0$ *such that* $pH < 1$ *we have*

$$E\left(\exp G^p\right) < \infty.$$

We return to the case of subsection 8 when the conditions of Theorem 5 are fulfilled.

Let the pair $(\tilde{B}^H, X)$ forms a weak solution of equation (2). Consider the function $h(s, X_s) := g(s)b(s, X_s)$.

In the case $H \in (0, 1/2)$ we have that by Gronwall inequality

$$X_T^* \leq (|x| + I_T^*(f) + CT)e^{CT}$$

and

$$|h(s, X_s)| \leq C(1 + X_T^*),$$

therefore

$$\left| \frac{d}{dt} \int_0^t l_H(t,s) h(s, X_s) ds \right| \leq C \left| \int_0^t s^{-\alpha}(t-s)^{-\alpha-1} \left| h(s, X_s) \right| ds \right|,$$

evidently, satisfies condition, similar to (7):

$$\sup_{0 \leq t \leq T} E \exp \left\{ \lambda t^{2\alpha} \left(\int_s^t s^{-\alpha}(t-s)^{-\alpha-1} |h(s, X_s)| ds \right)^2 \right\} < \infty$$

for some $\lambda > 0$. For $H \in (1/2, 1)$ the condition similar to (8) can be easily checked similarly to (iv) in Theorem 5.

So, $E\exp\{-\hat{L}_t-\frac{1}{2}\langle\hat{L}\rangle_t\}=1$ for $t\in[0,T]$, $\langle\hat{L}_t\rangle=\int_0^t s^{2\alpha}\delta_s^2 ds$ with $\int_0^t \delta_s ds=\int_0^t l_H(t,s)h(s,X_s)ds$. By Girsanov theorem, the process $\hat{B}_t^H:=\tilde{B}_t^H+\int_0^t h(s,X_s)ds$ is fBm w.r.t. measure $\hat{P}$ such that

$$\left.\frac{d\hat{P}}{dP}\right|_t=\exp\left\{-\hat{L}_t-\frac{1}{2}\langle\hat{L}\rangle_t\right\}. \tag{10}$$

It means that $X_t-X_0=\int_0^t f(s)d\hat{B}_s^H$ and as consequence, $X-x$ has the same distribution under measure $\hat{P}$ as $\int_0^\cdot f(s)d\tilde{B}_s^H$ under measure P.

Suppose now that X^1 and X^2 are two weak solutions defined on the same filtered probability spare $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$ with respect to the same fBm. Then $\max(X^1, X^2)$ and $\min(X^1, X^2)$ are also solutions and have the same distributions, whence $X^1 = X^2$.
We proved the following result.

Theorem 9

Under conditions of Theorem 5 any two weak solutions defined on the same filtered probability space coincide almost surely.

Let $H \in (1/2, 1)$, the function f is Hölder continuous of order $\beta > 1 - H$, and b is Lipschitz continuous. Then the conditions of main theorem are fulfilled, therefore the equation (2) has the unique strong solution. In the case when $b(s, x) = b(x)$ the equation (2) has for $f \in C^{\beta}[0, T]$, $\beta > 1 - H$ a strong solution, and it is unique due to Theorem 9. So, the case $H \in (1/2, 1)$ is not hard or interesting.
Now, let $H \in (0, 1/2)$. Consider a Krylov-type inequality as an auxiliary result.

Lemma 10

Let functions $f(s)$ and $b(s,x)$ are bounded, so $h(s,x)$ is bounded, X is a weak solution of (2), and for some $r>1$ the integral $\int_0^T \psi_r(t)dt<\infty$, where $\psi(t)=||f||^{-r}_{L_{\frac{1}{H}}[0,t]}$. Then there exists the constant C depending on $h:=\sup_{t\in[0,T],x\in\mathbb{R}}|h(t,x)|$ such that for any nonnegative measurable function $g(t,x):[0,T]\times\mathbb{R}\to\mathbb{R}$

$$E\int_0^T g(t,X_t)dt \le C\left(\int_0^T\int_{\mathbb{R}} g^2(t,x)dxdt\right)^{1/2}. \tag{11}$$

With the help of this inequality we obtain the following limit result.

Lemma 11

Let $b_n(t,x) = b_n(t,x)\mathbf{1}\{|x| \le C_1\}$ *be a sequence of measurable functions,* $|b_n(t,x)| \le C_2$, $\lim_{n\to\infty} b_n(t,x) = b(t,x)$, *for all* $(t,x) \in [0,T] \times \mathbb{R}$, *and the conditions of Lemma 10 hold. Let also the corresponding solutions* $X_t^{(n)}$ *of the equations*

$$X_t^{(n)} = X_0 + \int_0^t b_n(s, X_s^{(n)})ds + I_t(f), t \in [0,T],$$

converge a.s. to some process X_t *for all* $t \in [0,T]$. *Then the process* X *is a solution of equation* (2).

Theorem 12

Let the coefficients $h(t,x)$ and $b(t,x)$ satisfy the linear growth condition. Then the equation (2) has the unique strong solution.

Remark 13

The next condition is sufficient:

$$|b(s,x)| \leq C(|f(s)| \wedge 1)(1+|x|). \tag{12}$$

Let $\Omega = C_0([0,T],\mathbb{R})$ be the Banach space of continuous functions, null at time 0, equipped with the supremum norm, and P be the unique probability measure on Ω such that the canonical process is fBm with Hurst parameter $H \in (1/2, 1)$. Assume also that the canonical filtration is augmented with the P-negligible sets. We consider the following partial case of equation (2):

$$X_t = x + \int_0^t b(X_s)ds + B_t^H \tag{13}$$

with $b(x) = \operatorname{sign} x$, $H \in (1/2, H_0)$, $H_0 = \frac{1+\sqrt{5}}{4}$. According to Theorem 6, the equation (13) has a weak solution. Now we intend to prove the existence of its strong solution. For this purpose consider the following approximations of $b(x) = \operatorname{sign} x$:

$$b_n(x) = \begin{cases} -1, & x \le 0; \\ n^3x^2 - 1, & 0 < x \le \frac{1}{n^2}; \\ 2nx - 1, & 1/n^2 < x \le \frac{1}{n} - \frac{1}{n^2}; \\ 1 - n^3(x - \frac{1}{n})^2, & 1/n - 1/n^2 < x \le \frac{1}{n}; \\ 1, & x \ge \frac{1}{n}. \end{cases}$$

Then

$$b_n^{'}(x) = \begin{cases} 0, & x \le 0; \\ 2n^3x, & 0 < x \le \frac{1}{n^2}; \\ 2n, & 1/n^2 < x \le \frac{1}{n} - \frac{1}{n^2}; \\ 2n^3(x - \frac{1}{n}), & 1/n - 1/n^2 < x \le \frac{1}{n}; \\ 0, & x \ge \frac{1}{n}. \end{cases}$$

Evidently, $b_n^{'} \in C(\mathbb{R})$; moreover, it is Lipschitz:
$|b_n^{'}(x_1) - b_n^{'}(x_2)| \le 2n^3|x_1 - x_2|$.

Lemma 14

For any $x \in \mathbb{R}$ $b_{n+1}(x) > b_n(x),\ n \geq 1.$

Consider the approximating equation

$$X_t^n = x + \int_0^t b_n(X_s^n)ds + B_t. \tag{14}$$

The functions b_n are Lipschitz, therefore the equation (14) has unique strong solution X_t^n on $[0, T]$, and $X_t^n \leq X_t^{n+1}$ for any $t \in [0, T]$ a.s. Moreover, for any $0 < \varepsilon < H$

$$|X_{t_1}^n(\omega) - X_{t_2}^n(\omega)| \leq C(\omega)|t_2 - t_1|^{H-\varepsilon} + |t_2 - t_1|,$$

so, the set $\{X_n(\cdot, \omega), n \geq 1\}$ is tight for any $\omega \in \Omega',\ P(\Omega') = 1.$

We obtain that $X_t^n(\omega) \uparrow X_t(\omega)$, $\omega \in \Omega'$, where the limit process X is continuous in t. Further,

$$\left| \int_0^t b_n(X_s^n)ds - \int_0^t b(X_s)ds \right| \leq \int_0^t |b_n(X_s^n) - b_n(X_s)|ds + \int_0^t |b_n(X_s) - b(X_s)|ds. \quad (15)$$

Note that $|b_n(X_s^n) - b_n(X_s)| = b_n(X_s) - b_n(X_s^n) \leq 2$.

Consider the following cases.

(a) For $X_s^n < 0$, $X_s \in (0, \frac{1}{n}]$ $\quad b_n(X_s) - b_n(X_s^n) \le 2\mathbf{1}_{\{X_s \in (0, \frac{1}{n}]\}}$.

(b) For $X_s^n < 0$, $X_s > \frac{1}{n}$ $\quad b_n(X_s) - b_n(X_s^n) \le 2\mathbf{1}_{\{X_s > 0, X_s^n < 0\}} \to 0$ a.s., $n \to \infty$.

(c) For $X_s, X_s^n \in [0, \frac{1}{n}]$ $\quad b_n(X_s) - b_n(X_s^n) \le 2\mathbf{1}_{\{X_s \in [0, \frac{1}{n}]\}}$.

(d) For $X_s^n \in [0, \frac{1}{n}]$, $X_s > \frac{1}{n}$ $\quad |b_n(X_s) - b_n(X_s^n)| \le 2\mathbf{1}_{\{X_s > 0, X_s^n \in [0, \frac{1}{n}]\}} \to 0$ a.s., $n \to \infty$.

Further,

$$\int_0^t |b_n(X_s) - b(X_s)| ds \le 2 \int_0^t \mathbf{1}_{\{X_s \in [0, \frac{1}{n}]\}} ds. \tag{16}$$

We obtain from (15) – (16) and (a)–(d) that

$$\lim_{n\to\infty} \left| \int_0^t b_n(X_s^n) ds - \int_0^t b(X_s) ds \right|$$
$$\le 6 \lim_{n\to\infty} \int_0^t \mathbf{1}_{\{X_s \in (0, \frac{1}{n})\}} ds = 6 \int_0^t \mathbf{1}_{\{X_s = 0\}} ds.$$

Therefore, to prove the existence of strong solution of (13) it is sufficient to prove that $E\int_0^T \mathbf{1}_{\{X_s=0\}}ds = 0$, and in turn it is sufficient to establish the existence of bounded density $p_s(x)$, $x\in\mathbb{R}$, $s>0$ of the process X_s. For this purpose, return to X_s^n : since the functions b_n are continuously differentiable, then X_s^n has stochastic derivative, and on our probability space

$$D_sX_t^n = 1 + \int_s^t D_sX_u^n b_n^{'}(X_u^n)du,$$

whence $D_sX_t^n = \exp\{\int_s^t b_n^{'}(X_u^n)du\} \geq 1$, since $b_n^{'} \geq 0$.

Now we use the result of (Nual95): let the random variable $F \in D^{1,2}$, $h \in H$, $\langle DF, h\rangle_H \neq 0$ a.s. and $\frac{h}{\langle DF,h\rangle_H} \in Dom\,\delta$. Then F has continuous and bounded density

$$f(x) = E\left(\mathbf{1}_{\{F>x\}}\delta\left(\frac{h}{\langle DF, h\rangle_H}\right)\right).$$

Now we put $F := X_t^n$, $h_t(s) := \mathbf{1}_{\{0\le s\le t\}}$. Then

$$\begin{aligned}\langle DF, h\rangle_H &= H(2H-1)\int_0^t\int_0^t \exp\{\int_s^t b_n'(X_u^n)du\}\times \\ &\times \exp\{\int_v^t b_n'(X_u^n)du\}|v-s|^{2H-2}dvds \ge C_H t^{2H} > 0.\end{aligned}$$

Consider the function $\theta(s) = \frac{h_t(s)}{\langle DF, h_t \rangle_H} = h_t(s)\xi$, where ξ is bounded random variable, $\xi = \langle DF, h \rangle_H^{-1}$, $E\xi^2 = \int_0^T h_t^2(s)ds < \infty$. To establish that $\theta \in Dom\ \delta$, it is sufficient to verify that

$$E \int_0^T (D_s \xi)^2 ds < \infty. \tag{17}$$

But

$$\begin{aligned} D_s\xi &= D_s \Bigg(\Bigg(\int_0^t \int_0^t \exp\{\int_z^t b_n'(X_u^n)du\} \exp\{\int_v^t b_n'(X_u^n)du\}\times \\ &\times \quad |v-z|^{2H-2}dvdz\Bigg)^{-1}\Bigg) = \langle DF, h\rangle_H^{-2} \cdot \int_0^t \int_0^t \exp\{\int_z^t b_n'(X_u^n)du\} \times \\ &\times \quad \exp\{\int_v^t b_n'(X_u^n)du\}|v-z|^{2H-2} \int_z^t b_n''(X_u^n)dudvdz, \end{aligned}$$

where $|b_n''(x)| \le 2n^3$ (since $|b_n'(x_1) - b_n'(x_2)| \le 2n^3|x_1 - x_2|$). Therefore, $|D_s\xi| \le C_H^{-2} t^{-2H} \cdot 4n^3 \cdot C(H,n,t)$ (note that $|b_n'(x)| \le 2n$), and (17) holds. We obtain that $\theta \in Dom\ \delta$, and the density $p_t^n(x) := p_{X_t^n}(x)$ equals

$$p_t^n(x) = E\left\{\mathbf{1}_{\{X_t^n > x\}}\delta\left(\frac{h}{\langle DX_t^n, h_t\rangle_H}\right)\right\}.$$

Let $\psi(y) := \mathbf{1}_{[a,b]}(y)$. Then from Proposition 2.1.1 (Nua 95)

$$
\begin{aligned}
P\{a \le X_n(t) \le b\} &= \int_a^b p_t^n(x)dx \\
&= \int_a^b E\left\{\mathbf{1}_{\{X_t^n > x\}}\delta\left(\frac{h}{\langle DX_t^n, h_t\rangle_H}\right)\right\}dx \\
&= E\left(\left(\int_{-\infty}^{X_t^n}\psi(x)dx\right)\cdot\delta\left(\frac{h}{\langle DX_t^n, h_t\rangle_H}\right)\right) \\
&= E\left(\varphi(X_t^n)\delta\left(\frac{h}{\langle DX_t^n, h_t\rangle_H}\right)\right) = \left(\varphi(y) = \int_{-\infty}^{y}\psi(z)dz\right) \\
&= E\left(\left\langle D\varphi(X_t^n), \frac{h}{\langle DX_t^n, h_t\rangle_H}\right\rangle_H\right) \le \frac{1}{C_H t^{2H}} E\left(\langle D\varphi(X_t^n), h\rangle_H\right) \\
&= C_{1,H} t^{-2H}\int_a^b E\left(\mathbf{1}_{\{X_t^n > x\}}\delta(h)\right)dx \\
&\le C_{1,H} t^{-2H} E|\delta(h)|\int_a^b dx.
\end{aligned}
$$

Therefore, $p_n^t(x) \leq C_{2,H} t^{-2H}$, and $P\{a \leq X_t \leq b\} = \lim_{n\to\infty} P\{a \leq X_t^n \leq b\} = C_{2,H} t^{-2H}(b-a)$ for any continuous points of distribution function of X_t. Choosing $a \uparrow 0$, $b \downarrow 0$, we obtain density $p_t(0) \leq C_{2,H} t^{-2H}$. So, we proved the following result

Theorem 15

The equation (13) *with* $b(x) = \operatorname{sign} x$ *has a strong solution.*

Now we consider the case when $H \in (0, 1/2)$ and the condition (12) holds hence the equation (2) has the unique strong solution. Suppose in addition that $f \in L_p[0, T] \cap D_p^H[0, T]$ for some $p > \frac{1}{H}$, where $D_p^H[0, T] = \{f : [0, T] \to R| \int_0^T (\int_x^T \phi(x, t)dt)^p dx < \infty\}$ and $\phi(x, t) = \frac{|f(t)-f(x)|}{(t-x)^{1-\alpha}} 1_{0<x<t\leq T}$. Then the integral $I_t(f)$ is continuous on $[0, T]$. Evidently, the solution X_t is also continuous on $[0, T]$.

It can be proved that there exists constant $C(H, p)$ such that

$$\|I_t^*\|_r \le C(H,p)\left(\Gamma\left(\frac{r+1}{2}\right)\right)^{1/r} G_p^1(0,T,f), \tag{18}$$

where $G_p^1(0,T,f) := ||f||_{L_p[0,T]} T^{H-\frac{1}{p}} + T^{\frac{1}{2}-\frac{1}{p}} (\int_0^T (\int_x^T \phi(x,t)dt)^p dx)^{\frac{1}{p}}$. It can be proved with the help of these estimates that for any $0 \le t \le T$

$$E|X_t|^r \le K(3^r e)^{6CT+1},$$

where $K = 3^r(C_r(G_1(0,T,f))^r + (3|x|)^r) + 1$, $C_r = \frac{2^{r/2}+1}{\pi^{\frac{1}{2}}}\Gamma(\frac{r+1}{2})$, C is the linear growth constant of b.
The increments $E|X_t - X_{t'}|^r$, $0 \le t < t' \le T$ can be estimated similarly:

$$E|X_t - X_{t'}|^r \le (4C)^r D_r (t'-t)^r + 2^r C_r \left(C_{H,\beta,T}\right)^r (t'-t)^{Hr}. \tag{19}$$

Let the function $U(x) = \exp\{x^2\} - 1$, $\{\Omega, \mathcal{F}, P\}$ be some probability space.

Definition 16

Orlicz space $L_U(\Omega)$ generated by the function $U(x)$ is the space of random variables ξ on $(\Omega, \mathcal{F})$ such that for some constant $C_\xi > 0$ $EU(\frac{\xi}{C_\xi}) < \infty$.

The next result is proved in the book (BulKoz00).

Theorem 17

Orlicz space $L_U(\Omega)$ *is the Banach space with respect to Luxemburg norm*

$$\|\xi\|_U = \inf\{r > 0 : E\exp\{\frac{\xi^2}{r^2}\} \le 2\}.$$

Let $\mathbb{T}$ be a set of parameters.

Definition 18

The random process $Y = \{Y_t, t \in \mathbb{T}\}$ belongs to the space $L_U(\Omega)$, if for any $t \in \mathbb{T}$ the random variable Y_t belongs to this space.

Introduce the notations $a := (3e)^{6CT+1}$, $b := 3|x|a$, $c := 3aG^1(0, T, f)$, $c_1 = c\sqrt{2}$, $d := \max\{c, a\sqrt{e}, b\sqrt{e}\}$, $h := (3 + 2\sqrt{2})\exp\{\frac{d^2}{2c^2}\}$.

Theorem 19

Let the conditions of the Theorem 12 hold and $\{X_t, t \in [0, T]\}$ be the solution of equation (2). Then for any $\varepsilon > 0$

$$P\{|X_t| > \varepsilon\} \leq h \exp\{-\frac{\varepsilon^2}{2c^2}\}. \tag{20}$$

Theorem 20

Let the conditions of Theorem 12 hold and $\{X_t, t \in [0, T]\}$ be the solution of equation (2). Then the random variable X_t belongs to the Orlicz space $L_U(\Omega)$, and its norm in this space admits an estimate

$$\|X_t\|_U \leq \sqrt{2}(h+1)c.$$

Numerical solution via time discretization of SDEs driven by Brownian motion has long history. Concerning numerical solution of SDEs driven by fBm, we mention first the paper [Greksch and Anh (1998)], where equations with modified fBm that represents a special semimartingale are studied (recall that fBm itself is not a semimartingale). Papers [Nourdin (2005), Nourdin and Neunkirch (2007)] study Euler approximations for homogeneous one-dimensional SDEs with bounded coefficients having bounded derivatives up to third order, driven by fBm, and prove that error of approximation is a.s. equivalent to $\delta^{2H-1}\xi_t$, and the process ξ_t is given explicitly. These papers also discuss Crank–Nicholson and Milstein schemes for SDEs driven by fBm.

We consider the stochastic differential equation on R^d

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{ij}(s, X_s) dB_s^j + \int_0^t b^i(s, X_s) ds, \quad i = 1, ..., d, \quad t \in [0, T] \tag{21}$$

where the processes $B^i, i = 1, ..., m$ are fractional Brownian motions with Hurst parameter H, X_0 is a d-dimensional random variable, the coefficients $\sigma^{ij}, b^i : \Omega \times [0, T] \times R^d \to R$ are measurable functions. The integral in the right-hand side of (21) can be understood in the pathwise sense defined in [Zähle (1998), Nualart and Răşcanu (2000)] or in Wick–Skorohod sense [Alòs and Nualart (2002)].

We treat the pathwise case first. We remind that the pathwise integral w.r.t. a one-dimensional fBm B can be defined as

$$\int_a^b f dB = \int_a^b \left(D_{a+}^{\alpha} f\right)(s)\left(D_{b-}^{1-\alpha} B_{b-}\right)(s) ds,$$

where

$$\left(D_{a+}^{\alpha} f\right)(s) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(s)}{(s-a)^{\alpha}} + \alpha \int_a^s \frac{f(s)-f(u)}{(s-u)^{\alpha+1}} du\right] \mathbb{1}_{(a,b)}(s)$$

$$\left(D_{b-}^{1-\alpha} B_{b-}\right)(s) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left[\frac{B_{b-}(s)}{(b-s)^{1-\alpha}} + (1-\alpha)\int_s^b \frac{B_{b-}(s)-B_{b-}(u)}{(u-s)^{2-\alpha}} du\right] \mathbb{1}_{(a,$$

are fractional derivatives of corresponding orders, $B_{b-}(s) = \left(B_s - B_b\right)\mathbb{1}_{(a,b)}(s)$. The integral exists for any $\alpha \in (1-H, \nu)$ if, for example, $f \in C^{\nu}(a,b)$ with $\nu + H > 1$. Moreover, in this case pathwise integral admits an estimate

$$\left|\int_a^b f dB\right| \le C_0(\omega)\left[\int_a^b \frac{|f(s)|}{(s-a)^{\alpha}} ds + \int_a^b \int_a^s \frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}} du\, ds\right], \quad (22)$$

where $C_0(\omega) = C \cdot \sup_{a<s<b}\left|D_{b-}^{1-\alpha} B_{b-}(s)\right| < \infty$ a.s.

Denote $\sigma = (\sigma^{ij})_{d\times m}, b = (b^i)_{d\times 1}$ and for a matrix $A = (a^{ij})_{d\times m}$, and a vector $y = (y^i)_{d\times 1}$ denote $|A| = \sum_{i,j} |a^{ij}|, |y| = \sum_i |y^i|$. We suppose that the coefficients satisfy the following assumptions

(A) $\sigma(t, x)$ is differentiable in x and there exist such $M > 0, 1 - H < \beta \leq 1, \frac{1}{H} - 1 < \kappa \leq 1$ and for any $N > 0$ there exists such $M_N > 0$ that

1) $|\sigma(t, x) - \sigma(t, y)| \leq M|x - y|, x, y \in R^d, t \in [0, T]$;
2) $|\partial_{x_i}\sigma(t, x) - \partial_{x_i}\sigma(t, y)| \leq M_N|x - y|^{\kappa}, |x|, |y| \leq N,\ t \in [0, T]$;
3) $|\sigma(t, x) - \sigma(s, x)| + |\partial_{x_i}\sigma(t, x) - \partial_{x_i}\sigma(s, x)| \leq M|t - s|^{\beta}, x \in R^d, t, s \in [0, T]$.

(B) 1) for any $N > 0$ there exists $L_N > 0$ such that

$$|b(t, x) - b(t, y)| \leq L_N|x - y|,\ \ |x|, |y| \leq N,\ \ t \in [0, T];$$

2) $|b(t, x)| \leq L(1 + |x|)$.

As it was stated in [Nualart and Rășcanu (2000)], under conditions (A)–(B) the equation (21) has the unique solution $\{X_t,\ t \in [0, T]\}$, and for a.a. $\omega \in \Omega$ this solution belongs to $C^{H-\rho}[0, T]$ for any $0 < \rho < H$.

Now, let $t \in [0, T], \delta = \frac{T}{N}, \tau_n = \frac{nT}{N} = n\delta, n = 0, ..., N$. Consider discrete Euler approximations of solution of equation (21),

$$\widetilde{Y}^{i,\delta}_{\tau_{n+1}} = \widetilde{Y}^{i,\delta}_{\tau_n} + b^i(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\delta + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\Delta B^j_{\tau_n}, \quad \widetilde{Y}^{i,\delta}_0 = X^i_0,$$

and corresponding continuous interpolations

$$Y^{i,\delta}_t = \widetilde{Y}^{i,\delta}_{\tau_n} + b^i(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})(t-\tau_n) + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})(B^j_t - B^j_{\tau_n}), \quad t \in [\tau_n, \tau_{n+1}]. \tag{23}$$

Continuous interpolations satisfy the equation

$$Y^{i,\delta}_t = X^i_0 + \int_0^t b^i(t_u, Y^{\delta}_{t_u})du + \sum_{j=1}^{m} \int_0^t \sigma^{ij}(t_u, Y^{\delta}_{t_u})dB^j_u, \tag{24}$$

where $t_u = \tau_{n_u}$, $n_u = \max\{n : \tau_n \le u\}$.
For simplicity we denote the vector of solutions as $X_t = (X^i_t)_{i=1,...,d}$, vector of continuous approximations as $Y^{\delta}_t = (Y^{\delta,i}_t)_{i=1,...,d}$.

First consider growth and Hölder properties of the approximation process $\{Y_t^\delta, t \in [0, T]\}$. We need some additional notations. Denote $\varphi_{u,v} := \left|Y_{t_u}^\delta - Y_v^\delta\right| (u-v)^{-\alpha-1}$ for $0 < v < t_u < T$, $0 < \alpha < 1$, $X_t^* := \sup_{0\le s\le t} |X_s|$, $Y_t^{\delta,*} := \sup_{0\le s\le t} \left|Y_s^\delta\right|$. Further, for any $0 < \rho < H$ there exists such $C = C(\omega, \rho)$ that for any $0 < v < u$

$$|B_u - B_v| \le C(\omega, \rho)(u-v)^{H-\rho}. \tag{25}$$

We shall use the following statement [Nualart and Rășcanu (2000), Lemma 7.6]

Proposition 21

Let $0 < \alpha < 1$, $a, b > 0$, $x : \mathbb{R}_+ \to R_+$ *be a continuous function such that for each* t

$$x_t \le a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds.$$

Then $x_t \le ac_\alpha \exp\left\{d_\alpha t b^{1/(1-\alpha)}\right\}$, *where* $c_\alpha = 4e^2 \frac{\Gamma(1-\alpha)}{1-\alpha}$, $d_\alpha = 2\big(\Gamma(1-\alpha)\big)^{1/(1-\alpha)}$, $\Gamma(\cdot)$ *is Euler's Gamma function.*

Lemma 22

There exists such $C = C_\alpha > 0$ *that for any* $s \in [0, T]$, $s \neq t_s$ *and* $\delta \leq 1$, $\alpha \in (0, 1)$ *it holds*

$$J := \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_u} (v - t_v)^{-\alpha} dv\, du \leq C\delta^{-\alpha}.$$

Theorem 23

(i) Let the conditions (A)–(B) *hold and*

(C) 1) $\quad |\sigma(t,x)| \leq C(1+|x|)$.

Then for any $\varepsilon > 0$ *and* $0 < \rho < H$ *there exists* $\delta_0 > 0$ *and* $\Omega_{\varepsilon,\delta_0,\rho} \subset \Omega$ *such that* $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1 - \varepsilon$ *and for any* $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$ *one has* $\left|Y_t^\delta\right| \leq C(\omega)$, $\left|Y_{t_s}^\delta - Y_{t_r}^\delta\right| \leq C(\omega)(t_s - t_r)^{H-\rho}$, $0 \leq r < s \leq T$.

(ii) If, instead of (A), 2) *and* (C) *we assume that* b *and* σ *are bounded functions, then* $\left|Y_t^\delta\right| \leq C(\omega)$, $\left|Y_s^\delta - Y_r^\delta\right| \leq C(\omega)(s-r)^{H-\rho}$, $0 \leq r < s \leq T$.

Now we establish the estimates of the rate of convergence of our approximations (24). We establish even more: an estimate of convergence rate for the norm of the difference $X_t - Y_t^\delta$ in some Besov space. Denote $\Delta_{u,s}(X, Y^\delta) := \left|X_s - Y_s^\delta - X_u + Y_u^\delta\right|$.

Theorem 24

Let the conditions (A)–(C) *hold and also*

(D) 1) $|b(t,x) - b(s,x)| \le C|t-s|^\gamma$, $C > 0$, $2H - 1 < \gamma \le 1$;

2) *the exponent* β *from* (A) 3) *satisfies* $\beta > H$.

Then: (i) for any $\varepsilon > 0$ *and any* $\rho > 0$ *sufficiently small there exists* $\delta_0 > 0$ *and* $\Omega_{\varepsilon,\delta_0,\rho}$ *such that* $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1 - \varepsilon$ *and for any* $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$

$$U_\delta := \sup_{0 \le s \le T} \left(\left|X_s - Y_s^\delta\right| + \int_0^{t_s} \frac{\left|\Delta_{u,s}(X, Y^\delta)\right|}{(s-u)^{\alpha+1}} du \right) \le C(\omega) \cdot \delta^{2H-1-\rho},$$

where $C(\omega)$ *does not depend on* δ *and* ε *(but depends on* ρ*).*
If, in addition, b *and* σ *are bounded, then for any* $\rho \in (0, 2H-1)$ *there exists* $C(\omega) < \infty$ *a.s. such that* $U_\delta \le C(\omega)\delta^{2H-1-\rho}$.

Remark 25

In [Nourdin and Neunkirch (2007)] it is proved that $\left|X_t - Y_t^\delta\right| \delta^{1-2H}$ *almost surely converges to some stochastic process* ξ_t*, which means that the estimate of the rate of convergence in the previous Theorem is sharp.*

Now we assume that our probability space is the white noise space $(\Omega, \mathcal{F}, P) = (S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu)$, $\diamond$ is the Wick product, $B_t^0 = \langle \omega, \mathbb{1}_{[0,t]} \rangle$ is Brownian motion, $W^0 = \dot{B}^0$ is the white noise (see [Holden et al. (1996)] for definitions). Next, in order to introduce an fBm with Hurst parameter $H > 1/2$ on this space, we define for $f : [0, T] \to \mathbb{R}$ the fractional integral operator

$$Mf(x) = K \int_x^T (s - x)^{H-3/2} f(s)\, ds,$$

where K is some special constant, and set $M_t(x) = M\mathbb{1}_{[0,t]}(x)$. We also define for $f, g : [0, T] \to \mathbb{R}$ the scalar product and the norm

$$\langle f, g \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T f(t)g(s)\, |t-s|^{2H-2}\, dt\, ds, \quad \|f\|_{\mathcal{H}}^2 = \langle f, f \rangle_{\mathcal{H}}.$$

The process

$$B_t = \langle M_t, \omega \rangle, \quad t \in [0, T]$$

is the fBm with Hurst parameter H. Let also $W = \dot{B}$ be the fractional white noise. Detailed description of the white noise theory can be found in [Elliott and van der Hoek (2003)], [Hu and Øksendal (2003)].

Consider quasilinear Skorohod-type equation driven by fractional white noise

$$X(t) = X_0 + \int_0^t b(s, X(s), \omega)\, ds + \int_0^t \sigma(s) X(s) \diamond W(s)\, ds \tag{26}$$

with non-random initial condition X_0. Suppose that coefficients b and σ satisfy the following:

(E)

1) The linear growth condition and Lipschitz condition on b:

$$|b(t, x, \omega)| \le C(1 + |x|), \quad |b(t, x, \omega) - b(t, y, \omega)| \le C\, |x - y|\,;$$

2) "Smoothness" of b w.r.t. ω: for any $t \in [0, T]$ and for $h \in L^1(\mathbb{R})$

$$|b(t, x, \omega + h) - b(t, x, \omega)| \le C(1 + |x|) \int_{\mathbb{R}} |h(s)|\, ds.$$

3) Hölder continuity of b w.r.t. t or order H with constant that grows linearly in x:

$$|b(t, x, \omega) - b(s, x, \omega)| \le C(1 + |x|)\, |t - s|^H\,;$$

4) Hölder continuity of σ w.r.t. t or order H:

$$|\sigma(t) - \sigma(s)| \le C\, |t - s|^H\,.$$

Remark 26

The condition $(\mathrm{E})\ 2)$ *is true if, for example, the coefficient* b *has stochastic derivative growing at most linearly in* x*. It is obviously true if* b *is non-random.*

Define for $t \in [0, T]$ $\sigma_t(s) = \sigma(s)1\!\!1_{[0,t]}(s)$ and denote

$$J_\sigma(t) = \exp^\diamond \left\{ -\int_0^t \sigma(s) dB_s \right\} = \exp\left\{ -\int_{\mathbb{R}} M\sigma_t(s) dB^0(s) - \frac{1}{2} \|\sigma_t\|^2_{\mathcal{H}} \, ds \right\}$$

the fractional Wick exponent. It follows from [Mishura (2003), Theorem 2] that under assumptions (E) equation (26) has the unique solution that belongs to all L^p and can be represented in the form

$$X(t) = J_\sigma(t) \diamond Z(t),$$

where the process $Z(t)$ solves (ordinary) differential equation

$$Z(t) = X_0 + \int_0^t J_\sigma(s) b(s, J_\sigma^{-1}(s) Z(s), \omega + M\sigma_s) \, ds. \qquad (27)$$

This gives the following idea of constructing approximations of the solution of (26). Take the uniform partition $\{\tau_n = n\delta,\ n = 1, \dots, N\}$ of the segment $[0, T]$ and define first the approximations of Z recursively:

$$\begin{aligned} \widetilde{Z}(0) &= X_0, \\ \widetilde{Z}(\tau_{n+1}) &= \widetilde{Z}(\tau_n) + \widetilde{J}(\tau_n) b(\tau_n, \widetilde{J}^{-1}(\tau_n)\widetilde{Z}(\tau_n), \omega + M\widetilde{\sigma}_n)\delta, \end{aligned} \tag{28}$$

where

$$\widetilde{J}(t) = \exp\left\{-\int_0^t \widetilde{\sigma}(s) dB_s - \frac{1}{2}\left\|\widetilde{\sigma}\mathbb{1}_{[0,t]}\right\|^2_{\mathcal{H}}\right\},$$
$$\widetilde{\sigma}(s) = \sigma(t_s),\ \widetilde{\sigma}_n = \widetilde{\sigma}\mathbb{1}_{[0,\tau_n]}.$$

Note that both $\|\widetilde{\sigma}_n\|_{\mathcal{H}}$ and $M\widetilde{\sigma}_n$ are easily computable as finite sums of elementary integrals. Further, we interpolate continuously by

$$\widetilde{Z}(t) = X_0 + \int_0^t \widetilde{J}(t_s) b(t_s, \widetilde{J}^{-1}(t_s)\widetilde{Z}(t_s), \omega + M\widetilde{\sigma}_{n_s})\, ds, \tag{29}$$

where $n_s = \max\{n : \tau_n \le s\}$, and set $\widetilde{X}(t) = T_{-M\widetilde{\sigma}\mathbb{1}_{[0,t]}}\widetilde{J}^{-1}(t)\widetilde{Z}(t)$, where for $h \in S'(\mathbb{R})$ T_h is the shift operator, $T_hF(\omega) = F(\omega + h)$.

Lemma 27

Under the assumption (E) 1) *the following estimate is true*

$$\left|e^{\alpha_1} b(t, e^{-\alpha_1} x, \omega) - e^{\alpha_2} b(t, e^{-\alpha_2} x, \omega)\right| \leq C(1 + e^{\alpha_1} + e^{\alpha_2} + |x|)\, |\alpha_1 - \alpha_2|\, .$$

Theorem 28

Under conditions (E) *for any* $p \geq 1$ *the following estimate holds:*

$$\mathsf{E}\left|Z(t) - \widetilde{Z}(t)\right|^{2p} \leq C(p)\delta^{2pH}. \tag{30}$$

The main result follows.

Theorem 29

Under conditions (E) *approximations* $\widetilde{X}$ *defined by* (**??**) *converge to the solution* X *of* (26) *in the mean-square sense, and moreover*

$$\mathsf{E}\,(X(t) - \widetilde{X}(t))^2 \leq C\delta^{2H}.$$

Consider one-dimensional stochastic differential equation of the form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dB_s^H, \ \ t \in [0, T], \quad (31)$$

where X_0 is $\mathcal{F}_0$-measurable random variable with finite moments of any order, W is standard Brownian motion, B^H is fBm with Hurst index $H \in (\frac{1}{2}, 1)$, the first integral in the right-hand side is Lebesgue-Stieltjes integral, the second one is stochastic integral with respect to standard Brownian motion, and the third one is the generalized Lebesgue-Stieltjes integral. We suppose that Brownian diffusion coefficient satisfies usual Lipschitz and growth conditions and fractional diffusion satisfies the same assumptions as before.

Theorem 30

Let assumptions (A)–(D) hold. Then the equation (31) *has the unique solution X. Also, the Euler approximations X^δ converge to X in Besov space $W_\gamma[0,T]$ for any $0<\gamma<1/2$ in the following sense: for any η there exists $C=C_\eta$ such that*

$$||X^\delta - X||_{\gamma,N} \le \exp\{CN^2\}\delta^{\kappa+H-1-\eta},$$

where $\kappa = \min(\frac{1}{2},\beta)$,

$$W_\gamma([0,T]) := \{Y = Y_t(\omega) : (t,\omega)\in[0,T]\times\Omega, ||Y||_\gamma < \infty\},$$

with the norm

$$||Y||_\gamma^2 := \sup_{t\in[0,T]}\left(\mathrm{E}\, Y_t^2 + \mathrm{E}\left(\int_0^t \frac{|Y_t - Y_s|}{(t-s)^{1+\gamma}}ds\right)^2\right), \qquad (32)$$

and index N corresponds to the stopping time connected to the fractional norms of fBm.

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