Absolute continuous approximations for multifractional processes and fields

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Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition

The fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with stationary increments and the covariance function

$$EB_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

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Remark

Since $E(B_t^H - B_s^H)^2 = |t - s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

Let $H\colon [0,+\infty) \to \left(\frac{1}{2},1\right)$ be a function which satisfies Hölder condition: there exist $C_1 > 0$ and $\gamma > \frac{1}{2}$ such that for all $t_1, t_2 \in [0,+\infty)$

$$\left|H_{t_1} - H_{t_2}\right| \le C_1 \left|t_1 - t_2\right|^{\gamma}.$$

Let $H_{\min} := \min\left\{\gamma, \min_{t\in[0,T]} H_t\right\}$.

- **Moving average mBm:** $Y_t = B_t^{H_t}$, where

$$B_t^H = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}\right]dW_s + \int_0^t (t-s)^{H-\frac{1}{2}}dW_s\right\},$$

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- **Volterra-type mBm:** $Y_t = B_t^{H_t}$, where

$$B_t^H = \int_0^t K_H(t,s)\,dW_s, \quad t \ge 0,$$

$$K_H(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}}\,dv,$$

$$C_H = \left(\frac{H(2H-1)}{\mathrm{B}\left(2-2H, H-\frac{1}{2}\right)}\right)^{\frac{1}{2}}.$$

- **Harmonizable mBm**

  Let $W(\cdot)$ be a comlex-valued random measure on $\mathbb{R}$ such that

  1) for all $A, B \in \mathcal{B}(\mathbb{R})$
  $$\mathsf{E} W(A)\overline{W(B)} = \lambda(A \cap B),$$
  where λ is Lebesgue measure,
  2) for any sequence $\{A_1, A_2, \dots\} \subset \mathcal{B}(\mathbb{R})$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, $\{W(A_i), i \geq 1\}$ are centered and normal,
  $$W\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i \geq 1} W(A_i),$$
  3) for all $A \in \mathcal{B}(\mathbb{R})$ $\;W(A) = \overline{W(-A)}$,
  4) for all $\theta \in \mathbb{R}$
  $$\left\{e^{i\theta} W(A), A \in \mathcal{B}(\mathbb{R})\right\} \stackrel{d}{=} \left\{W(A), A \in \mathcal{B}(\mathbb{R})\right\}.$$

  Harmonizable mBm is defined as $Y_t = B_t^{H_t}$ where
  $$B_t^H = \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{\frac{1}{2}+H}} W(dx).$$

We consider generalizations of fBm of the form $Y_t = B_t^{H_t}$, where $\left\{B_t^H, t \in [0, T], H \in \left(\frac{1}{2}, 1\right)\right\}$ is a set of random variables such that

(i) for each $H \in \left(\frac{1}{2}, 1\right)$ $\left\{B_t^H, t \in [0, T]\right\}$ is fBm with Hurst parameter H;

(ii) for all $t \in [0, T]$, $H_1, H_2 \in \left(\frac{1}{2}, 1\right)$

$$\mathrm{E}\left(B_t^{H_1} - B_t^{H_2}\right)^2 \leq C_2(H_1 - H_2)^2.$$

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Remark

The process $Y_t = B_t^{H_t}$ has a continuous modification, according to the Kolmogorov theorem.

Let for $0 < \beta < 1$

$$\varphi_f^\beta(t) := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{1+\beta}} ds,$$

and $W_0^\beta = W_0^\beta[0, T]$ be the space of measurable functions $f\colon [0, T] \to \mathbb{R}$ with

$$\|f\|_{0,\beta} := \sup_{t \in [0,T]} \varphi_f^\beta(t) < \infty.$$

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Also let $W_1^\beta = W_1^\beta[0, T]$ be the space of functions $f\colon [0, T] \to \mathbb{R}$ with

$$\|f\|_{1,\beta} := \sup_{0 \le s < t \le T} \left(\frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} du \right) < \infty$$

Let us consider such an approximation:

$$B_t^{H_t,\varepsilon} := \frac{1}{\phi_t(\varepsilon)} \int_t^{t+\phi_t(\varepsilon)} B_s^{H_s}\, ds = \frac{1}{\phi_t(\varepsilon)} \int_0^{\phi_t(\varepsilon)} B_{u+t}^{H_{u+t}}\, du,$$

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where $\phi_t(\varepsilon) = \phi(t,\varepsilon)\colon [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a set of measurable functions such that

1. $\sup_{t\in[0,T]} \phi_t(\varepsilon) \to 0,\ \varepsilon \to 0+$;
2. for all $t, s \in [0, T]$ and for all $\varepsilon > 0$

$$\left| \frac{\phi_s(\varepsilon) - \phi_t(\varepsilon)}{\phi_s(\varepsilon)} \right| \le C_3 \left| t - s \right|^{H_{\min}}, \qquad (1)$$

C_3 is a constant which does not depend on ε.

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C_3 is a constant which does not depend on ε.

For example, one can consider functions of the form $\phi_t(\varepsilon) = \psi(t)\varepsilon$, where $\psi(t)$ satisfies the following conditions:

1) $\psi(t) > c > 0$,

2) $|\psi(t) - \psi(s)| \le C \left| t - s \right|^{H_{\min}}$.

Theorem

For any $\beta \in (0, H_{\min})$ *one has the convergence in Besov space* W_1^{β}

$$\left\| B^{H,\varepsilon} - B^{H} \right\|_{1,\beta} \xrightarrow{\mathrm{P}} 0, \qquad \varepsilon \to 0+ .$$

Corollary

Let $\left\{B_t^H, t \in [0,T]\right\}$ *be fBm with Hurst parameter* $H \in \left(\frac{1}{2}, 1\right)$*, functions* $\phi_t(\varepsilon)$ *satisfy conditions:*

1) $\sup_{t\in[0,T]} \phi_t(\varepsilon) \to 0,\ \varepsilon \to 0+$;
2) *for all* $t, s \in [0,T]$ *and for all* $\varepsilon > 0$

$$\left|\frac{\phi_s(\varepsilon) - \phi_t(\varepsilon)}{\phi_s(\varepsilon)}\right| \le C_3 \left|t - s\right|^H .$$

Then for approximations

$$B_t^{H,\varepsilon} = \frac{1}{\phi_t(\varepsilon)} \int\limits_t^{t+\phi_t(\varepsilon)} B_s^H \, ds$$

one has the convergence

$$\left\|B^{H,\varepsilon} - B^H\right\|_{1,\beta} \xrightarrow{\mathsf{P}} 0, \qquad \varepsilon \to 0+$$

Consider the SDE with mBm:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s^{H_s}, \quad t \in [0, T].$$

The stochastic integral here is understood in the pathwise sense.

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The stochastic integral here is understood in the pathwise sense.
We construct approximations for the solution of this equation as solutions of

$$X_t^\varepsilon = X_0 + \int_0^t b(s, X_s^\varepsilon) ds + \int_0^t \sigma(s, X_s^\varepsilon) dB_s^{H_s,\varepsilon}, \quad t \in [0, T].$$

I. $\sigma(t,x)$ is differentiable in x, and there exist some constants $1 - H_{\min} < \varkappa \le 1$ and $\frac{1}{H_{\min}} - 1 < \delta \le 1$, and for every $N > 0$ there exists $M_N > 0$ such that the following properties hold:

(i) $\forall x \in \mathbb{R}$, $\forall t \in [0, T]$

$$|\sigma(t,x) - \sigma(t,y)| \le M_0 \, |x - y| \, ;$$

(ii) $\forall |x|, |y| \le N$, $\forall t \in [0, T]$

$$\left| \frac{\partial}{\partial x}\sigma(t,x) - \frac{\partial}{\partial x}\sigma(t,y) \right| \le M_N \, |x - y|^{\delta} \, ;$$

(iii) $\forall x \in \mathbb{R}$, $\forall t, s \in [0, T]$

$$|\sigma(t,x) - \sigma(s,x)| + \left| \frac{\partial}{\partial x}\sigma(t,x) - \frac{\partial}{\partial x}\sigma(s,x) \right| \le M_0 \, |t - s|^{\varkappa} \, .$$

II. There exists $b_0 \in L^\rho(0, T)$, where $\rho \geq 2$, and for every $N > 0$ there exists $L_N > 0$ such that the following properties hold:

(iv) $\forall |x|, |y| \leq N$, $\forall t \in [0, T]$

$$|b(t,x) - b(t,y)| \leq L_N \, |x - y| \, ;$$

(v) $\forall x \in \mathbb{R}$, $\forall t \in [0, T]$

$$|b(t,x)| \leq L_0 |x| + b_0(t).$$

Let

$$\alpha_0 = \min\left\{\frac{1}{2}, \varkappa, \frac{\delta}{1+\delta}\right\}.$$

Theorem

Suppose that $\alpha \in (1 - H_{\min}, \alpha_0)$, X_0 is a random variable, the coefficients $\sigma(t,x)$ and $b(t,x)$ satisfy assumptions (i)–(v) with $\rho \geq 1/\alpha$. Then there exists a unique solution $\{X_t, t \in [0,T]\}$ of the equation

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s^{H_s}, \quad t \in [0,T],$$

$X \in L^0(\Omega, \mathcal{F}, \mathrm{P}, W_0^{\alpha}[0,T])$, with trajectories from $C^{1-\alpha}[0,T]$ a. s.

Theorem

Suppose that $\alpha \in (1 - H_{\min}, \alpha_0)$*,* X_0 *is a random variable, the coefficients* $\sigma(t,x)$ *and* $b(t,x)$ *satisfy assumptions (i)–(v) with* $\rho \geq 1/\alpha$*.*
Then one has the uniform convergence in probability

$$\sup_{t\in[0,T]} |X_t - X_t^{\varepsilon}| \xrightarrow{\mathsf{P}} 0, \qquad \varepsilon \to 0+.$$

Let $T = (T_1, T_2) \in (0, \infty)^2$, $[0, T] = [0, T_1] \times [0, T_2]$.
Let $s, t \in [0, T]$, $s = (s_1, s_2)$, $t = (t_1, t_2)$, $f : [0, T] \to \mathbb{R}$.

$$\Delta_s f(t) := f(s_1, s_2) - f(s_1, t_2) - f(t_1, s_2) + f(t_1, t_2)$$

$$f_{t-}(s) := f(s_1, s_2) - f(s_1, t_2-) - f(t_1-, s_2) + f(t_1-, t_2-)$$

$$s < t \Leftrightarrow \begin{cases} s_1 < t_1 \\ s_2 < t_2 \end{cases}$$

$$s \le t \Leftrightarrow \begin{cases} s_1 \le t_1 \\ s_2 \le t_2 \end{cases}$$

Let $W_1^{\beta_1,\beta_2} = W_1^{\beta_1,\beta_2}([0,T])$ be a space of measurable functions $f\colon [0,T] \to \mathbb{R}$ with

$$\|f\|_{1,\beta_1,\beta_2} = \sup_{0\le s<t\le T} \left(\frac{|\Delta_s f(t)|}{(t_1-s_1)^{\beta_1}(t_2-s_2)^{\beta_2}} + \frac{1}{(t_2-s_2)^{\beta_2}} \int_{s_1}^{t_1} \frac{|f_{t-}(u,s_2)-f_{t-}(s)|}{(u-s_1)^{1+\beta_1}}\,du + \frac{1}{(t_1-s_1)^{\beta_1}} \int_{s_2}^{t_2} \frac{|f_{t-}(s_1,v)-f_{t-}(s)|}{(v-s_2)^{1+\beta_2}}\,dv + \int_{[s,t]} \frac{|\Delta_s f(r)|}{(r_1-s_1)^{1+\beta_1}(r_2-s_2)^{1+\beta_2}}\,dr \right) < \infty,$$

Let $\{B_t, t \in [0, T]\}$ be a random field which satisfy the following conditions

1) B_t is Gaussian field;
2) there exists constants $C > 0$ and $\lambda > 1$ such that for all $s, t \in [0, T]$

$$E(\Delta_s B_t)^2 \leq C(|t_1 - s_1||t_2 - s_2|)^{\lambda};$$

3) the trajectories of B_t are continuous with probability one.

We consider the following approximation for B_t:

$$B_t^\varepsilon = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B_s \, ds = \frac{1}{\varepsilon^2} \int_{[0,\varepsilon]^2} B_{s+t} \, ds.$$

Theorem

For all $\beta_1, \beta_2 \in (0, \lambda/2)$

$$\|B^\varepsilon - B\|_{1,\beta_1,\beta_2} \xrightarrow{\mathrm{P}} 0, \qquad \varepsilon \to 0+ .$$

Definition

A random field $\{B_t^H, t \in \mathbb{R}_+^2\}$ is called a fractional Brownian field with Hurst index $H = (H_1, H_2) \in (0,1)^2$, if

1) B_t^H is a Gaussian field such that $B_t^H = 0$, $t \in \partial\mathbb{R}_+^2$,

2) $\mathsf{E}B_t^H = 0$, $\mathsf{E}B_t^H B_s^H = \frac{1}{4} \prod_{i=1,2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right)$,

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The increments of fBf satisfy the following equality

$$\mathrm{E}\left(\Delta_s B_t^H\right)^2 = |t_1 - s_1|^{2H_1} |t_2 - s_2|^{2H_2} .$$

Corollary

For $\beta_1, \beta_2 \in (0, H_1 \wedge H_2)$ the following convergence holds:

$$\left\| B^{H,\varepsilon} - B^H \right\|_{1,\beta_1,\beta_2} \xrightarrow{\mathrm{P}} 0, \qquad \varepsilon \to 0+,$$

where

$$B_t^{H,\varepsilon} = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B_s^H \, ds.$$

Let $H(t) = (H_1(t), H_2(t))\colon [0, T] \to (1/2, 1)^2$ be a continuous function such that

$$\frac{1}{2} < \mu < \min_{t\in[0,T]} H_i(t) \le \max_{t\in[0,T]} H_i(t) < \nu < 1.$$

Suppose that for all $t, s \in [0, T]$

(H1) $|H_i(t) - H_i(s)| \le c_1 \left(|t_1 - s_1|^{\nu} + |t_2 - s_2|^{\nu}\right)$,

(H2) $|\Delta_s H_i(t)| \le c_2 \left(|t_1 - s_1|\,|t_2 - s_2|\right)^{\nu}$.

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(H2) $|\Delta_s H_i(t)| \le c_2 \left(|t_1 - s_1|\,|t_2 - s_2|\right)^\nu$.

Definition ([Meerschaert, Wu, and Xiao (2008)])

Multifractional Brownian sheet with Hurst index $H(t)$ is defined as follows

$$B_t^{H(t)} := \int_{\mathbb{R}^2} \prod_{i=1,2} \left[(t_i - u_i)_+^{H_i(t)-1/2} - (-u_i)_+^{H_i(t)-1/2}\right] dW_u, \qquad t \in [0, T],$$

where $s_+ = \max\{s, 0\}$, $W = \{W_s, s \in \mathbb{R}^2\}$ is a Wiener field.

Theorem

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There exists $C > 0$ *such that for all* $s, t \in [0, T]$

$$\mathrm{E}(\Delta_s Y_t)^2 \leq C(|t_1 - s_1|\,|t_2 - s_2|)^{2\mu}.$$

Theorem

For any $\beta_1, \beta_2 \in (0, \mu)$ *one has the convergence in Besov space* $W_1^{\beta_1,\beta_2}$

$$\left\| B_t^{H(t),\varepsilon} - B_t^{H(t)} \right\|_{1,\beta_1,\beta_2} \xrightarrow{\mathrm{P}} 0, \qquad \varepsilon \to 0+,$$

where

$$B_t^{H(t),\varepsilon} = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B_s^{H(s)}\, ds.$$

T.O. Androshchuk.
Approximation of a stochastic integral with respect to fractional Brownian motion by integrals with respect to absolutely continuous processes.
Theor. Probability and Math. Statist., 73:19–29, 2006.

A. Benassi, S. Jaffard, and D. Roux.
Gaussian processes and pseudodifferential elliptic operators.
Revista Mathematica Iberoamericana, 13(1):19–89, 1997.

B. Boufoussi, M. Dozzi, and R. Marty.
Local time and Tanaka formula for a Volterra-type multifractional Gaussian process.
2009.

M. Meerschaert, D. Wu, and Y. Xiao.
Local times of multifractional Brownian sheets.
Bernoulli, 14(3):865–898, 2008.

D. Nualart and A. Răşcanu.
Differential equations driven by fractional Brownian motion.
Collectanea Mathematica, 53(1):55–81, 2002.

R. F. Peltier and J. Lévy Véhel.
Multifractional Brownian motion: definition and preliminary results.
INRIA research report, (2645), 1995.

K. V. Ralchenko.
Absolute continuous approximations for multifractional Brownian motion.
Teor. Ĭmovir. ta Matem. Statyst., 82:115–127, 2010.

K. V. Ralchenko and G. M. Shevchenko.
Path properties of the multifractional Brownian motion.
Teor. Ĭmovir. ta Matem. Statyst., 80:106–116, 2009.