Differential Geometry over General Base Fields and Rings. Part III: Lie Theory

Wolfgang Bertram
Institut Elie Cartan – U.M.R. 7502
Université Henri Poincaré (Nancy I)
B.P. 239
54506 Vandœuvre-lès-Nancy Cedex
France
e-mail: bertram@iecn.u-nancy.fr

Abstract. We develop the basic theory of Lie groups and symmetric spaces over general base fields and -rings and without any restriction on the dimension, leading to a theory that has much similarity with the theory of *formal groups*. The great degree of generality is achieved by replacing the concept of formal power series by the use of iterated tangent- and jet-functors.

Contents.

AMS subject classification (2000):
Primary: 22E20, 22E65, 53C35
Secondary: 14L05, 22E25

Key words: Campbell-Hausdorff formula, exponential map, formal group, Lie group, symmetric space

Introduction

The present and third part of *Differential Geometry over General Base Fields and Rings* contains (elements of) the general theory of Lie groups and symmetric spaces. Most parts can be read independently of Part II and even of Part I (in this case the reader should read $\mathbb{R}$ or $\mathbb{C}$ instead of $\mathbb{K}$ and consider V as some $\mathbb{K}^n$.) For convenience of the reader, we start with a slightly revised version of Chapter 5 (basic facts on Lie groups and symmetric spaces), already included in the preprint version of Part I (prépublication Institut Elie Cartan 2003/47) ; but, in contrast to the preprint version just mentioned, in the sequel of this paper we develop the theory of Lie groups independently from the theory of symmetric spaces (thus basic Lie group theory works in completely arbitrary characteristic, whereas symmetric space theory works only in characteristic different from 2).

1. *Lie groups versus formal groups.* In Part II of this work we tried to lay foundations of what one might call *formal differential geometry.* In this spirit, our approach to Lie groups really is a sort of "formal group theory", but instead of the power series approach from usual formal group theory (see [Haz78]), we use a "differential" or "tangent functor" approach: the higher order tangent groups T^kG (or, more precisely, their nilpotent subgroups $(T^kG)_e$ and $(J^kG)_e$), are a sort of "k-th order Taylor expansions of G" and hence can be considered as good approximations of G itself. Following our general philosophy of multilinear connections, we try to linearize $(T^kG)_e$ by introducing a suitable "canonical chart" and to describe its group structure by a polynomial formula. Since $(T^kG)_e$ is nilpotent, one immediately thinks of the exponential map and the Campbell-Hausdorff formula in order to describe the group structure – if the integers are invertible in $\mathbb{K}$, then this is indeed possible, see Chapter 25. However, this is not the most elementary description of $(T^kG)_e$: there are two canonical charts Φ^L and Φ^R of $(T^kG)_e$ which are given by left-, resp. right-trivialization of tangent bundles. These charts can be defined in arbitrary characteristic, and we have *left* and *right product formulae* describing the group law of $(T^kG)_e$,

$$u *^F v = (\Phi^F)^{-1}(\Phi^F(u) \cdot \Phi^F(v)), \qquad F = L, R$$

(see Theorem 24.7 for the explicit formula). Just as the Campbell-Hausdorff formula for nilpotent groups, these formulae are given in terms of the Lie bracket and are polynomial in u, v, and they have the advantage that no division by scalars is needed. On the other hand, inversion in the chart Φ^F is more complicated than just multiplication by -1. Conversely, to any $\mathbb{K}$-Lie algebra $\mathfrak{g}$, we may associate a group modelled on a nilpotent Lie algebra denoted by $E = (T^k\mathfrak{g})_0$ which represents the k-jet of a virtual Lie group associated to $\mathfrak{g}$. A more detailed study of the underlying combinatorial and geometric structures of this construction is left for later work – this should also be interesting for the theory of formal groups in arbitrary, especially in positive characteristic.

2. *The Lie algebra of a Lie group.* As is well-known, the Lie algebra of a Lie group can be defined in various ways: one may define the Lie bracket via the bracket of (left or right) invariant vector fields (as is done in most textbooks), or express it in a more or less formal way by the group commutator $[g, h] = ghg^{-1}h^{-1}$ (the definition by deriving the adjoint representation is just another version of this). For convenience of the reader, we start with the first definiton (Chapter 5), and show (Theorem 23.2) that it agrees with a version of the second definition which we state in terms of the two-step nilpotent group $(TTG)_e$ (kernel of the canonical projection $TTG \to G$). Strictly speaking, invariant vector fields are not needed at all for the formal theory of Lie groups, and also for the analytic theory of Lie groups over base fields such as the p-adic numbers they are less useful than in the real case – see work of H. Glöckner [Gl03], [Gl04] for recent progress in the analytic theory of general Lie groups. In particular, in [Gl04] large classes of finite and infinite dimensional examples of general Lie groups are constructed, and in [Gl03] the existence of a compatible analytic structure on finite-dimensional Lie groups over fields such as $\mathbb{Q}_p$ is proved. A crucial step is the construction of an exponential map for such groups – we feel that a combination with our methods could lead to a unified construction of exponential mappings for these and other classes of groups.

3. *Symmetric spaces.* The definition and basic theory of *symmetric spaces* follows the one given by O. Loos in [Lo69] (see also [BeNe04], where examples are constructed): a symmetric space is a manifold M equipped with a family of symmetries $\sigma_x : M \to M$, $x \in M$, such that the map $(x,y) \mapsto \mu(x,y) := \sigma_x(y)$ is smooth and satisfies certain natural axioms (Chapter 5). In [Lo69], the higher order theory of symmetric spaces is developed by using the *linear* bundles $T^{(k)}M$ whose sections are k-th order differential operators. Again following our general philosophy, we replace these bundles by the iterated tangent bundles T^kM which are not linear, but have the advantage to be themselves symmetric spaces in a natural way. In particular, we construct the *canonical connection* of M in terms of the symmetric space TTM (Chapter 26) and express its curvature by the symmetric space structure on T^3M (Chapter 27). In later work we hope to add an intrinsic theory of the higher order bundles T^kM and of the "exponential jet" (cf. Section 27.7).

Notational remarks on the preprint version of Part I

Sign of the Lie bracket. In Part I (Section 5.3), the Lie bracket was defined via *left*-invariant vector fields. It turns out that this definition causes unnecessary signs, and hence from now on we will use the definition of the Lie bracket of $X, Y \in \mathfrak{g} = T_eG$ by

$$[X,Y] := [X^R, Y^R]_e$$

where Z^R is the *right*-invariant vector field corresponding such that $(Z^R)(e) = Z$. Then $Z^R(x) = Z \cdot x$ ($x \in G$; product in the group TG) is given by *left* multiplication with Z in TG, and this coincides with the usual conventions that let act $\mathfrak{gl}(V)$ and $\mathrm{Gl}(V)$ on V from the left (as mappings in general). Thus on $\mathfrak{gl}(V)$ (both considered as an algebra of vector fields on V and as the Lie algebra of $\mathrm{Gl}(V)$) we get the usual Lie bracket $[X,Y] = XY - YX$.

5. Lie groups and symmetric spaces: basic facts

5.1. *Manifolds with multiplication.* A *product* or *multiplication map* on a manifold M is a smooth binary map $m : M \times M \to M$, and *homomorphisms of manifolds with multiplication* are smooth maps that are compatible with the respective multiplication maps. *Left and right multiplication operators*, defined by $l_x(y) = m(x,y) = r_y(x)$, are partial maps of m and hence smooth self maps of M. Applying the tangent functor to this situation, we see that (TM, Tm) is again a manifold with multiplication, and tangent maps of homomorphisms are homomorphisms of the respective tangent spaces. The tangent map Tm is given by the formula

$$T_{(x,y)}m(\delta_x, \delta_y) = T_{(x,y)}m((\delta_x, 0_y) + (0_x, \delta_y)) = T_x(r_y)\delta_x + T_y(l_x)\delta_y. \tag{5.1}$$

Formula (5.1) is nothing but the rule on partial derivatives (Section 1.3 (6)) written in the language of manifolds. In particular, (5.1) shows that the canonical projection and the zero section,

$$\pi : TM \to M, \quad \delta_p \to p, \qquad z : M \to TM, \quad p \mapsto 0_p \tag{5.2}$$

are homomorphisms of manifolds with multiplication. We will always identify M with the subspace $z(M)$ of TM. Then (5.1) implies that the operator of left multiplication by $p = 0_p$ in TM is nothing but $T(l_p) : TM \to TM$, and similarly for right multiplications.

5.2. *Lie groups.* A *Lie group over* $\mathbb{K}$ is a smooth $\mathbb{K}$-manifold G carrying a group structure such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are smooth. Homomorphisms of Lie groups are smooth group homomorphisms. Clearly, Lie groups and their homomorphisms form a category in which direct products exist.

Applying the tangent functor to the defining identities of the group structure (G, m, i, e), it is immediately seen that then $(TG, Tm, Ti, 0_{T_eG})$ is again a Lie group such that $\pi : TG \to G$ becomes a homomorphism of Lie groups and such that the zero section $z : G \to TG$ also is a homomorphism of Lie groups.

5.3. *The Lie algebra of a Lie group.* A vector field $X \in \mathfrak{X}(G)$ is called *left invariant* if, for all $g \in G$, $X \circ l_g = Tl_g \circ X$, and similarly we define *right invariant vector fields.* If X is right invariant, then $X(g) = X(r_g(e)) = T_e r_g \cdot X(e)$; thus X is uniquely determined by the value $X(e)$, and thus the map

$$\mathfrak{X}(G)^{r_G} \to T_eG, \quad X \mapsto X(e) \tag{5.3}$$

from the space of right invariant vector fields into T_eG is injective. It is also surjective: if $v \in T_eG$, then left multiplication with v in TG, $Tl_v : TG \to TG$, preserves fibers (by (5.1)) and hence defines a vector field $v^R : G \to TG$, $g \mapsto T_g l_v(0_g) = Tm(v, 0_g) = T_e r_g(v)$ which is right invariant since right multiplications commute with left multiplications. Now, the space $\mathfrak{X}(G)^{r_G}$ is a Lie subalgebra of $\mathfrak{X}(M)$; this follows immediately from Lemma 4.3 because X is right invariant iff the pair (X, X) is r_g-related for all $g \in G$. The space $\mathfrak{g} := T_eG$ with the Lie bracket defined by

$$[v, w] := [v^R, w^R]_e$$

is called the *Lie algebra of* G.

Theorem 5.4.

(i) *The Lie bracket* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is of class* C^0.

(ii) *For every homomorphism* $f : G \to H$, *the tangent map* $\dot{f} := T_e f : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of Lie algebras.*

Proof. (i) Pick a chart $\varphi : U \to V$ of G such that $\varphi(e) = 0$. Since $w^R(x) = Tm(w, x)$ depends smoothly on (x, w), it is represented in the chart by a smooth map (which again will be denoted by $w^R(x)$). But this implies that $[v^R, w^R](x) = d(v^R)(x)w^R(x) - d(w^R)(x)v^R(x)$ depends smoothly on v, w and x and hence $[v, w]$ depends smoothly on v, w.

(ii) First one checks that the pair of vector fields $(v^R, (\dot{\varphi}v)^R)$ is f-related, and then one applies Lemma 4.3 in order to conclude that $\dot{f}[v, w] = [\dot{f}v, \dot{f}w]$. ■

The functor from Lie groups over $\mathbb{K}$ to C^0-Lie algebras over $\mathbb{K}$ will be called the *Lie functor (for $\mathbb{K}$-Lie groups and $\mathbb{K}$-Lie algebras).* We say that a C^0-Lie algebra over $\mathbb{K}$ is *enlargible* if there is a Lie group over $\mathbb{K}$ such that $\mathfrak{g} = \mathrm{Lie}(G)$. The following is called the *enlargibility problem for $\mathbb{K}$-Lie algebras*;

E1. Which $\mathbb{K}$-Lie algebras are enlargible ?

E2. If a $\mathbb{K}$-Lie algebra is enlargible, how can we describe all equivalence classes of Lie groups belonging to this Lie algebra ?

These are difficult problems which fall out of the scope of the present work. Let us just mention that all matrix groups $\mathrm{Gl}(n,\mathbb{K})$ are Lie groups (with atlas given by the natural chart), as well as all orthogonal groups $\mathrm{O}(b,\mathbb{K}^n)$ corresponding to non-degenerate forms $b : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ – in these cases the manifold structure can be defined by "Cayley's rational chart", cf. [Be00] and [BeNe04]. For $\mathrm{SL}(n,\mathbb{K})$ one can define an atlas via the Bruhat decomposition; thus all "classical groups over $\mathbb{K}$" are Lie groups, and their Lie algebras are calculated in the usual way. For the construction of much more general classes of Lie groups over topological fields we refer to [Gl03e].

5.5. *Symmetric spaces.* A *reflection space (over $\mathbb{K}$)* is a smooth manifold with a multiplication map $m : M \times M \to M$ such that, for all $x, y, z \in M$,

(M1) $m(x,x) = x$

(M2) $m(x, m(x,y)) = y$, i.e. $l_x^2 = \mathrm{id}_M$,

(M3) $m(x, m(y,z)) = m(m(x,y), m(x,z))$, i.e. $l_x \in \mathrm{Aut}(M,m)$.

(Reflection spaces – "Spiegelungsräume" in German – have been introduced by O. Loos in [Lo67] in the finite dimensional real case.) The left multiplication operator l_x is, by (M1)–(M3), an automorphism of order two fixing x; it is called the *symmetry around* x and will usually be denoted by σ_x, so that $m(x,y) = \sigma_x(y)$. The "trivial reflection space" $\sigma_x = \mathrm{id}_M$ for all x is not excluded by the axioms (M1)–(M3). We say that (M,m) is a *symmetric space (over $\mathbb{K}$)* if (M,m) is a reflection space such that 2 is invertible in $\mathbb{K}$ and the property

(M4) for all $x \in M$, $T_x(\sigma_x) = -\mathrm{id}_{T_x M}$

holds. The assumption that 2 is invertible in $\mathbb{K}$ guarantess that 0 is the only fixed point of the differential $T_x(\sigma_x) : T_x M \to T_x M$, and without this assumption (M4) would be useless. In the finite dimensional case over $\mathbb{K} = \mathbb{R}$, (M4) implies by the implicit function theorem that x is an *isolated fixed point* of σ_x and hence our definition contains the one from [Lo69] as a special case (see [Ne02a, Lemma 3.2] for the Banach case). The group $G(M)$ generated by all $\sigma_x \sigma_y$ is a (normal) subgroup of $\mathrm{Aut}(M,m)$, called the *group of displacements.* A distinguished point $o \in M$ is called a *base point.* With respect to the base point, one defines the *quadratic representation*

$$Q := Q_o : M \to G(M), \quad x \mapsto \sigma_x \sigma_o \tag{5.4}$$

and the *powers* (for $n \in \mathbb{Z}$)

$$x^{-1} := \sigma_o(x), \quad x^{2n} := Q(x)^n.o, \quad x^{2n+1} := Q(x)^n.x. \tag{5.5}$$

By a straightforward calculation, one proves then the *fundamental formula*

$$Q(Q(x)y) = Q(x)Q(y)Q(x) \tag{5.6}$$

and the *power associativity rules* (cf. [Lo69] or [Be00, Lemma I.5.6])

$$m(x^n, x^m) = x^{2n-m}, \quad (x^m)^n = x^{mn}. \tag{5.7}$$

A symmetric space is called *abelian* or *commutative* if the group $G(M)$ is commutative. For instance, any topological $\mathbb{K}$-module with the product

$$m(u,v) = u - v + u = 2u - v \tag{5.8}$$

is a commutative symmetric space.

Proposition 5.6. *Assume (M,m) is a symmetric space over $\mathbb{K}$.*

(i) *The tangent bundle (TM,Tm) of a reflection space is again a reflection space.*

(ii) *The tangent bundle (TM,Tm) of a symmetric space is again a symmetric space.*

Proof. (i) We express the identities (M1)–(M3) by commutative diagrams to which we apply the tangent functor T. Since T commutes with direct products, we get the same diagrams and hence the laws (M1)–(M3) for Tm (cf. [Lo69] for the explicit form of the diagrams).

(ii) We have to prove that (M4) holds for (TM,Tm). First of all, note that the fibers of $\pi : TM \to M$ (i.e. the tangent spaces) are stable under Tm because π is a homomorphism. We claim that for $v,w \in T_pM$ the explicit formula $Tm(v,w) = 2v - w$ holds (i.e. the structure induced on tangent spaces is the canonical "flat" symmetric structure (5.8) of an affine space). In fact, from (M1) for Tm we get $v = Tm(v,v) = T_p(\sigma_p)v + T_p(r_p)v = -v + T_p(r_p)v$, whence $T_p(r_p)v = 2v$ and

$$Tm(v,w) = T_p(\sigma_p)w + T_p(r_p)v = 2v - w.$$

Now fix $p \in M$ and $v \in T_pM$. We choose 0_p as base point in TM. Then $Q(v) = \sigma_v\sigma_{0_p}$ is, by (M3), an automorphism of (TM,Tm) such that $Q(v)0_p = \sigma_v(0_p) = 2v$. But

$$\frac{1}{2} : TM \to TM, \quad \delta_x \mapsto \frac{1}{2}\delta_x$$

also is an automorphism of (TM,Tm), as shows Formula (5.1). Therefore the automorphism group of TM acts transitively on fibers, and after conjugation of σ_v with $(\frac{1}{2}Q(v))^{-1}$ we may assume that $v = 0_p$. But in this case the proof of our claim is easy: we have $\sigma_{0_p} = T\sigma_p$, and since $T_p\sigma_p = -\operatorname{id}_{T_pM}$, the canonical identification $T_{0_p}(TM) \cong T_pM \oplus T_pM$ yields $T_{0_p}(\sigma_{0_p}) = (-\operatorname{id}_{T_pM}) \times (-\operatorname{id}_{T_pM}) = -\operatorname{id}_{T_{0_p}TM}$, whence (M4). ■

The alert reader may have noticed that our proof of (M4) already contains the construction of a canonical connection on TM: in fact, the argument of the proof shows that the "vertical space" $V_v := T_v(T_pM) \subset T_v(TM)$ has a canonical complement $H_v = \frac{1}{2}Q(v)H_0$, where H_0 is one of the factors of the canonical decomposition $T_{0_p}(TM) \cong T_pM \oplus T_pM$ – see Chapter 21 for the further theory concerning this.

5.7. *The algebra of derivations of M.* A vector field $X : M \to TM$ on a symmetric space M is called a *derivation* if X is also a homomorphism of symmetric spaces. This can be rephrased by saying that $(X \times X, X)$ is m-related. Lemma 4.3 therefore implies that the space $\mathfrak{g}$ of derivations is stable under the Lie bracket. It is also easily checked that it is a $\mathbb{K}$-submodule of $\mathfrak{X}(M)$, and hence $\mathfrak{g} \subset \mathfrak{X}(M)$ is a Lie-subalgebra.

5.8. *The Lie triple system of a symmetric space with base point.* We fix a base point $o \in M$. The map $X \mapsto T\sigma_o \circ X \circ \sigma$ is a Lie algebra automorphism of $\mathfrak{X}(M)$ of order 2 which stabilizes $\mathfrak{g}$. We let

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-, \quad \mathfrak{g}^{\pm} = \{X \in \mathfrak{g} \,|\, T\sigma_o \circ X \circ \sigma_o = \pm X\}$$

be its associated eigenspace decomposition (recall our assumption on $\mathbb{K}$!). The space $\mathfrak{g}^+$ is a Lie subalgebra of $\mathfrak{X}(M)$, whereas $\mathfrak{g}^-$ is only closed under the triple bracket

$$(X,Y,Z) \mapsto [X,Y,Z] := [[X,Y],Z].$$

Proposition 5.9.

(i) *The space $\mathfrak{g}^+$ is the kernel of the evaluation map* $\operatorname{ev}_o : \mathfrak{g} \to T_oM$, $X \mapsto X(o)$.

(ii) *Restriction of* ev_o *yields a bijection* $\mathfrak{g}^- \to T_oM$, $X \mapsto X(o)$.

Proof. (i) Assume $X \in \mathfrak{g}^+$. Then $T_o\sigma X(o) = X(\sigma_o(o)) = X(o)$ implies $-X(o) = X(o)$ and hence $X(o) = 0$. On the other hand, if $X(o) = 0$, then $X(\sigma_o(p)) = X(m(o,p)) = Tm(X(o),X(p)) = Tm(0_o,X(p)) = T\sigma_oX(p)$, whence $X \in \mathfrak{g}^+$.

(ii) By (i), $\mathfrak{g}^- \cap \ker(\operatorname{ev}_o) = \mathfrak{g}^- \cap \mathfrak{g}^+ = 0$, and hence $\operatorname{ev}_o : \mathfrak{g}^- \to T_oM$ is injective. It is also surjective: let $v \in T_oM$. Consider the map

$$\widetilde{v} = \frac{1}{2}Q(v) \circ z : M \to TM, \quad p \mapsto \frac{1}{2}Q(v)0_p = \frac{1}{2}Tm(v, Tm(0_o, 0_p)). \tag{5.9}$$

It is a composition of homomorphisms and hence is itself a homomorphism from M into TM. Moreover, as seen in the proof of Proposition 5.7, $\widetilde{v}(o) = v$. Thus we will be done if can show that $\widetilde{v} \in \mathfrak{g}^-$. First of all, $\widetilde{v}$ is a vector field since $Q(v)\delta_p \in T_{m(o,m(o,p))}M = T_pM$ for all $p \in M$. Finally,

$$\begin{aligned} T\sigma_o \circ \widetilde{v} \circ \sigma_o &= \frac{1}{2} T\sigma_o \circ Q(v) \circ z \circ \sigma_o \\ &= \frac{1}{2} Q(T\sigma_o v) \circ z = \frac{1}{2} Q(-v) \circ z = -\widetilde{v}. \end{aligned}$$

■

The space $\mathfrak{m} := T_oM$ with triple bracket given by

$$[u,v,w] := [[\widetilde{u},\widetilde{v}],\widetilde{w}](o)$$

is called the *Lie triple system (Lts) associated to* (M,o). It satisfies the identities of an abstract Lie triple system over $\mathbb{K}$ (cf. [Lo69, p. 78/79] or [Be00]).

Theorem 5.10. *Let M be a symmetric space over $\mathbb{K}$ with base point o.*

(i) *The triple Lie bracket of the Lts $\mathfrak{m}$ associated to (M,o) is od class C^0.*

(ii) *If $\varphi : M \to M'$ is a homomorphism of symmetric spaces such that $\varphi(o) = o'$, then $\dot{\varphi} := T_o\varphi : \mathfrak{m} \to \mathfrak{m}'$ is a homomorphism of associated Lts.*

Proof. One uses the same arguments as in the proof of Theorem 5.4. ■

5.11. *Lie functor, group case, and the classical spaces.* The functor described by the preceding theorem will be called the *Lie functor for symmetric spaces and Lie triple systems (defined over* $\mathbb{K}$*)*, and the same enlargibility problem as for Lie groups arises. It generalizes the enlargibility problem for Lie groups because every Lie group can be turned into a symmetric space by letting

$$\mu : G \times G \to G, \quad (x,y) \mapsto xy^{-1}x;$$

then (G,μ) is a symmetric space, called a *group case* (Properties (M1)–(M3) are immediate; for (M4) one proves as usual in Lie group theory that $T_ej = -\mathrm{id}_{T_eG}$ for the inversion map $j : G \to G$); the Lie triple system of (G,μ,e) is then $\mathfrak{g}$ with the triple bracket

$$[X,Y,Z] = \frac{1}{4}[[X,Y],Z]$$

– for proving this one can use the same arguments as in [Lo69, p. 81]. Note that, in the group case, $\mathrm{Aut}(G,\mu)$ acts transitively on the space since all left and right translations are automorphisms; however, the action of the transvection group $G(G,\mu)$ is in general not transitive – e.g., take the Lie group $\mathrm{Gl}(n,\mathbb{K})$; we have $\sigma_x\sigma_y = l_x r_x l_{y^{-1}} r_{y^{-1}}$ in terms of left and right translations, and hence the determinant of $\sigma_x\sigma_y(z)$ is congruent to the determinant of z modulo a square in $\mathbb{K}$. This shows that the orbit structure of $G = \mathrm{Gl}(n,\mathbb{K})$ under the action of $G(G,\mu)$ is at least as complicated as the structure of $\mathbb{K}^\times/(\mathbb{K}^\times)^2$.

The "classical spaces" can all be constructed in the following way: assume G is a Lie group and σ an involution of G; we assume moreover that the *space of symmetric elements*

$$M := \{g \in G \mid \sigma(g) = g^{-1}\} \tag{5.11}$$

is a submanifold of G in the sense of 2.3. (The last assumption is not automatically satisfied since we do not have an exponential map at our disposition; however, in all "classical cases" mentioned below, it is easily checked either directly or by remarking that in "Jordan coordinates" it is indeed automatic, cf. [Be00].) Then M is a subsymmetric space of (G,μ) in the obvious sense. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ into $+1$- and -1-eigenspaces of $T_e\sigma$, then, as in [Lo69, p. 82], it is seen that the Lie triple system of (M,μ,e) is $\mathfrak{q}$ with $[X,Y,Z] = [[X,Y],Z]$. Similarly, for an arbitray element $g \in M$, the Lie triple system is described in the same way, but replacing σ by the involution $I_g \circ \sigma$ where $I_g(x) = gxg^{-1}$ is conjugation by g. In contrast to the group case, here the action of the automorphism group $\mathrm{Aut}(M)$ is in general no longer transitive on M. For instance, if $G = \mathrm{Gl}(n, \mathbb{K})$ and $\sigma = \mathrm{id}$, then the orbits in $M = \{g \in G | \, g^2 = \mathbf{1}\}$ are described by the matrices $I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$, $p + q = n$. Similarly, taking for G a classical matrix group (in the sense explained after Theorem 5.4) and for σ a "classical involution" (see [Be00, Chapter I.6] for a fairly exhaustive list in the real case) we get analogs of all matrix series from Berger's classification of irreducible real symmetric spaces [B57], but where the number of non-isomorphic types may be considerably bigger in case of base-fields or rings different from $\mathbb{R}$ or $\mathbb{C}$. All these symmetric spaces are "Jordan symmetric spaces" in the sense of [BeNe04] where a very general construction of symmetric spaces (of classical or non-classical type) is described.

23. The three canonical connections of a Lie group

23.1. *The Lie bracket revisited.* In the following theorem we will give another characterization of the Lie bracket of a Lie group. One might as well use it as *definition* of the Lie bracket; this has the advantage that one can develop Lie theory without speaking about vector fields, and in this way the theory becomes simpler and more transparent. In the following, G is a Lie group with multiplication $m : G \times G \to G$. Then TTG is a Lie group with multiplication TTm, and the fiber $(TTG)_e$ is a normal subgroup. We let $\mathfrak{g} := T_eG$; then the three "axes" $\varepsilon_1\mathfrak{g}$, $\varepsilon_2\mathfrak{g}$, $\varepsilon_1\varepsilon_2\mathfrak{g}$ of $(TTG)_e$ are canonical, and the fiber of the axes-bundle over the origin is $(A^2G)_e = \varepsilon_1\mathfrak{g} \oplus \varepsilon_2\mathfrak{g} \oplus \varepsilon_1\varepsilon_2\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$.

Theorem 23.2. (Canonical commutation rules in $(TTG)_e$.) *The group commutator $[g,h] = ghg^{-1}h^{-1}$ of an element $g = \varepsilon_1 v$ from the first and an element $h = \varepsilon_2 w$ from the second axis in TTG can be expressed by the Lie bracket $[v,w]$ in the following way:*

$$[\varepsilon_1 v, \varepsilon_2 w] = \varepsilon_1\varepsilon_2[v,w], \tag{23.1}$$

and the third axis $\varepsilon_1\varepsilon_2\mathfrak{g}$ is central in $(TTG)_e$, i.e. for $j = 1,2$ and all $u,v \in \mathfrak{g}$,

$$[\varepsilon_1\varepsilon_2 u, \varepsilon_j v] = 0. \tag{23.2}$$

Proof. First we prove (23.2). For any Lie group H, the tangent space T_eH is an abelian subgroup of TH whose group law is vector addition; this follows from Equation (5.1). Now, taking $H := TG$, $\varepsilon_1\varepsilon_2 u$ and $\varepsilon_j v$ both belong to the fiber over the origin in $T_{\varepsilon_j}TH$ and hence commute. Moreover, we see that

$$\varepsilon_i v \cdot \varepsilon_i w = \varepsilon_i(v+w) \quad (i=1,2), \qquad \varepsilon_1\varepsilon_2 v \cdot \varepsilon_1\varepsilon_2 w = \varepsilon_1\varepsilon_2(v+w). \tag{23.3}$$

Next, we prove the fundamental relation (23.1): by definition, the Lie bracket of $v,w \in \mathfrak{g}$ is given by evaluating $[v^R, w^R]$ et e, where v^R, w^R are the right-invariant vector fields extending v,w (cf. our convention concerning the sign of the Lie bracket from Section 5.3). Recall that v^R is given by left multiplication by v in TG, so the corresponding infinitesimal automorphism is left translation $l_v : TG \to TG$, $u \mapsto v \cdot u$. The two versions of its tangent map from Section 14.2 are then given by $(Tl_v)_1 = l_{\varepsilon_1 v}$ and $(Tl_v)_2 = l_{\varepsilon_2 v}$ with $\varepsilon_j v \in (TTG)_e$, $j = 1,2$, defined as above. Now it follows from Theorem 14.3 that

$$\begin{aligned}\varepsilon_1\varepsilon_2[v,w] &= \varepsilon_1\varepsilon_2[v^R, w^R](e)\\ &= [(Tl_v)_1, (Tl_w)_2](e) = [l_{\varepsilon_1 v}, l_{\varepsilon_2 w}](e)\\ &= (\varepsilon_1 v)(\varepsilon_2 w)(\varepsilon_1 v)^{-1}(\varepsilon_2 w)^{-1} = [\varepsilon_1 v, \varepsilon_2 w].\end{aligned}$$

■

We will often use (23.1) in the following form:

$$\varepsilon_1 v_1 \cdot \varepsilon_2 w_2 = \varepsilon_1\varepsilon_2[v_1, w_2] \cdot \varepsilon_2 w_2 \cdot \varepsilon_1 v_1. \tag{23.4}$$

Another way of stating (23.1) is by saying that the second order tangent map of the commutator map

$$c := c_G : G \times G \to G, \quad (g,h) \mapsto [g,h] = ghg^{-1}h^{-1}$$

is determined by

$$T^2c(\varepsilon_1 v, \varepsilon_2 w) = \varepsilon_1\varepsilon_2[v,w];$$

this follows simply by noting that $c_{TTG} = TTc_G$. One may also work with the conjugation map $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$; then the fundamental relation (23.1) can also be written in the following way

$$\mathrm{Ad}(\varepsilon_1 v)\varepsilon_2 w = \varepsilon_2 w \cdot \varepsilon_1\varepsilon_2[v,w] = (\mathrm{id} + \varepsilon_1 \,\mathrm{ad}(v))w;$$

which is equivalent to the the fact that the differential of $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ is $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$ (cf. [Ne02b, Prop. III.1.6] for the real locally convex case).

23.3. *Left- and right trivialization of TG and of TTG.* So far we have avoided to use the well-known fact that the tangent bundle TG of a Lie group G can be trivialized. Let $\mathfrak{g} := T_eG$ be imbedded in TG and imbed G in TG via the zero-section. Recall that (TG, Tm) is a Lie group. The map

$$\Psi_1^R : \mathfrak{g} \times G \to TG, \quad (v,g) \mapsto v \cdot g = Tm(v, 0_g) = T_e(r_g)v = l_v(0_g) \tag{23.5}$$

is a smooth bijection with smooth inverse $TG \to \mathfrak{g} \times G$, $u \mapsto (u\pi(u)^{-1}, \pi(u))$. The diffeomorphism $TG \cong \mathfrak{g} \times G$ thus obtained is called the *right trivialization of the tangent bundle.* Similarly, the left trivialization Ψ_1^L is defined. The kernel of the canonical projection $TG \to G$ is $\mathfrak{g} = T_eG$ on which G acts via the *adjoint representation*

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad (g,v) \mapsto \mathrm{Ad}(g)v = gvg^{-1}$$

(product taken in TG; then as usual $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the differential of the conjugation by g at the origin). From Formula (5.1) we see that the normal subgroup $\mathfrak{g}$ of TG is abelian with product being vector addition. It follows that the group structure of TG in the right trivialization is given by

$$(v,g) \cdot (w,h) = v \cdot g \cdot w \cdot h = v \cdot (gwg^{-1}) \cdot gh = (v + \mathrm{Ad}(g)w, gh), \tag{23.6}$$

i.e., TG is a semidirect product of G with the vector group $\mathfrak{g}$ via the representation Ad. Simlilarly, TTG is described as an iterated semidirect product: the right trivialization map of the group $TTG = T_{\varepsilon_2}T_{\varepsilon_1}G$ is obtained by replacing in the right trivialization map for TG, $\Psi_1^R : \mathfrak{g} \times G \to TG$, $(\varepsilon v, g) \mapsto \varepsilon v \cdot v$, the group G by $TG = T_{\varepsilon_1}G$ and letting $\varepsilon = \varepsilon_2$. We get a diffeomorphism

$$\begin{aligned} \Psi_2^R : \varepsilon_1\varepsilon_2\mathfrak{g} \times \varepsilon_2\mathfrak{g} \times \varepsilon_1\mathfrak{g} \times G \to TTG, \\ (\varepsilon_2\varepsilon_1 v_{12}, \varepsilon_2 v_2, \varepsilon_1 v_1, g) \mapsto \varepsilon_2\varepsilon_1 v_{12} \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 v_1 \cdot g, \end{aligned} \tag{23.7}$$

where the products are taken in the group TTG. Similarly, we get the left trivialization

$$\begin{aligned} \Psi_2^L : G \times \varepsilon_1\mathfrak{g} \times \varepsilon_2\mathfrak{g} \times \varepsilon_1\varepsilon_2\mathfrak{g} \to TTG, \\ (g, \varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_2\varepsilon_1 v_{12}) \mapsto g \cdot \varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_2\varepsilon_1 v_{12}. \end{aligned} \tag{23.8}$$

Clearly, both trivializations define linear structures on the fiber $(TTG)_e$ and hence on all other fibers. We will show below (Theorem 23.5) that both linear structures indeed define linear connections on TG.

Theorem 23.4. *With respect to the right trivialization Ψ_2^R, the group structure of the subgroup $(TTG)_e \cong \varepsilon_1\varepsilon_2\mathfrak{g} \times \varepsilon_2\mathfrak{g} \times \varepsilon_1\mathfrak{g}$ is given by the product*

$$\begin{aligned} &\Psi_2^R(\varepsilon_1\varepsilon_2 v_{12}, \varepsilon_2 v_2, \varepsilon_1 v_1) \cdot \Psi_2^R(\varepsilon_1\varepsilon_2 w_{12}, \varepsilon_2 w_2, \varepsilon_1 w_1) \\ &= \Psi_2^R(\varepsilon_1\varepsilon_2(v_{12} + w_{12} + [v_1, w_2]), \varepsilon_2(v_2 + w_2), \varepsilon_1(v_1 + w_1)) \end{aligned} \tag{23.9}$$

and inversion

$$(\Psi_2^R(\varepsilon_1\varepsilon_2 v_{12}, \varepsilon_2 v_2, \varepsilon_1 v_1))^{-1} = \Psi_2^R(\varepsilon_1\varepsilon_2([v_1, v_2] - v_{12}), -\varepsilon_2 v_2, -\varepsilon_1 v_1). \tag{23.10}$$

Proof. Using the Commutation Rules (23.1) and (23.2) we get

$$\begin{aligned} \Psi_2^R(\varepsilon_1\varepsilon_2 v_{12}, \varepsilon_2 v_2, \varepsilon_1 v_1) \cdot \Psi_2^R(\varepsilon_1\varepsilon_2 w_{12}, \varepsilon_2 w_2, \varepsilon_1 w_1) &= \varepsilon_1\varepsilon_2 v_{12} \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 v_1 \cdot \varepsilon_1\varepsilon_2 w_{12} \cdot \varepsilon_2 w_2 \cdot \varepsilon_1 w_1 \\ &= \varepsilon_1\varepsilon_2(v_{12} + w_{12}) \cdot \varepsilon_2 v_2 \cdot \varepsilon_1\varepsilon_2[v_1, w_2] \cdot \varepsilon_2 w_2 \cdot \varepsilon_1 v_1 \cdot \varepsilon_1 w_1 \\ &= \varepsilon_1\varepsilon_2(v_{12} + w_{12} + [v_1, w_2]) \cdot \varepsilon_2(v_2 + w_2) \cdot \varepsilon_1(v_1 + w_1)) \\ &= \Psi_2^R(\varepsilon_1\varepsilon_2(v_{12} + w_{12} + [v_1, w_2]), \varepsilon_2(v_2 + w_2), \varepsilon_1(v_1 + w_1)) \end{aligned}$$

and

$$\begin{aligned}(\Psi_2^R(\varepsilon_1\varepsilon_2 v_{12},\varepsilon_2 v_2,\varepsilon_1 v_1))^{-1} &= (\varepsilon_1\varepsilon_2 v_{12}\cdot\varepsilon_2 v_2\cdot\varepsilon_1 v_1)^{-1}\\ &= \varepsilon_1(-v_1)\cdot\varepsilon_2(-v_2)\cdot\varepsilon_1\varepsilon_2(-v_{12})\\ &= \varepsilon_2(-v_2)\cdot\varepsilon_1(-v_1)\cdot\varepsilon_1\varepsilon_2([-v_1,-v_2]-v_{12})\\ &= \varepsilon_1\varepsilon_2([v_1,v_2]-v_{12})\cdot\varepsilon_2(-v_2)\cdot\varepsilon_1(-v_1)\\ &= \Psi_2^R(\varepsilon_1\varepsilon_2([v_1,v_2]-v_{12}),\varepsilon_2(-v_2),\varepsilon_1(-v_1))\end{aligned}$$

■

The formula for group structure of the whole group TTG in the right trivialization is easily deduced from the theorem: considering Ψ_2^R as an identification, it reads

$$\begin{aligned}&(\varepsilon_1\varepsilon_2 v_{12},\varepsilon_2 v_2,\varepsilon_1 v_1,g)\cdot(\varepsilon_1\varepsilon_2 w_{12},\varepsilon_2 w_2,\varepsilon_1 w_1,h)\\ =&(\varepsilon_1\varepsilon_2(v_{12}+\mathrm{Ad}(g)w_{12}+[v_1,\mathrm{Ad}(g)w_2]),\varepsilon_2(v_2+\mathrm{Ad}(g)w_2),\varepsilon_1(v_1+\mathrm{Ad}(g)w_1),gh)\end{aligned} \tag{23.11}$$

In particular, the group G acts (by conjugation) *linearly* with respect to the linear structure on $(T^2G)_e$, and hence we can transport it by left- or right translations to all other fibers $(T^2g)_x$, $x\in G$. Clearly, this depends smoothly on x, and hence we have defined two linear structures on TTG which will be denoted by L^L and L^R, called the *left* and *right linear structures.*

Theorem 23.5. *The left and right linear structures L^L and L^R are connections on TG, i.e. they are bilinearly related to all chart structures. The difference L^L-L^R is the tensor field of type $(2,1)$ on G given by the Lie bracket, and this is also the torsion of L^L.*

Proof. Let us prove first that L^L and L^R are bilinearly related to all chart structures. Recall from Section 17.6 that, for any manifold M, a trivialization $TM\cong M\times V$ of the tangent bundle defines a connection on TM. More specifically, in the present case, deriving $\Psi_1^R:\mathfrak{g}\times G\to TG$,

$$T\Psi_1^R: T\mathfrak{g}\times TG\to TTG,\quad (X,h)\mapsto X\cdot h,$$

and composing again with Ψ_1^R, we get a trivialization of TTG:

$$T(\Psi_1^R)\circ(\mathrm{id}\times\Psi_1^R):\varepsilon_1\varepsilon_2\mathfrak{g}\times\varepsilon_2\mathfrak{g}\times\varepsilon_1\mathfrak{g}\times G = T\mathfrak{g}\times(\mathfrak{g}\times G)\to TTG, \tag{23.12}$$

and the linear structure L induced by this trivialization is in fact a connection (Lemma17.7). We claim that $L=L^L$. In fact, writing $\Psi_1 = Tm\circ(\iota\times z)$, where $\iota:\mathfrak{g}\to TG$ and $z:G\to TG$ are the natural inclusions, we get $T\Psi_1 = TTm\circ(T\iota\times Tz)$, which means with $T=T_{\varepsilon_2}$,

$$\begin{aligned}T(\Psi_1^R)(\varepsilon_1\varepsilon_2 v_{12},\varepsilon_2 v_2,\Psi_1^R(\varepsilon_1 v_1,g)) &= (\varepsilon_1\varepsilon_2 v_{12}\cdot\varepsilon_1 v_1)\cdot(\varepsilon_2 v_2\cdot g)\\ &= \varepsilon_1 v_1\cdot\varepsilon_2 v_2\cdot\varepsilon_1\varepsilon_2 v_{12}\cdot g.\end{aligned}$$

For $g=e$, this agrees with Ψ_2^L, and hence we have $L_e=(L^L)_e$. But both linear structures are invariant under left- and right translations and hence agree on all fibers. We have proved that L^L is a connection. In the same way we see that L^R is a connection.

We show that, on the fiber $(TTG)_e$, the difference between L^L and L^R is the Lie bracket. In fact, using (23.1), we can calculate the map relating both linear structures:

$$\begin{aligned}(\Psi_2^L)^{-1}\circ\Psi_2^R(\varepsilon_2\varepsilon_1 v_{12},\varepsilon_1 v_2,\varepsilon_1 v_1) &= (\Psi_2^L)^{-1}(\varepsilon_2\varepsilon_1 v_{12}\cdot\varepsilon_2 v_2\cdot\varepsilon_1 v_1)\\ &= (\Psi_2^L)^{-1}(\varepsilon_1 v_1\cdot\varepsilon_2 v_2\cdot\varepsilon_2\varepsilon_1(v_{12}+[v_2,v_1]))\\ &= (\varepsilon_1 v_1,\varepsilon_2 v_2,\varepsilon_1\varepsilon_2(v_{12}+[v_2,v_1])).\end{aligned} \tag{23.13}$$

Identifying $\varepsilon_1\mathfrak{g}\times\varepsilon_2\mathfrak{g}\times\varepsilon_1\varepsilon_2\mathfrak{g}$ and $\varepsilon_1\varepsilon_2\mathfrak{g}\times\varepsilon_2\mathfrak{g}\times\varepsilon_1\mathfrak{g}$ in the canonical way, this means that L^L and L^R are related via the map

$$f_b(u,v,w) = (u,v,w+[u,v]), \tag{23.14}$$

i.e. they are bilinearly related via the Lie bracket.

In order to prove that $L^R - L^L$ is the torsion tensor of L^R, we have to show that $L^R = \kappa \cdot L^L$, where $\kappa : TTG \to TTG$ is the canonical flip. But, since κ commutes with second tangent maps, κ is a group automorphism of (TTG, TTm), whence

$$\begin{aligned} \kappa \circ \Psi_2^L(\varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12}) &= \kappa(\varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1\varepsilon_2 v_{12}) \\ &= \kappa(\varepsilon_1 v_1) \cdot \kappa(\varepsilon_2 v_2) \cdot \kappa(\varepsilon_1\varepsilon_2 v_{12}) \\ &= \varepsilon_2 v_1 \cdot \varepsilon_1 v_2 \cdot \varepsilon_1\varepsilon_2 v_{12} \\ &= \Psi_2^R(\varepsilon_1 v_2, \varepsilon_2 v_1, \varepsilon_1\varepsilon_2 v_{12}) \\ &= \Psi_2^R(\kappa(\varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12})) \end{aligned} \tag{23.15}$$

and hence $\Psi_2^R = \kappa \circ \Psi_2^L \circ \kappa$. ■

23.6. *Remarks on the symmetric connection.* If $\frac{1}{2} \in \mathbb{K}$, then the connection $L^S := \frac{L^R + L^L}{2}$ is called the *symmetric connection of* G; it is torsionfree and invariant under left- and right-translations. Moreover, it is invariant under the inversion map $j : G \to G$ because $j.L^L = L^R$. In fact, this is the canonical connection of G seen as a symmetric space, cf. Chapter 26.

Theorem 23.7. (Structure of $(J^2G)_e$.) *For $F = L, R, S$ (the latter in case $\frac{1}{2} \in \mathbb{K}$), the bijections $\Psi_2^F : \varepsilon_1\mathfrak{g} \times \varepsilon_2\mathfrak{g} \times \varepsilon_1\varepsilon_2\mathfrak{g} \to (TTG)_e$ restrict to bijections*

$$J\Psi_2^F : \delta\mathfrak{g} \times \delta^{(2)}\mathfrak{g} \to (J^2G)_e.$$

These three maps coincide and are given by the explicit formula

$$\Psi_2(\delta^{(2)}v, \delta w) = \delta^{(2)}v \cdot \delta w = \varepsilon_1\varepsilon_2 v \cdot \varepsilon_1 w \cdot \varepsilon_2 w,$$

and the group structure of $(J^2G)_e$ is given by

$$\Psi_2(\delta^{(2)}v', \delta v) \cdot \Psi_2(\delta^{(2)}w', \delta w) = \Psi_2(\delta^{(2)}(v' + w' + [v,w]), \delta(v+w)).$$

Inversion in $(J^2G)_e$ is multiplication by the scalar -1.

Proof. We have remarked above (Equation (23.15)) that the canonical flip κ acts on $(TTG)_e$ by $\kappa(\varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1\varepsilon_2 v_{12}) = \varepsilon_1\varepsilon_2 v_{12} \cdot \varepsilon_2 v_1 \cdot \varepsilon_1 v_2$. Now all claims follow from Formulae (23.7) – (23.10) by observing that $[v,v] = 0$. ■

Finally, for the sake of completeness we add some results that are well-known to hold in the real (say, Banach) case ; these results will not be needed in the sequel.

Lemma 23.8. *The covariant derivatives associated to the three canonical connections of the Lie group G can be described as follows: fix $x \in G$; for $v \in T_xG$ denote by v^L, resp. v^R the unique left- (resp. right-) invariant vector field taking value v at x. Then, for all vector fields $X, Y \in \mathfrak{X}(G)$,*

$$(\nabla_X^L Y)(x) = [Y, (X(x))^R](x), \quad (\nabla_X^R Y)(x) = [Y, (X(x))^L](x),$$

$$(\nabla_X^S Y)(x) = \frac{1}{2}[Y, (X(x))^L + (X(x))^R](x).$$

Proof. Note that v^R is constant in the right trivialized picture, and hence $K^R \circ Tv^R = 0$ for the connector of L^R. It follows that

$$(\nabla_v^R Y)(x) = K^R \circ TY \circ v^R(x) = K^R \circ (TY \circ v^R - Tv^R \circ Y)(x) = K^R \circ [Y, v^R](x) = [Y, v^R](x).$$

Similarly for ∇^L. The last claim follows by taking the arithmetic mean. ■

Theorem 23.9. *The curvature tensor of the connections L^L and L^R vanhishes.*

Proof. Let R^R be the curvature tensor of the connection L^R. We use the "classical" characterization of R^R via $R^R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$ (Theorem 18.6). Taking values at the origin e and using Lemma 23.8, we see that the right hand side equals $(([\mathrm{ad}(X^R), \mathrm{ad}(Y^R)] - \mathrm{ad}[X^R, Y^R])Z^R)(e)$ which is zero since the space of right-invariant vector fields is a Lie algebra.∎

Since L^L and L^R have torsion, the preceding result should not be interpreted in the sense that L^L and L^R be flat.

23.10. *The Maurer-Cartan form.* We denote by ω the tensor field of type $(2,1)$ on G given at each point by the Lie bracket (torsion of L^R). Using the right trivialization of TG, it can also be seen as a $\mathrm{End}(\mathfrak{g})$-valued one-form

$$\omega : TG \cong G \times \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad (g,v) \mapsto \mathrm{ad}(v).$$

(Our definition of the Maurer-Cartan ω deviates slightly from other ones, see e.g. [Sh97, p. 108], in that our form is $\mathrm{End}(\mathfrak{g})$-valued, not $\mathfrak{g}$-valued; i.e. our form is obtained from the one defined there by composing with $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.)

Theorem 23.11. *The Maurer-Cartan form satisfies the structure equation*

$$\mathrm{d}\omega(X,Y) + [\omega_X, \omega_Y] = 0$$

for all $X, Y \in \mathfrak{X}(G)$ (where the bracket is a commutator in $\mathrm{End}(\mathfrak{g})$*). The curvature R^S of the connection L^S is given by taking pointwise triple Lie brackets:*

$$R^S(X,Y)Z = \frac{1}{2}\mathrm{d}\omega(X,Y)Z = \frac{1}{4}\omega(\omega(X,Y),Z) = \frac{1}{4}[[X,Y],Z].$$

Proof. We have seen that the curvatures R^L and R^R of the connections L^L, resp. L^R vanish. Thus we can apply the structure equation (18.9), with the difference term $A = \omega$, which leads to the first claim.

Next we apply the structure equation (18.10), but this time with $R = R^L = 0$ and $R' = R^S$, the symmetric curvature. The difference term is now $A = \frac{1}{2}\omega$. Thus $R^S = \frac{1}{2}\mathrm{d}\omega + \frac{1}{8}[\omega,\omega] = -\frac{1}{8}[\omega,\omega] = -\frac{1}{4}\mathrm{d}\omega$. ∎

24. The structure of higher order tangent groups

We retain the assumptioms from the preceding chapter: G is a Lie group with multiplication $m : G \times G \to G$. We are going to investigate the structure of the groups T^kG for $k \geq 3$.

24.1. *Fundamental commutation relations for elements of the axes.* Let $\varepsilon^\alpha v_\alpha$, $\varepsilon^\beta w_\beta \in (T^kG)_e$ be elements of axes in $(T_kG)_e$. Then the commutator of these two elements in the group $(T^kG)_e$ is given by the following fundamental commutation rules:

$$[\varepsilon^\alpha v_\alpha, \varepsilon^\beta w_\beta] = \begin{cases} \varepsilon^{\alpha+\beta}[v_\alpha, w_\beta] & \text{if } \alpha \perp \beta, \\ 0 & \text{else.} \end{cases} \tag{24.1}$$

In fact, if $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) \neq \emptyset$, then both elements commute: $[\varepsilon^\alpha v_\alpha, \varepsilon^\beta v_\beta] = 0$, because both belong to the tangent space $T_{\varepsilon_i}(T^{k-1}G)$ at the origin for some ε_i (namely for i such that $e_i \in \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$), and the group structure on the tangent space is just vector addition. This proves the second relation. If $\alpha \perp \beta$, then we apply (23.1) with ε_1 replaced by ε^α and ε_2 replaced by ε^β, and we get the first relation. In a unified way, (24.1) may be written

$$[\varepsilon^\alpha v_\alpha, \varepsilon^\beta w_\beta] = \varepsilon^\alpha \varepsilon^\beta [v_\alpha, w_\beta] \tag{24.2}$$

because $\varepsilon^\alpha \varepsilon^\beta = 0$ if $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) \neq \emptyset$. This can also be written

$$\varepsilon^\alpha v_\alpha \cdot \varepsilon^\beta w_\beta = \varepsilon^\alpha \varepsilon^\beta [v_\alpha, w_\beta] \cdot \varepsilon^\beta w_\beta \cdot \varepsilon^\alpha v_\alpha = \varepsilon^\beta w_\beta \cdot \varepsilon^\alpha v_\alpha \cdot \varepsilon^\alpha \varepsilon^\beta [v_\alpha, w_\beta]. \tag{24.3}$$

24.2. *Remark: $(T^kG)_e$ as a filtered group.* Let $G_1 := (T^kG)_e$ and $G_j := \langle \varepsilon^\alpha v_\alpha | \, |\alpha| > j, \, v_\alpha \in \mathfrak{g} \rangle$ be the "vertical subgroup" generated by elements from axes with total degree bigger than j. Then it follows from (24.1) that

$$\mathbf{1} = G_k \subset G_{k-1} \subset \ldots \subset G_1$$

is a *filtration* of G_1 in the sense of [Se65, LA 2.2] or [Bou72, II. 4.4]. Recall from [Se65, LA 2.3] or [Bou72] that the associated *graded group* carries a canonical structure of a Lie algebra. This Lie algebra structure is compatible with ours, but in our setting we can go further by constructing special bijections (coming from the three canonical connections) between the graded and the filtered objects.

24.3. *Right and left trivialization of T^kG.* Note that, according to our conventions, T^kG is viewed as obtained by successive scalar extensions via $\varepsilon_1, \ldots, \varepsilon_k$, i.e. $T^{k+1}G = T_{\varepsilon_{k+1}}(T^kG)$. The iterated right and left trivialization of T^kG are defined inductively: for $k = 1, 2$ they are given by (23.2) – (23.4); then the right trivialization of T^kG is given by appliying Formula (23.3) to the right trivialization of $H := T^{k-1}G$, with $TH = T_{\varepsilon_k}H$. For instance, the third order right trivialization is

$$\begin{aligned}\Psi_3^R :& (\varepsilon_1\varepsilon_2\varepsilon_3 v_{123}, \varepsilon_3\varepsilon_2 v_{23}, \varepsilon_3\varepsilon_1 v_{13}, \varepsilon_3 v_3, \varepsilon_1\varepsilon_2 v_{12}, \varepsilon_2 v_2, \varepsilon_1 v_1, g) \mapsto \\ & \varepsilon_1\varepsilon_2\varepsilon_3 v_{123} \cdot \varepsilon_3\varepsilon_2 v_{23} \cdot \varepsilon_3\varepsilon_1 v_{13} \cdot \varepsilon_3 v_3 \cdot \varepsilon_1\varepsilon_2 v_{12} \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 v_1 \cdot g\end{aligned}$$

(products taken in T^3G). The important feature is the order in which the product in T^3G has to be taken: here it is the "antilexographic" order (opposite order of the lexicographic order). An easy induction shows that the general formulae are

$$\begin{aligned}\Psi_k^R &: \bigoplus_{\alpha>0} \varepsilon^\alpha \mathfrak{g} \times G \to T^kG, \quad (\varepsilon^\alpha v_\alpha)_{\alpha>0}, g) \mapsto \prod_{\alpha>0}^{\downarrow} \varepsilon^\alpha v_\alpha \cdot g, \\ \Psi_k^L &: \bigoplus_{\alpha>0} \varepsilon^\alpha \mathfrak{g} \times G \to T^kG, \quad (\varepsilon^\alpha v_\alpha)_{\alpha>0}, g) \mapsto g \cdot \prod_{\alpha>0}^{\uparrow} \varepsilon^\alpha v_\alpha,\end{aligned} \tag{24.4}$$

where $\prod^{\uparrow}$ is the product of the elements labelled by $\alpha > 0$ in $(T^kG)_e$ taken with respect to the lexicographic order of the index set and $\prod^{\downarrow}$ is the product taken in $(T^kG)_e$ in the opposite ("antilexicographic") order.

24.4. *The case $k = 3$ and the Jacobi identity.* As an application of our formalism, let us now give an independent proof of the Jacobi identity (it is essentially equivalent to the arguments used in [Se65]): we calculate both sides of $\varepsilon_3 v_3 \cdot (\varepsilon_2 v_2 \cdot \varepsilon_1 v_1) = (\varepsilon_3 v_3 \cdot \varepsilon_2 v_2) \cdot \varepsilon_1 v_1$ in $(T^3G)_e$ and express the result in the left trivialization :

$$\begin{aligned}
&\varepsilon_3 v_3 \cdot (\varepsilon_2 v_2 \cdot \varepsilon_1 v_1) = \varepsilon_3 v_3 \cdot (\varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1\varepsilon_2 [v_2, v_1]) \\
&\quad = \varepsilon_1 v_1 \cdot \varepsilon_3 v_3 \cdot \varepsilon_1 \varepsilon_3 [v_3, v_1] \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 \varepsilon_2 [v_2, v_1] \\
&\quad = \varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_3 v_3 \cdot \varepsilon_1 \varepsilon_3 [v_3, v_1] \cdot \varepsilon_2 \varepsilon_3 [v_3, v_2] \cdot \varepsilon_1 \varepsilon_2 \varepsilon_3 [[v_3, v_1], v_2] \cdot \varepsilon_1 \varepsilon_2 [v_2, v_1] \\
&\quad = \varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 \varepsilon_2 [v_2, v_1] \cdot \varepsilon_3 v_3 \cdot \varepsilon_1 \varepsilon_3 [v_3, v_1] \cdot \varepsilon_2 \varepsilon_3 [v_3, v_2] \cdot \varepsilon_1 \varepsilon_2 \varepsilon_3 ([[v_3, v_1], v_2] + [v_3, [v_2, v_1]]) \\
&\quad = \Psi_3^L (\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1 \varepsilon_2 [v_2, v_1] + \varepsilon_3 v_3 \cdot \varepsilon_1 \varepsilon_3 [v_3, v_1] + \varepsilon_2 \varepsilon_3 [v_3, v_2] + \\
&\qquad\quad \varepsilon_1 \varepsilon_2 \varepsilon_3 ([[v_3, v_1], v_2] + [v_3, [v_2, v_1]]) \\
&(\varepsilon_3 v_3 \cdot \varepsilon_2 v_2) \cdot \varepsilon_1 v_1 = \varepsilon_2 v_2 \cdot \varepsilon_3 v_3 \cdot \varepsilon_2 \varepsilon_3 [v_3, v_2] \cdot \varepsilon_1 v_1 \\
&\quad = \varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1 \varepsilon_2 [v_2, v_1] \cdot \varepsilon_3 v_3 \cdot \varepsilon_1 \varepsilon_3 [v_3, v_1] \cdot \varepsilon_2 \varepsilon_3 [v_3, v_2] \cdot \varepsilon_1 \varepsilon_2 \varepsilon_3 [[v_3, v_2], v_1]. \\
&\quad = \Psi_3^L (\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1 \varepsilon_2 [v_2, v_1] + \varepsilon_3 v_3 + \varepsilon_1 \varepsilon_3 [v_3, v_1] + \varepsilon_2 \varepsilon_3 [v_3, v_2] + \varepsilon_1 \varepsilon_2 \varepsilon_3 [[v_3, v_2], v_1]).
\end{aligned}$$

Applying $(\Psi_3^L)^{-1}$ and comparing, we get

$$[[v_3, v_1], v_2] + [v_3, [v_2, v_1]] = [[v_3, v_2], v_1]$$

which is the Jacobi identity. Note that this proof does not make any use of the interpretation of the Lie bracket via left or right invariant vector fields.

24.5. *Action of the symmetric group.* The natural action of the symmetric group Σ_k by automorphisms of $(T^kG)_e$ is described by the following formula: for $\sigma \in \Sigma_k$,

$$\sigma \cdot \prod_{\alpha}^{\uparrow} \varepsilon^{\alpha} v_{\alpha} = \prod_{\alpha}^{\uparrow} \varepsilon^{\sigma.\alpha} v_{\alpha}. \tag{24.4}$$

As a consequence, the push-forward $\Psi^{L,\sigma} = \sigma \circ \Psi^L \circ \sigma^{-1}$ of Ψ^L by σ is given by

$$\Psi^{L,\sigma}((\varepsilon^{\alpha} v_{\alpha})_{\alpha > 0}, g) = g \cdot \prod_{\alpha > 0}^{\uparrow} \varepsilon^{\sigma.\alpha} v_{\sigma.\alpha}. \tag{24.5}$$

In fact, Formula (24.4) is proved by exactly the same arguments as in case $k = 2$ for $\sigma = \kappa$ (cf. Equation (23.15)). Re-ordering the terms and using the commutation rules (24.1), one can calculate the curvature forms. If G is commutative, then by a change of variables the left-hand side can be re-written $\prod^{\uparrow} \varepsilon^{\alpha} v_{\sigma^{-1}\alpha}$, and this can be interpreted by saying that Ψ^L is Σ_k-invariant. Clearly, then Ψ^L and $\Psi^{L,\sigma}$ coincide. However, if G is non-commutative, then we cannot re-write the formula in the way just mentioned since the order would be disturbed. In particular, for $k > 2$, Ψ_k^L and Ψ_k^R are no longer related via a permutation since reversing the lexicographic order on I_k is not induced by an element of Σ_k acting on I_k.

24.6. *Left and right group structures.* We have defined two canonical charts Ψ_k^F, $F = R, L$, of the group $(T^kG)_e$, and we want to describe the group structure in these charts. The model space is $E = \bigoplus_{\alpha > 0} \varepsilon^{\alpha} \mathfrak{g}$, which is just the tangent space $T_0((T^kG)_e)$ – being a linear space, the tangent space is canonically isomorphic to the direct sum of the axes. Via Ψ_k^F $(F = R, L)$ we transfer the group structure to E:

$$\begin{aligned}
*^F : E \times E \to E, \quad & \Psi_k^F v \cdot \Psi_k^F w = \Psi_k^F (v *^F w), \\
J^F : E \to E, \quad & (\Psi_k^F(v))^{-1} = \Psi_k^F (J^F v),
\end{aligned} \tag{24.6}$$

and call this the *right*, resp. *left group structure on* E. In Theorem 23.3 we have given explicit formulae for these maps in case $k=2$. For the general case we use the same strategy: just apply the Commutation Rules 24.1 along with $\varepsilon^\alpha v \cdot \varepsilon^\alpha w = \varepsilon^\alpha (v+w)$ in order to re-order the product. Before coming to the general result, we invite the reader to check, using the commutation rules, that for $k=3$ we have the following: with $v=\sum_{\alpha\in I_3, \alpha>0} \varepsilon^\alpha v_\alpha$, $w=\sum_{\alpha\in I_3, \alpha>0} \varepsilon^\alpha w_\alpha$,

$$\begin{aligned}
&\Psi_3^L(v)\cdot \Psi_3^L(w)\\
&\quad= \varepsilon^{001}v_{001}\cdot\varepsilon^{010}v_{010}\cdot\varepsilon^{011}v_{011}\cdot\varepsilon^{100}v_{100}\cdot\varepsilon^{101}v_{101}\cdot\varepsilon^{110}v_{110}\cdot\varepsilon^{111}v_{111}\cdot\\
&\qquad \varepsilon^{001}w_{001}\cdot\varepsilon^{010}w_{010}\cdot\varepsilon^{011}w_{011}\cdot\varepsilon^{100}w_{100}\cdot\varepsilon^{101}w_{101}\cdot\varepsilon^{110}w_{110}\cdot\varepsilon^{111}w_{111}\\
&\quad= \varepsilon^{001}(v_{001}+w_{001})\cdot\varepsilon^{010}(v_{010}+w_{010})\cdot\varepsilon^{011}(v_{011}+w_{011}+[v_{010},w_{001}])\cdot\\
&\qquad \varepsilon^{100}(v_{100}+w_{100})\cdot\varepsilon^{101}(v_{101}+w_{101}+[v_{100},w_{001}])\cdot\varepsilon^{110}(v_{110}+w_{110}+[v_{100},w_{010}])\cdot\\
&\qquad \varepsilon^{111}(v_{111}+w_{111}+[v_{110},w_{001}]+[v_{101},w_{010}]+[v_{100},w_{011}]+[[v_{100},w_{001}],w_{010}])\\
&\quad= \Psi_3^L(v+w+\varepsilon^{011}[v_{010},w_{001}]+\varepsilon^{101}[v_{100},w_{001}]+\varepsilon^{110}[v_{100},w_{010}]+\\
&\qquad \varepsilon^{111}([v_{110},w_{001}]+[v_{101},w_{010}]+[v_{100},w_{011}]+[[v_{100},w_{001}],w_{010}]))
\end{aligned}$$

and for the inversion, using the same kind of arguments,

$$\begin{aligned}
&(\Psi_3^L v)^{-1}\\
&= (\varepsilon^{001}v_{001}\cdot\varepsilon^{010}v_{010}\cdot\varepsilon^{011}v_{011}\cdot\varepsilon^{100}v_{100}\cdot\varepsilon^{101}v_{101}\cdot\varepsilon^{110}v_{110}\cdot\varepsilon^{111}v_{111})^{-1}\\
&= \varepsilon^{111}(-v_{111})\cdot\varepsilon^{110}(-v_{110})\cdot\varepsilon^{101}(-v_{101})\cdot\varepsilon^{100}(-v_{100})\cdot\varepsilon^{011}(-v_{011})\cdot\varepsilon^{010}(-v_{010})\cdot\varepsilon^{001}(-v_{001})\\
&= \varepsilon^{001}(-v_{001})\cdot\varepsilon^{111}(-v_{111}+[v_{110},v_{001}])\cdot\varepsilon^{110}(-v_{110})\cdot\varepsilon^{101}(-v_{101}+[v_{100},v_{001}])\cdot\\
&\quad \varepsilon^{100}(-v_{100})\cdot\varepsilon^{011}(-v_{011}+[v_{010},v_{001}])\cdot\varepsilon^{010}(-vv_{010})\\
&= \varepsilon^{001}(-v_{001})\cdot\varepsilon^{010}(-v_{010})\cdot\varepsilon^{011}(-v_{011}+[v_{010},v_{001}])\cdot\varepsilon^{100}(-v_{100})\cdot\\
&\quad \varepsilon^{101}(-v_{101}+[v_{100},v_{001}])\cdot\varepsilon^{110}(-v_{110}+[v_{100},v_{010}])\cdot\\
&\quad \varepsilon^{111}(-v_{111}+[v_{110},v_{001}]+[v_{101},v_{010}]+[v_{100},v_{011}]-[v_{100},[v_{010},v_{001}]])\\
&= \Psi_L(-v+\varepsilon^{011}[v_{010},v_{001}]+\varepsilon^{101}[v_{100},v_{001}]+\varepsilon^{110}[v_{100},v_{010}]+\\
&\quad \varepsilon^{111}([v_{110},v_{001}]+[v_{101},v_{010}]+[v_{100},v_{011}]-[v_{100},[v_{010},v_{001}]]))
\end{aligned}$$

Theorem 24.7. (Left and right product formulae for $(T^kG)_e$.) *With respect to the left trivialization Ψ_k^L of T^kG, we have*

$$\sum_\alpha \varepsilon^\alpha v_\alpha *^L \sum_\beta \varepsilon^\beta w_\beta = \sum_\gamma \varepsilon^\gamma z_\gamma$$

with

$$z_\gamma = v_\gamma + w_\gamma + \sum_{m=2}^{|\gamma|} \sum_{\lambda\in\mathcal{P}_m(\gamma)} [\ldots[[v_{\lambda^m},w_{\lambda^1}],w_{\lambda^2}],\ldots,w_{\lambda^{m-1}}].$$

The left inversion formula is

$$J^L(\sum_\alpha \varepsilon^\alpha v_\alpha) = \sum_\gamma \varepsilon^\gamma(-v_\gamma + \sum_{m=2}^{|\gamma|}(-1)^m \sum_{\lambda\in\mathcal{P}_m(\gamma)} [v_{\lambda^m},[v_{\lambda^{m-1}},\ldots[v_{\lambda^2},v_{\lambda^1}]]]),$$

and the right product is given by the formula

$$(v *^R w)_\gamma = v_\gamma + w_\gamma + \sum_{m=2}^{|\gamma|} \sum_{\lambda\in\mathcal{P}_m(\gamma)} [v_{\lambda^{m-1}},\ldots,[w_{\lambda^1},v_{\lambda^m}]].$$

In these formulae, partitions λ are always considered as ordered partitions: $\lambda^1 < \ldots < \lambda^m$.

Proof. According to (24.4), we have to show that

$$\prod_{\alpha>0}^{\uparrow} \varepsilon^\alpha v_\alpha \cdot \prod_{\beta>0}^{\uparrow} \varepsilon^\beta w_\beta = \prod_{\gamma>0}^{\uparrow} \varepsilon^\gamma z_\gamma$$

with z_γ as in the claim. Using the Commutation Rules 24.1, we will move all terms of the form $\varepsilon^\beta w_\beta$ to the left until we reach $\varepsilon^\beta v_\beta$; then both terms give rise to a new term $\varepsilon^\beta(v_\beta + w_\beta)$. But every time we exchange $\varepsilon^\beta w_\beta$ on its way to the left with an element $\varepsilon^\alpha v^\alpha$ with $\alpha > \beta$ and $\alpha \perp \beta$, applying the commutation rule we get a term $\varepsilon^{\alpha+\beta}[v_\alpha, w_\beta]$. Before doing the same thing with the next element from the second factor, we may re-order the first factor: since $\alpha > \beta$ and $\alpha \perp \beta$, the term $\varepsilon^{\alpha+\beta}[v_\alpha, w_\beta]$ commutes with all terms between position α and $\alpha+\beta$, and hence it can be pulled to position $\alpha+\beta$ without giving rise to new commutators.

We start the procedure just described with the first element $\varepsilon^{0\ldots01} w_{0\ldots01}$ of the second factor, then take the second element $\varepsilon^{0\ldots10} w_{0\ldots10}$, and so on up to the last element. Each time we get commutators of the type $\varepsilon^{\alpha+\beta}[v'_\alpha, w_\beta]$ with v'_α being the sum of v_α and all commutators produced in the preceding steps. This gives us precisely all iterated commutators of the form

$$[\ldots[v_{\mu^1}, w_{\mu^2}]\ldots, w_{\mu^l}]$$

with $\mu_i \perp \mu_j$ ($\forall i \neq j$) and the order conditions

$$\mu^2 < \ldots < \mu^l, \qquad \mu^1 > \mu^2, \quad \mu^1 + \mu^2 > \mu^3, \ldots, \mu^1 + \ldots + \mu^{l-1} > \mu^l.$$

These conditions imply $\mu_1 > \mu_j$ for all $j = 1, \ldots, l-1$ because the supports of the μ_i are disjoint and hence the biggest of the μ_j is bigger than the sum of all the others. Hence $(\lambda^1, \ldots, \lambda^{l-1}, \lambda^l) := (\mu^2, \ldots, \mu^l, \mu^1)$ is a partition, and every partition is obtained in this way. Thus, by a change of variables $\lambda \leftrightarrow \mu$, we get the formula for the left product. Similarly for the right product.

The inversion formula is proved by the same kind of arguments: we write

$$(\Psi_k^L(\sum_\alpha \varepsilon^\alpha v_\alpha))^{-1} = (\prod^{\uparrow} \varepsilon^\alpha v_\alpha)^{-1} = \prod^{\downarrow} \varepsilon^\alpha(-v_\alpha)$$

and start to re-order this product from the right. ■

Comparing our formulae with the "usual" Campbell-Hausdorff formula (say, for a nilpotent Lie group), one remarks that, whereas in the usual formula, inversion is just multiplication by -1, this is not the case here. More generally, the powers $X^n = X * \ldots * X$ in the usual formula correspond to multiples nX, whereas here we have, e.g., the following *(left) squaring formula*:

$$(\Psi_k^L(v))^2 = \Psi_k^L(2v + \sum_{\substack{\gamma \\ |\gamma|\geq 2}} \varepsilon^\gamma(\sum_{\substack{\lambda\in\mathcal{P}(\gamma)\\ l(\gamma)\geq 2}} [\ldots[v_{\lambda^l}, v_{\lambda^1}]\ldots, v_{\lambda^{l-1}}])).$$

Moreover, our formulae for J^L and $*^F$ are multilinear (resp. bi-multilinear) in the sense of Section MA.5. In the following we will give a conceptual explanation of this fact.

Theorem 24.8. *Denote by $\kappa \in \Sigma_k$ the permutation that reverses the order, i.e. $\kappa(i) = k+1-i$, $i = 1, \ldots, k$. (For $k = 2$, this is the canonical flip, and for $k = 3$, it is the transposition (13).) Then left- and right trivialization Ψ_k^L and Ψ_k^R are multilinear connections on $T^k G$, and, up to the permutation κ, they coincide with the derived connections $D^{k-1}L^L$, resp. $D^{k-1}L^R$ defined in* Chapter 17. *Put differently, the Dombrowski Splitting Maps Φ_k^F associated to the multilinear connections $D^{k-1}L^F$, $F = L, R$, are related with the trivialization maps via*

$$\Phi_k^F = \Psi_k^{F,\kappa}$$

where $\Psi_k^{F,\kappa}$ *is as in* Equation (24.5).

Proof. We proceed as in the proof of Theorem 23.5 where the case $k=2$ of the claim has been proved. Starting with the left-trivialization $\Psi_1^L : G \times \mathfrak{g} \to TG$, we have two possibilities to iterate the procedure:

(a) we replace G by $H := TG$ and write the left-trivialization for TG: $TG \times T\mathfrak{g} \to TTG$, compose again with Ψ_1 and obtain $\Psi_2 : G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to TTG$.

(b) we take the tangent map $T\Psi_1^L : TG \times T\mathfrak{g} \to TTG$, compose again with Ψ_1 and obtain another map $\widetilde{\Psi}_2 : G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to TTG$.

Both procedures yield essentially the same result. However, the precise relation depends on the labelling of the copies of TG we use: as to convention (a) we agreed in Section 23.3 to write $H = T_{\varepsilon_1}G$ so that $TTG = TH = T_{\varepsilon_2}T_{\varepsilon_1}G$. Thus, in (b), we must take $T = T_{\varepsilon_2}$, whence the copy TG appearing in the formula really is $T_{\varepsilon_2}G$, but up to exchanging the order of ε_1 and ε_2 both constructions are the same. Now, when iterating the construction, in the k-th step, we must take the *last* tangent functor in (b) with respect to ε_1, i.e. we must use the order of scalar extensions $\varepsilon_k, \ldots, \varepsilon_1$ instead of the usual order $\varepsilon_1, \ldots, \varepsilon_k$, and then constructions (a) and (b) coincide. According to Lemma 17.7, construction (b) corresponds to the construction of the derived linear structure $D^{k-1}L^L$, where due to the change of order the permutation κ appears in the final result. ■

Corollary 24.9. *Left- and right products* $*^L$ *and* $*^R$ *can be interpreted as higher covariant differentials in the sense of* Section 16.3*:* $*^F$ *is the higher differential of the group multiplication* $m : G \times G \to G$ *with respect to the linear structures derived from* $L^F \times L^F$ *and* L^F*,* $F = L, R$*, and inversion* J^F *is the higher covariant differential of group inversion.*

Proof. Since left, resp. right trivialization are the linearizations associated to the connection L, the claim follows directly from the definition of the higher order covariant differentials in Section 16.3. ■

The corollary gives an *a priori* proof of the fact that the left and right product formulae are bi-multilinear (i.e. multilinear in the sense of MA.5 in both arguments) and that J^F is multilinear.

Theorem 24.10. (Restriction to jet groups.) *Left and right trivialization of* T^kG *induce, by restriction, two trivializations of the group* J^kG*; in other words, the restrictions*

$$\Psi_k^F|_{E^{\Sigma_k}} : E^{\Sigma_k} = \oplus_{j=1}^k \delta^{(j)}\mathfrak{g} \to (J^kG)_e,$$

$F = L, R$*, are well-defined.*

Proof. For $k=2$, an elementary proof has been given in the preceding chapter (Theorem 23.7). Let us give, for $k=3$, an elementary proof in the same spirit: we get from the explicit form of the left product formula stated before Theorem 24.5, by letting $v_1 := v_{001} = v_{010} = v_{100}$, $v_2 := v_{011} = v_{101} = v_{110}$, $v_3 := v_{111}$, $v = \delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3$ (and similarly for w):

$$\begin{aligned}
\Psi_3^L(v) \cdot \Psi_3^L(w) &= \varepsilon^{001}(v_1 + w_1) \cdot \varepsilon^{010}(v_1 + w_1) \cdot \varepsilon^{011}(v_2 + w_2 + [v_1, w_1]) \cdot \\
&\qquad \varepsilon^{100}(v_1 + w_1) \cdot \varepsilon^{101}(v_2 + w_2 + [v_1, w_1]) \cdot \varepsilon^{110}(v_2 + w_2 + [v_1, w_1]) \cdot \\
&\qquad \varepsilon^{111}(v_3 + w_3 + [v_2, w_1] + [v_2, w_1] + [v_1, w_2] + [[v_1, w_1], w_1]) \\
&= \Psi_3^L(v + w + (\varepsilon^{011} + \varepsilon^{101} + \varepsilon^{110})[v_1, w_1] + \varepsilon^{111}(2[v_2, w_1] + [v_1, w_2] + [[v_1, w_1], w_1])) \\
&= \Psi_3^L(v + w + \delta^{(2)}[v_1, w_1] + \delta^{(3)}(2[v_2, w_1] + [v_1, w_2] + [[v_1, w_1], w_1]))
\end{aligned}$$

which shows that the group structure of $(J^3G)_e$ in the left trivialization is given by

$$\begin{aligned}
(\delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3) *^L (\delta w_1 + \delta^{(2)} w_2 + \delta^{(3)} w_3) &= \delta(v_1 + w_1) + \delta^{(2)}(v_2 + w_2 + [v_1, w_1]) + \\
&\qquad \delta^{(3)}(w_3 + v_3 + [v_1, w_2] + 2[v_2, w_1] + [v_1, [w_1, w_1]]).
\end{aligned}$$

Similarly, we have for the inversion:

$$(\Psi_3^L(\delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3))^{-1} = \Psi_3^L(-\delta v_1 - \delta^{(2)} v_2 + \delta^{(3)}(-v_3 + [v_2, v_1])) .$$

For general k, the combinatorial structure of the corresponding calculation becomes rather involved. For this reason, we prefer to invoke Theorem 19.2 which in a sense contains the relevant combinatorial arguments: by Theorem 24.8, Ψ_k^L and Ψ_k^R are the Dombrowski splitting maps associated to the connection $(D^{k-1}L^F)^\kappa$ $(F = L, R)$. According to Theorem 19.2, such connections are weakly symmetric, i.e. they preserve the Σ_k-fixed subbundles. ■

24.11. *The left and right product formulae for jets.* According to the preceding result, the product formulae from Theorem 24.5 can be restricted to spaces of Σ_k-invariants: there is a relation of the kind

$$\sum_{i=1}^k \delta^{(i)} v_i *^L \sum_{j=1}^k \delta^{(j)} w_j = \sum_{r=1}^k \delta^{(r)} z_r$$

with $z_r \in \mathfrak{g}$ depending on v and w. For $k = 3$ we have seen in the preceding proof that

$$\begin{aligned} z_1 &= v_1 + w_1, \\ z_2 &= v_2 + w_2 + [v_1, w_1], \\ z_3 &= v_3 + w_3 + 2[v_2, w_1] + [v_1, w_2] + [[v_1, w_1], w_1]. \end{aligned}$$

We will not dwell here on the combinatorial details of the formula in the general case, but content ourself with the case of multiplying first order jets in $J^k G$: we claim that for $v, w \in \mathfrak{g}$,

$$\delta v *^L \delta w = \delta(v + w) + \delta^{(2)}[v, w] + \ldots + \delta^{(k)}[[v, w], \ldots w]. \tag{24.8}$$

In fact, $\delta v = \sum_{j=1}^k \varepsilon_j v$, and with the notation of Theorem 24.5, we have $v_\alpha = 0 = w_\beta$ if $|\alpha|, |\beta| > 1$. Thus only partitions λ with $|\lambda^i| \leq 1$ contribute to z_γ, i.e. we have

$$z_\gamma = v_\gamma + w_\gamma + \sum_{l=2}^k \sum_{\lambda \in \mathcal{P}_l(\gamma)} [[v, w], \ldots w].$$

For $|\gamma| = 1$, there are no partitions of length at least 2, whence $z_\gamma = v_\gamma + w_\gamma$. This gives rise to the terms $\delta v + \delta w$. For $|\gamma| = j > 1$, just one partition contributes to z_γ, giving rise to a term $\delta^{(j)}[[v, w], \ldots w]$. This proves (24.8). – Note that, since $[v, v] = 0$,

$$\Psi_k^L(\delta v) = \varepsilon_1 v \cdot \ldots \cdot \varepsilon_k v = \varepsilon_k v \cdot \ldots \cdot \varepsilon_1 v = \Psi_k^R(\delta v),$$

i.e. on $\delta\mathfrak{g}$, left and right trivialization coincide; for elements of the form $\delta^{(j)} v$ with $j > 1$ this will no longer be the case.

– It would be interesting to investigate further the combinatorial structures involved in the product and inversion formulae. In the case of arbitrary characteristic, these formulae are in a way the best results one can get. In characteristic zero, one can use a "better" trivialization of $(T^k G)_e$ given by the *exponential map.*

25. Exponential map and Campbell-Hausdorff formula for jet groups

25.1. *$(T^kG)_e$ as a polynomial group.* Recall from Chapter PG, Example PG.2 (2), that $(T^kG)_e$ satisfies the properties of a *polynomial group*: there is chart (given by left or right trivialization) such that, in this chart, group multiplication is polynomial, and the iterated multiplication maps

$$m^{(j)} : (T^kG)_e \times \ldots \times (T^kG)_e \to (T^kG)_e, \quad (v_1, \ldots, v_j) \mapsto v_1 \cdots v_j$$

are polynomial maps of degree bounded by k.

Theorem 25.2. *Assume the integers are invertible in $\mathbb{K}$. Then there exists a unique map* $\exp : (T^k\mathfrak{g})_0 \to (T^kG)_e$ *such that:*

(1) *the representation of* $\exp$ *with respect to left trivialization,* $\exp_L := (\Psi_k^L)^{-1} \circ \exp : (T^k\mathfrak{g})_0 \to (T^k\mathfrak{g})_0$, *is a polynomial map,*
(2) $T_0 \exp = \mathrm{id}_{(T^k\mathfrak{g})_0}$,
(3) *for all* $n \in \mathbb{Z}$ *and* $v \in (T^k\mathfrak{g})_0$, $\exp(nv) = (\exp(v))^n$.

The map $\exp$ *is bijective and has a polynomial inverse* $\log$. *Explicitly, the polynomials* $\exp_L$ *and* $\log_L := \log \circ \Psi_k^L$ *are given by*

$$(\exp_L(v))_\gamma = v_\gamma + \sum_{j=2}^{k} \frac{1}{j!} \sum_{\lambda \in \mathcal{P}_j(\gamma)} [v_{\lambda^j}, [v_{\lambda^{j-1}}, \ldots [v_{\lambda^2}, v_{\lambda^1}]]]$$

$$(\log_L(v))_\gamma = v_\gamma + \sum_{j=2}^{k} \frac{(-1)^{j-1}}{j} \sum_{\lambda \in \mathcal{P}_j(\gamma)} [v_{\lambda^j}, [v_{\lambda^{j-1}}, \ldots [v_{\lambda^2}, v_{\lambda^1}]]].$$

Proof. Existence and uniqueness of log and exp follow from Theorem PG.6. The explicit formulae are obtained by identifying the terms $\psi_p(x)$ and $\psi_{p,p}(x)$ appearing in the explicit formulae from Theorem PG.6 : by definition, $\psi_1(v) = v$, and comparing the Left Inversion Formula from Theorem 24.7 with Formula (PG.8) for the inversion, $v^{-1} = \sum_j (-1)^j \psi_j(v)$, we get by induction, using that $\psi_{p,m} = 0$ for $m < p$ (i.e., ψ_p vanishes at 0 of order at least p),

$$(\psi_j(v))_\gamma = (\psi_{j,j}(v))_\gamma = \sum_{\lambda \in \mathcal{P}_j(\gamma)} [v_{\lambda^j}, [v_{\lambda^{j-1}}, \ldots [v_{\lambda^2}, v_{\lambda^1}]]].$$

Inserting this in the formulae from Theorem PG.6, we get the claim. ∎

25.3. It is clear that exp can also be expressed by a polynomial with respect to the right trivialization. It would be interesting to have also an "absolute formula" for exp, i.e., a formula that does not refer to any trivialization. For $k = 2$, such a formula is given by

$$\begin{aligned} \exp(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) &= \varepsilon_1 v_1 \cdot \varepsilon_2 v_2 \cdot \varepsilon_1\varepsilon_2 (v_{12} + \frac{1}{2}[v_2, v_1]) \\ &= \varepsilon_2 \frac{v_2}{2} \cdot \varepsilon_1 v_1 \cdot \varepsilon_1\varepsilon_2 v_{12} \cdot \varepsilon_2 \frac{v_2}{2} \\ &= \varepsilon_2 \frac{v_2}{2} \cdot \varepsilon_1 \frac{v_1}{2} \cdot \varepsilon_1\varepsilon_2 v_{12} \cdot \varepsilon_1 \frac{v_1}{2} \cdot \varepsilon_2 \frac{v_2}{2}, \end{aligned}$$

where the last expression appears to be fairly symmetric (it is a special case of the formula for the canonical connection of a symmetric space, see Chapter 26). It is not at all obvious how to generalize this formula to the case of general k.

Theorem 25.4. *The exponential map commutes with Lie group automorphisms $\varphi : (T^kG)_e \to (T^kG)_e$ in the sense that $\varphi \circ \exp = \exp \circ T_0\varphi$, and it can be extended in G-invariant way to a trivialization of T^kG, again denoted by $\exp : G \times (\oplus_\alpha \varepsilon^\alpha \mathfrak{g}) \to T^kG$. This trivialization is a totally symmetric multilinear connection on T^kG.*

Proof. One easily checks that both exp and $\varphi \circ \exp \circ (T_0\varphi)^{-1}$ satisfy Properties (1), (2), (3) of an exponential map and hence agree by the uniqueness statement of Theorem 25.2. Applying this to the inner automorphisms $(T^k c_g)_e$ (where $c_g : G \to G$ is conjugation by $g \in G$), we see that exp can be transported in a well-defined way to any fiber of T^kG over G, thus defining the trivialization map of T^kG. On the fiber over e, it is multilinearly related to all chart structures (by the explicit formula from Theorem 25.2, which clearly is multilinear), and hence the same is true in any other fiber. Thus exp is a multilinear connection. This connection is totally symmetric because the symmetric group Σ_k acts by automorphisms of T^kG, and we have just seen that the exponential map is invariant under automorphisms. ■

Theorem 25.5. *With respect to the exponential connection from the preceding theorem, the group structure of $(T^kG)_e$ and the one of $(J^kG)_e$ is given by the Campbell-Hausdorff multiplication with respect to the nilpotent Lie algebras $(T^k\mathfrak{g})_0$, resp. $(J^k\mathfrak{g})_0$.*

Proof. The fact that the exponential map restricts to spaces of Σ_k-invariants follows from the preceding theorem, and Theorem PG.8 implies that the group structure with respect to the exponential map is given by the Campbell-Hausdorff formula. ■

Recall that the classical Campbell-Hausdorff formula for Lie groups is given by

$$X * Y := \log(\exp(X) \cdot \exp(Y)) = X + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots q_k + 1)} \cdot \frac{(\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \cdots (\operatorname{ad} X)^{p_k} (\operatorname{ad} Y)^{q_k} (\operatorname{ad} X)^m}{p_1! q_1! \cdots p_k! q_k! m!} Y,$$

and the first four terms are:

$$X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \frac{1}{24}[X,[Y,[X,Y]]] + \ldots$$

(cf. [T04] for the last term). In particular, we may apply this to the nilpotent Lie algebra

$$(J^k\mathfrak{g})_0 = \delta\mathfrak{g} \oplus \delta^{(2)}\mathfrak{g} \oplus \ldots \oplus \delta^{(k)}\mathfrak{g} \subset \mathfrak{g} \otimes_{\mathbb{K}} J^k\mathbb{K}$$

where $\mathfrak{g}$ is an arbitrary $\mathbb{K}$-Lie algebra. Elements are multiplied by the rule

$$\delta^{(i)}\delta^{(j)} = \binom{i+j}{i}, \qquad [\delta^{(i)}X, \delta^{(j)}Y] = \binom{i+j}{i} \delta^{(i+j)}[X,Y].$$

We re-obtain for $k = 2$ the result from Theorem 23.7:

$$(\delta v_1 + \delta^{(2)} v_2) * (\delta w_1 + \delta^{(2)} w_2) = \delta(v_1 + w_1) + \delta^{(2)}(v_2 + w_2 + [v_1, w_1])),$$

and for $k = 3$ we have:

$$(\delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3) * (\delta w_1 + \delta^{(2)} w_2 + \delta^{(3)} w_3)) = \delta(v_1 + w_1) + \delta^{(2)}(v_2 + w_2 + [v_1, w_1]) + \delta^{(3)}(w_3 + v_3 + \frac{3}{2}[v_1, w_2] + \frac{3}{2}[v_2, w_1] + \frac{1}{2}([v_1,[v_1,w_1]] + [w_1,[w_1,v_1]])).$$

Note that this formula makes sense whenever 2 is invertible in $\mathbb{K}$. Compare also with the formula from the proof of Theorem 24.10 which is valid in arbitrary characteristic.

25.6. *Example: Associative continuous inverse algebras.* An associative algebra $(V, \cdot)$ over $\mathbb{K}$ is called a *continuous inverse algebra (CIA)* if multiplication $V \times V \to V$ continuous, if the set $V^\times$

of invertible elements is open in V and inversion $i : V^\times \to V$ is continuous. We assume that V has a unit element $\mathbf{1}$. It is easily proved that multiplication and inversion are actually smooth, and hence $G = V^\times$ is a Lie group. We consider V as a global chart of G and identify $(T^kG)_e$ with $\mathbf{1} + (T^kV)_0$ where $(T^kV)_0 = \oplus_{\alpha>0}\varepsilon^\alpha V$. The group (T^kG) is just the group of invertible elements in the associative algebra $T^kV = V \otimes_{\mathbb{K}} T^k\mathbb{K}$. The inverse of $\mathbf{1} + \varepsilon v$ is $\mathbf{1} - \varepsilon v$, and hence we get the group commutator

$$[\mathbf{1} + \varepsilon_1 v, \mathbf{1} + \varepsilon_2 w] = (\mathbf{1} + \varepsilon_1 v)(\mathbf{1} + \varepsilon_2 w)(\mathbf{1} - \varepsilon_1 v)(\mathbf{1} - \varepsilon_2 v) = \varepsilon_1\varepsilon_2(vw - wv),$$

which implies by the commutation relations (23.1) that the Lie algebra of G is V with the usual commutator bracket.

Proposition 25.7. *Let $G = V^\times$ be the group of invertible elements in a continuous inverse algebra. Then the group $(T^kG)_e = (T^kV)_0^\times$ is a polynomial group with respect to the natural chart given by the ambient $\mathbb{K}$-module $(T^kV)_0$, and the exponential map with respect to this polynomial group structure is given by the usual exponential formula*

$$\exp(\sum_{\alpha>0} \varepsilon^\alpha v_\alpha) = \sum_{j=0}^{k} \frac{1}{j!}(\sum_{\alpha>0} \varepsilon^\alpha v_\alpha)^j.$$

Proof. The group $(T^kG)_e$ is polynomial in the natural chart since the algebra $(T^kV)_0$ is nilpotent of order k, and hence the degree of the iterated product maps $m^{(j)}$ is bounded by k. Clearly, there is just one homogeneous component of $m^{(j)}$ and we have $\psi_j(x) = \psi_{j,j}(x) = x^j$. Inserting this in the formulae from Theorem PG6, we get the usual formulae for the exponential map and the logarithm. ∎

25.8. *Comparing the exponential map with right trivialization.* Developing all terms in the formula for exp from the proposition, we get

$$\begin{aligned}
\exp(\sum_{\alpha>0} \varepsilon^\alpha v_\alpha) &= \sum_{\alpha} \varepsilon^\alpha \sum_{l} \frac{1}{l!} \sum_{\Lambda\in\mathcal{P}_l(\alpha)} \sum_{\sigma\in\Sigma_l} v_{\sigma.\Lambda} \\
&= \sum_{l} \frac{1}{l!} \sum_{\Lambda\in\mathrm{Part}_l(I_k)} \varepsilon^{\underline{\Lambda}} \sum_{\sigma\in\Sigma_l} v_{\sigma.\Lambda} \\
&= \mathbf{1} + \sum_{\alpha} \varepsilon^\alpha v_\alpha + \frac{1}{2} \sum_{\substack{\Lambda: \\ l(\Lambda)=2}} \varepsilon^{\underline{\Lambda}}(v_{\Lambda^1}v_{\Lambda^2} + v_{\Lambda^2}v_{\Lambda^1}) + \frac{1}{6} \sum_{\substack{\Lambda: \\ l(\Lambda)=3}} \varepsilon^{\underline{\Lambda}}(v_{\Lambda^1}v_{\Lambda^2}v_{\Lambda^3} + \ldots) + \ldots
\end{aligned} \tag{25.3}$$

On the other hand, the trivialization map Ψ_k^R is in the chart T^kV described by

$$\begin{aligned}
\Psi_k^R(\sum_{\alpha>0} \varepsilon^\alpha v_\alpha) &= \prod_{\alpha}^{\downarrow}(\mathbf{1} + \varepsilon^\alpha v_\alpha) = \mathbf{1} + \sum_{\alpha} \varepsilon^\alpha \sum_{\lambda\in P(\alpha)} v_{\lambda^1} \cdot \ldots \cdot v_{\lambda^l} \\
&= e + \sum \varepsilon^\alpha v_\alpha + \sum_{l(\Lambda)=2} \varepsilon^{\underline{\Lambda}} v_{\Lambda^1}v_{\Lambda^2} + \sum_{l(\Lambda)=3} \varepsilon^{\underline{\Lambda}} v_{\Lambda^1}v_{\Lambda^2}v_{\Lambda^3} + \ldots,
\end{aligned} \tag{25.4}$$

where partitions are considered as ordered partitions. Comparing (25.3) and (25.4), we see that Ψ_k^R is defined by making one particular choice among the $l!$ terms in the sum in exp and forgetting division by $l!$, and in turn exp is obtained by symmetrizing each term of Ψ_k^R. Here, "symmetrization" is to be understood with respect to the natural linear structure of T^kV. In the case of a general Lie group, it seems much more difficult to understand the passage from the right (or left) trivialization to the exponential connection in terms of a suitable symmetrization procedure.

25.9. *Relation with an exponential map of* G. We say that a Lie group G *has an exponential map* if the following holds: for every $v \in \mathfrak{g}$ there exists a unique Lie group homomorphism $\gamma_v : \mathbb{K} \to G$ such that $\gamma_v'(0) = v$ and such that the map $\exp_G$ defined by

$$\exp_G : \mathfrak{g} \to G, \quad v \mapsto \gamma_v(1)$$

is smooth. In general, a Lie group will not have an exponential map; but if it does, then the higher differentials of exp are given by the exponential map $\exp =: \exp^{(k)}$ from Theorem 25.2, i.e.

$$(T^k \exp_G)_o = \exp^{(k)} : (T^k \mathfrak{g})_0 \to (T^k G)_e. \tag{25.5}$$

In fact, it follows immediately from the definitions that $(T^k \exp_G)_0$ has the properties (1), (2), (3) from Theorem 25.2 and hence agrees with $\exp^{(k)}$. It is instructive to check (25.5) directly in the case of a CIA, using the Derivation Formula (7.19): let $p^j(x) = x^j$ be the j-th power in V. Then $d^i p_j(0) = 0$ if $i \neq j$ and

$$d^j p_j(0)(v_1, \ldots, v_j) = \sum_{\sigma \in \Sigma_j} v_{\sigma(1)} \cdots v_{\sigma(j)}$$

is the total polarization of p_j. Thus $T^k \exp(0)(v) = \sum_\alpha \varepsilon^\alpha \sum_{l=1}^{|\alpha|} \sum_{\Lambda \in \mathcal{P}_l(\alpha)} d^l \frac{p_l}{l!}(v_\Lambda)$ agrees with (25.3).

26. The canonical connection of a symmetric space

26.1. *Abelian symmetric spaces.* We return to the study of symmetric spaces and start by some general (in fact, purely algebraic) remarks on abelian symmetric spaces(Chapter 5). Recall that a symmetric space is called *abelian* if the group $G(M)$ generated by all $\sigma_x \circ \sigma_y$, $x, y \in M$, is abelian. In particular, if $o \in M$ is a base point, the operators $Q(x) = \sigma_x \circ \sigma_o$, $x \in M$, commute among each other. For instance, if $(V, \cdot)$ is a commutative group, then

$$\mu(u,v) = uv^{-1}u = u^2 v^{-1} \tag{26.1}$$

defines on V the structure of an abelian symmetric space, and $Q(v)x = v^2 x$ is translation by v^2. We say that a group M (resp. a symmetric space M with base point o) *admits unique square roots* if, for all $v \in M$ there is a unique element $\sqrt{v} \in M$ such that $(\sqrt{v})^2 = v$ (resp. $\mu(\sqrt{v}, o) = v$, which is the same as $Q(\sqrt{v})o = v$).

Proposition 26.2. *There is a bijection between*

(1) *abelian groups $(V, \cdot)$ admitting unique square roots, and*

(2) *abelian reflection spaces with base point (V, μ, o) admitting unique square roots.*

The bijection is given by associating to $(V, \cdot)$ the symmetric space structure (26.1), *and by associating to (V, μ, o) the group structure given by*

$$u \cdot v = \mu(\sqrt{u}, \mu(e,v)) = Q(\sqrt{u})v = Q(\sqrt{u})Q(\sqrt{v})o. \tag{26.2}$$

Proof. In this proof and on the following pages, the "fundamental formula" (5.6) will be frequently used without further comments. – We have already seen that (26.1) is an (abelian) reflection space structure ; the square roots are unique since the squaring operation is the same as the one for the group. Let us prove that (26.2) defines an abelian group law on V : we have $u \cdot v = v \cdot u$ since the operators $Q(\sqrt{u})$ and $Q(\sqrt{v})$ commute by our assumption. Clearly, $v \cdot o = o \cdot v = v$. In order to establish associativity, note first that

$$\sqrt{Q(x)y} = Q(\sqrt{x})\sqrt{y}. \tag{26.3}$$

In fact, we have $Q(\sqrt{x})^2 = Q(Q(\sqrt{x})o) = Q(x)$ and hence $Q(Q(\sqrt{x})\sqrt{y})o = Q(\sqrt{x})^2 Q(\sqrt{y})o = Q(x)y$, whence (26.3). Using this, associativity follows:

$$\begin{aligned} u \cdot (v \cdot w) &= Q(\sqrt{u})Q(\sqrt{v})w, \\ (u \cdot v) \cdot w &= Q(\sqrt{Q(\sqrt{u})v})w = Q(Q(\sqrt{\sqrt{u}})\sqrt{v})w \\ &= Q(\sqrt{\sqrt{u}})^2 Q(\sqrt{v})w = Q(\sqrt{u})Q(\sqrt{v})w. \end{aligned}$$

Similarly, we see that the inverse of x is $x^{-1} = Q(x)^{-1}x$, and thus $(V, \cdot, o)$ is an abelian group; it admits unique square roots since the squaring operation is the same as the one of (V, μ, o). Finally, it is clear that both constructions are inverse to each other. ■

In the sequel, abelian groups will mainly be written additively, so that v^2 corresponds to $2v$ and $\sqrt{v}$ corresponds to $\frac{1}{2}v$. Then (26.1) and (26.2) read

$$\mu(u,v) = 2u - v, \qquad u + v = \mu(\frac{1}{2}u, \mu(0,v)) = Q(\frac{u}{2})v = Q(\frac{u}{2})Q(\frac{v}{2})o. \tag{26.4}$$

Now let (M, μ) be an arbitrary symmetric space over $\mathbb{K}$. Recall that the higher tangent bundles $(T^k M, T^k \mu)$ are again symmetric spaces. The purpose of this chapter is to prove the following theorem:

Theorem 26.3. *Assume (M,μ) is a symmetric space over $\mathbb{K}$ with base point $o \in M$. There exists a linear connection L on TM which is uniquely determined by one of the following two equivalent properties:*

(a) *L is invariant under all symmetries σ_x, $x \in M$.*

(b) *For all $x \in M$, the symmetric space structure on $(TTM)_x$ is the canonical flat structure associated to the $\mathbb{K}$-module $((TTM)_x, L_x)$: for all $u,v \in (TTM)_x$, $TT\mu(u,v) = 2u - v$.*

The torsion of L vanishes.

Proof. Let us first prove uniqueness. Assume two linear connections L, L' with Property (a) given. Then $A := L - L'$ is a tensor field of type $(2,1)$ that is again invariant under all symmetries. From $A_x \sigma_x = -\mathrm{id}_{T_xM}$ it follows then that $-A_x(u,v) = A_x(-u,-v) = A_x(u,v)$ and $A_x(u,v) = 0$ for all $x \in M$ and $u,v \in T_xM$. Hence $A = 0$. The same argument also shows that the torsion of L must vanish.

Now assume L is a linear structure on TM satisfying (b). Then multiplication by 2 in the fiber $(TTM)_o$ is given by $2u = TT\mu(u, 0_o) = Q(u).0_o$; it is a bijection whose inverse is denoted by 2^{-1}. Thus $(TTM)_o$ has unique square roots, and therefore addition in the fiber $(TTM)_o$ is necessarily given by (26.4): $u + v = TT\mu(2^{-1}u, TT\mu(0,v)) = Q(\frac{u}{2})Q(\frac{v}{2}).0_o$. Thus the addition is uniquely determined by the symmetric space structure of $(TTM)_o$. But then also the bijection

$$\varepsilon_1 T_oM \times \varepsilon_2 T_oM \times \varepsilon_1\varepsilon_2 T_oM \to (TTM)_o, \quad (\varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12}) \mapsto \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}$$

depends only on the symmetric space structure. Since the axes are $\mathbb{K}$-modules in a canonical way, it follows that also the action of scalars on $(TTM)_o$ is uniquely determined. Summing up, there is at most one linear structure on TTM satisfying (b), and moreover it follows that such a linear structure will be invariant under all automorphisms of M since it is defined by the symmetric space structure alone (more generally, it will depend functorially on (M,μ)).

Existence of a linear structure L satisfying (b) can be proved in several ways. In the following we give a proof by using charts, which shows also that the linear structure is a connection. The proof relies on the following

Lemma 26.4. *If U is a chart domain of M around x with model space V, then for all $z, w \in (TTM)_x = V \times V \times V$,*

$$TT\mu(z,w) = 2_b z -_b w,$$

(notion for bilinearly related structures being as in Section BA.1*) where*

$$b := b_x := \frac{1}{2} d^2\sigma_x(x) : V \times V \to V$$

is given by the ordinary second differential of the symmetry σ_x at x in the chart $U \subset V$.

Proof. (Cf. [Pos01, p. 64–66] for a coordinate version of the following proof in the finite-dimensional real case.) First of all, if V_1, V_2, V_3 are $\mathbb{K}$-modules and $b \in \mathrm{Bil}(V_1 \times V_2, V_3)$, then the symmetric structure with respect to the linear structure L^b is given by the "Barycenter Formula" (BA.4), with $r = -1$:

$$2_b(u_1,u_2,u_3) -_b (v_1,v_2,v_3) = (2u_1 - v_1, 2u_2 - v_2, 2u_3 - v_3 + 2b(u_1 - v_1, u_2 - v_2)) \qquad (26.5)$$

Now choose a chart U around o. From the general expression of the second tangent map of f,

$$TTf(x,u_1,u_2,u_3) = (f(x), df(x)u_1, df(x)u_2, df(x)u_3 + d^2 f(x)(u_1,u_2)),$$

for $f = m$, we get the chart formula for $TTm : TT(U \times U) \to TTM$ at the point (x,x):

$$\begin{aligned} TTm((x,x),(u_1,v_1),(u_2,v_2),(u_3,v_3)) &= (m(x,x), dm(x,x)(u_1,v_1), dm(x,x)(u_2,v_2), \\ &\qquad dm(x,x)(u_3,v_3) + d^2m(x,x)((u_1,v_1),(u_2,v_2)) \\ &= (x, 2v_1 - u_1, 2v_2 - u_2, \\ &\qquad 2v_3 - u_3 + d^2m(x,x)((u_1,v_1),(u_2,v_2))) \end{aligned}$$

Comparing with (26.5), we see that the claim will be proved if we can show that

$$d^2m(x,x)((u_1,v_1),(u_2,v_2)) = d^2\sigma_x(x)(u_1-v_1,u_2-v_2). \tag{26.6}$$

In order to prove (26.6), we assume that $\varphi_i(x)=0$ is the origin of V and write Taylor expansions of order 2 at the origin for $\sigma_x=\sigma_0$ and for μ,

$$\begin{aligned}\sigma_0(tv) &= -tv + \frac{t^2}{2}d^2\sigma_0(0)(v,v) + t^2O(t),\\ \mu(tu,tv) &= t(2u-v) + \frac{t^2}{2}d^2\mu(0,0)((u,v),(u,v)) + t^2O(t) = t(2u-v) + t^2c(u,v) + t^2O(t)\end{aligned}$$

with $c(u,v) := \frac{1}{2}d^2\mu(0,0)((u,v),(u,v))$ (this is not bilinear as function of (u,v) but is a quadratic form in (u,v)). The defining identity (M2) of a symmetric space, $p=\mu(x,\mu(x,p))$, implies

$$\begin{aligned}tv = \mu(0,\sigma_0(tv)) &= \mu(0,t(-v+\frac{t}{2}d^2\sigma_0(0)(v,v)+tO(t)))\\ &= t(v-\frac{t}{2}d^2\sigma_0(0)(v,v)-tO(t)) + t^2c((0,-v+O(t))) + t^2O(t)\\ &= tv + t^2(c(0,-v)-\frac{1}{2}d^2\sigma_0(0)(v,v)) + t^2O(t).\end{aligned}$$

Comparing quadratic terms and letting $t=0$ we get

$$d^2\sigma_0(0)(v,v) = 2c((0,-v)) = d^2\mu(0,0)\,((0,-v),(0,-v)).$$

This is a special case of (26.6). The general formula now follows: from the identity $\mu(p,p)=p$ we get

$$\partial_{v,v}\mu(x,x) = \frac{\mu(x+tv,x+tv)-\mu(x,x)}{t}|_{t=0} = \frac{tv}{t}|_{t=0} = v$$

and hence

$$d^2\mu(0,0)((u,w),(v,v)) = \partial_{(u,w)}\partial_{(v,v)}\mu(0,0) = \partial_{(u,w)}v = 0.$$

We write $(u,v)=(v,v)+(u-v,0)$, and so on, and get

$$\begin{aligned}d^2\mu(0,0)((u_1,v_1),(u_2,v_2)) &= d^2\mu(0,0)((v_1,v_1)+(u_1-v_1,0),(v_2,v_2)+(u_2-v_2,0))\\ &= d^2\mu(0,0)((u_1-v_1,0),(u_2-v_2,0)) = d^2\sigma_0(0)(u_1-v_1,u_2-v_2)\end{aligned}$$

whence (26.6), and the claim is proved. ■

Returning to the proof of the existence part of the theorem, using a chart U as in the lemma, we define a linear structure on $(TTF)_x = T_xM\times T_xM\times T_xM$ by $L_x := L^{b_x}$ with $b_x = \frac{1}{2}d^2\sigma_x(x)$. By the lemma, this linear structure satisfies Property (b) from the theorem. We have already seen that there is at most one linear structure on $(TTM)_x$ satisfying (b), and therefore L_x does not depend on the choice of the chart. Thus $(L_x)_{x\in M}$ is a well-defined linear structure on TTM, and it is a connection since, again by the lemma, it is bilinearly related to all chart structures via bilinear maps b_x depending smoothly on x. We have already remarked that this connection is invariant under all automorphisms and hence satisfies also Property (a). ■

Corollary 26.5. *If (φ_i,U_i) is a chart of M around x such that $\varphi_i(x)=0$ and $\sigma_x(y)=-y$ for all $y\in U$, then in this chart the symmetric space structure on $(TTM)_x$ is simply the usual flat structure of the vector space $V\times V\times V$.*

Proof. We have $d^2(-\mathrm{id})(x)=0$ and hence $b_x=0$. ■

For example, this may applied to the exponential map of a symmetric space, if it exists (cf. [BeNe04] or Section 27.7), or to Jordan coordinates of a symmetric space with twist ([Be00]). (In fact, it it is well-known that in the real finite dimensional case and for any torsionfree connection one can find a chart such that the Christoffel symbols at a given point x vanish and hence the linear structure of $(TTM)_x$ is the canonical flat one from the chart. This generalizes to all contexts in which one has an inverse function theorem at disposition.)

26.6. *Dombrowski splitting and the associated spray.* The Dombrowski splitting (Theorem 10.5) for the canonical connection of a symmetric space is given by

$$\begin{aligned}\Phi_1 :& \varepsilon_1 TM \times_M \varepsilon_2 TM \times_M \varepsilon_1\varepsilon_2 TM \to TTM, \\ &(x;\varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12}) \mapsto \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12} = \varepsilon_2 v_2 + (\varepsilon_1 v_1 + \varepsilon_1\varepsilon_2 v_{12}) \\ &= Q_x(\varepsilon_2 \frac{v_2}{2}) Q_x(\varepsilon_1 \frac{v_1}{2}) \varepsilon_1\varepsilon_2 v_{12} = Q_x(\varepsilon_2\frac{v_2}{2}) Q_x(\varepsilon_1\frac{v_1}{2}) Q_x(\varepsilon_1\varepsilon_2 \frac{v_{12}}{2}).0_x,\end{aligned} \tag{26.7}$$

where $Q_x(y) = s_y s_x$ is the quadratic map in TTM with respect to the base point $x = 0_x$ (origin in $(TTM)_x$). Restricting to invariants under the canonical flip κ, we get a bijection of vector bundles over M

$$\delta TM \times_M \delta^{(2)} TM \to J^2 M, \quad (x;\delta u, \delta^{(2)} w) \mapsto u + \kappa(u) + w = Q_x(\varepsilon_2 \frac{u}{2}) Q_x(\varepsilon_1 \frac{u}{2}) Q_x(\varepsilon_1\varepsilon_2\frac{w}{2}).0_x$$

which for $w = 0$ gives the spray

$$TM \to J^2 M, \quad (x;u) \mapsto u + \kappa(u) = Q_x(\varepsilon_2 \frac{u}{2})\varepsilon_1 u = Q_x(\varepsilon_1\frac{u}{2})\varepsilon_2 u.$$

The last expression can be rewritten: writing $z : M \to TM$ for the zero section, we get

$$u + \kappa(u) = TT\mu(\frac{u}{2}, -\kappa(u)) = T\sigma_{\frac{u}{2}}(-\kappa(u)) = T\sigma_{\frac{u}{2}} Tz(-u)$$

since $TT\mu(v,\cdot) = T\sigma_v$ and $\kappa(v) = Tz(v)$. Compare with [Ne02a, Th. 3.4].

Corollary 26.7. *The covariant derivative associated to the canonical connection of a symmetric space M can be expressed in the following way: for $x \in M$ and $v \in T_x M$ let $l_x(v) := \widetilde{v} = \frac{1}{2} \circ Q(v) \circ z \in \mathfrak{X}(M)$ be the vector field extending v constructed in the proof of* Proposition 5.9. *Then we have, for all $X, Y \in \mathfrak{X}(M)$ and $x \in M$,*

$$(\nabla_X Y)(x) = [l_x(X(x)), Y](x).$$

Proof. Chose a chart such that x corresponds to the origin 0 and let $v := X(0)$. Then the value of

$$T\widetilde{v} = \frac{1}{2} TQ(v) \circ Tz = \frac{1}{2} T\sigma_v \circ T\sigma_0 \circ Tz$$

at the origin is given, in the chart, by

$$d\widetilde{v}(0)w = -\frac{1}{2} d^2\mu(0,0)((v,0),(w,0)) = \frac{1}{2} d^2\sigma_0(0)(v,w)$$

where for the last equality we used (26.6). Hence

$$\begin{aligned}[l_0(X(0)), Y](0) &= [\widetilde{v}, Y](0) = dY(0)v - d\widetilde{v}(0)Y(0) \\ &= dY(0)v + \frac{1}{2} d^2\sigma_0(0)(v, Y(0)) \\ &= dY(0)v + b_x(v, Y(0)) \\ &= (\nabla_X Y)(0)\end{aligned}$$

according to Lemma 12.2. ■

27. The higher order tangent structure of symmetric spaces

27.1. In this chapter we study the structure of tangent bundles T^kM of a symmetric space for $k \geq 3$. In order to simplify notation, if $v \in (T^kM)_o$, we will write $Q(v)$ for the quadratic map $Q_o(v) : T^kM \to T^kM$. If the base point $o \in M$ is fixed and if we use the notation

$$Q := Q^{(M)} : M \times M \to M, \quad (x,y) \mapsto Q(x)y = \sigma_x \sigma_o(y)$$

for the quadratic map of M w.r.t. o, then the quadratic map of T^kM w.r.t. 0_o is given by $Q^{(T^kM)} = T^k(Q^{(M)})$.

Theorem 27.2. *Assume (M,μ) is a symmetric space over $\mathbb{K}$ with base point $o \in M$. For elements from the axes of $(T^3M)_o$, the following "fundamental commutation relations" hold: for all $u,v,w \in T_oM$ and $i,j \in \{1,2,3\}$,*

$$[Q(\varepsilon_i u), Q(\varepsilon_i v)]\varepsilon_j w = \varepsilon_j w,$$
$$[Q(\varepsilon_i u), Q(\varepsilon_j v)]\varepsilon_j w = \varepsilon_j w,$$
$$[Q(\varepsilon_1 u), Q(\varepsilon_2 v)]\varepsilon_3 w = \varepsilon_3 w + \varepsilon_1\varepsilon_2\varepsilon_3[u,v,w],$$

where the commutator is taken in the group $G(T^3M)$ and $[u,v,w]$ is the Lie triple product of (M,o).

Proof. The first two relations simply take account of the fact that the fibers of $TTM = T_{\varepsilon_i}T_{\varepsilon_j}M$ are abelian symmetric spaces (Theorem 26.3), i.e. the quadratic operators commute on the fiber.

In order to prove the third relation, we need a lemma. We say that a diffeomorphism $\varphi : TM \to TM$ is an *infinitesimal automorphism* if it is a fiber-preserving automorphism: $\pi \circ \varphi = \pi$. Clearly then $X := \varphi \circ z : M \to TM$ is a homomorphism and a vector field, i.e. a derivation. Conversely, one can show that every derivation $X : M \to TM$ gives rise to an infinitesimal automorphism $\tau_{\varepsilon X} : TM \to TM$, $\delta_x \mapsto \delta_x + X(x)$; this will not be needed in the sequel. In general, it is not true that every infinitesimal automorphism is of this form (it is true in the real finite-dimensional case, cf. [Lo69, p. 53]), but the infinitesimal automorphisms $Q(v) : TM \to TM$ for $v \in T_oM$ are always of this form:

Lemma 27.3. *For all $v \in T_oM$, the map $Q(v) : TM \to TM$ acts by translation in fibers: for all $w \in T_pM, p \in M$:*

$$Q(v).w = Q(v).0_p + w,$$

and hence $Q(\frac{v}{2}) = \tau_{\varepsilon\widetilde{v}}$ is the infinitesimal automorphism induced by the derivation $\widetilde{v}$ introduced in Equation (5.9).

Proof. We apply (5.1) to the binary map

$$Q(x,y) := Q(x)y = \sigma_x\sigma_o(y) =: \lambda_x(y) =: \rho_y(x).$$

Note that $\lambda_x = \sigma_x\sigma_o$ and in particular $\lambda_o = \mathrm{id}_M$. Now (5.1) implies

$$\begin{aligned} Q(v)w &= T_{(o,p)}Q(v,w) = T_o(\rho_p)v + T_p(\lambda_o)w \\ &= T_o(\rho_p)v + w = Q(v)0_p + w. \end{aligned}$$

■

Using the lemma, we can give another proof (not depending on the preceding chapter) of the first two relations from the theorem: on TTM, the quadratic operator is given by taking the tangent map the infinitesimal automorphism from the preceding lemma,

$$Q(\varepsilon_1 \frac{v}{2}) = T_{\varepsilon_2}\tau_{\varepsilon_1\widetilde{v}},$$

and similarly for ε_1 and ε_2 exchanged. Then Theorem 14.3 on the Lie bracket implies

$$[Q(\varepsilon_1 \frac{u}{2}), Q(\varepsilon_2 \frac{v}{2})](w) = [T_{\varepsilon_2}\tau_{\varepsilon_1 \widetilde{u}}, T_{\varepsilon_1}\tau_{\varepsilon_2 \widetilde{v}}](w) = \varepsilon_1\varepsilon_2[\widetilde{u},\widetilde{v}](o) + w = 0_o + w = w$$

since $[\widetilde{u},\widetilde{v}]$ vanishes at o (Prop. 5.9). In a similar way, again with Theorem 14.3, we get the third relation: the triple bracket $[[\widetilde{u},\widetilde{v}],\widetilde{w}]$ is given by

$$\varepsilon_1\varepsilon_2\varepsilon_3[[\widetilde{u},\widetilde{v}],\widetilde{w}] = [[Q(\frac{\varepsilon_1 u}{2}), Q(\frac{\varepsilon_2 v}{2})], Q(\frac{\varepsilon_3 w}{2})]|_{\varepsilon_1\varepsilon_2\varepsilon_3 TM}.$$

Evaluating at the base point, using that the vector field $[\widetilde{u},\widetilde{v}]$ vanishes at o (Prop. 5.9) and that $Q(\frac{\varepsilon_3 w}{2}).0 = \varepsilon_3 w$, we get

$$\varepsilon_1\varepsilon_2\varepsilon_3[u,v,w] = \varepsilon_1\varepsilon_2\varepsilon_3[[\widetilde{u},\widetilde{v}],\widetilde{w}].0 = \varepsilon_1\varepsilon_2[\widetilde{u},\widetilde{v}]\cdot\varepsilon_3 w - \varepsilon_3\widetilde{w}(0) = [Q(\frac{\varepsilon_1 u}{2}), Q(\frac{\varepsilon_2 v}{2})].\varepsilon_3 w - \varepsilon_3 w,$$

which had to be shown. ∎

Another way of stating the preceding relations goes as follows: let us introduce (with respect to a fixed base point $o \in M$) the map

$$\begin{aligned}\rho := \rho_M : M \times M \times M \to M,&\\ (x,y,z) \mapsto [Q(x),Q(y)]z = Q(x)Q(y)Q(x)^{-1}Q(y)^{-1}z = \sigma_x\sigma_o\sigma_y\sigma_x\sigma_o\sigma_y.z&\end{aligned}$$

Then, writing $w = \sum_\alpha \varepsilon^\alpha w_\alpha$, etc., we have

$$(T^3\rho)_{o,o,o}(u,v,w) = w + \varepsilon^{111}[u_{100}, v_{010}, w_{001}]. \tag{27.1}$$

This follows by noting that $T^3\rho_M = \rho_{T^3M}$.

27.4. *Fundamental commutation relations for elements in the axes of a symmetric space.* For elements $\varepsilon^\alpha u_\alpha$, $\varepsilon^\beta v_\beta$, $\varepsilon^\gamma w_\gamma$ of the axes of $(T^kM)_o$ with $k \geq 3$ we have the following "commutation relations":

$$[Q(\varepsilon^\alpha \frac{u_\alpha}{2}), Q(\varepsilon^\beta \frac{v_\beta}{2})]\varepsilon^\gamma w_\gamma = \begin{cases} \varepsilon^\gamma w_\gamma + \varepsilon^{\alpha+\beta+\gamma}[u_\alpha, v_\beta, w_\gamma] & \text{if } \alpha \perp \beta, \beta \perp \gamma, \alpha \perp \gamma, \\ \varepsilon^\gamma w_\gamma & \text{else,} \end{cases} \tag{27.2}$$

where the bracket on the left hand side denotes the commutator in the group $\mathrm{Diff}(T^kM)$. In fact, the general case is reduced to the case $k = 3$; if α, β, γ are not disjoint, then we may further reduce to the case $k = 2$ in which the fibers are abelian symmetric spaces (Theorem 26.3), whence the claim in this case; in the other case we apply Theorem 27.2.

27.5. *The derived multilinear connections on T^kM.* Starting with the canonical connection L on TM, we may define a sequence of derived multilinear connections on T^kM (Chapter 17). We will give an explicit formula for the derived linear structure on T^3M: let $L_1 := L$ be the canonical connection of the symmetric space M and $L_k := D^{k-1}L$ be the sequence of derived linear structures, and $\Phi_k : A^{k+1}M \to T^{k+1}M$ be the corresponding linearization map. The map $\Phi_1 : A^2M \to T^2M$ is given by (26.7). We are going to calculate $\Phi_2 : A^3M \to T^3M$. First of all, we have to calculate the tangent map $T\Phi_1$. It is a bundle map over the base space $TM = T_{\varepsilon_3}M$; a typical point in the base is $\varepsilon_3 v_3 \in T_xM$. In the formula for $T\Phi_1$, the Q-operators in T^2M are replaced by the corresponding ones in T^3M, but taken with respect to the base point $\varepsilon_3 v_3 \in T_xM$. The factor $\frac{1}{2}$ in (26.7) really stands for the map

$$(\frac{1}{2})_{TTM} : TTM \to TTM, \quad v \mapsto \frac{v}{2}. \tag{27.3}$$

When taking its derivative with respect to ε_3, the base point $\varepsilon_3 v_3$ is fixed. In the following, the sign $+$ without index stands for $+_x$ (addition in fiber of TTM over x), and Q without index stands for Q_{0_x}; the index $\varepsilon_3 v_3$ labels the corresponding operations in the fiber over $\varepsilon_3 v_3$. In the following calculation, we pass from one base point to another via the "isotopy formula"

$$Q_x(y) = s_y s_o s_o s_x = Q_o(y) Q_o(x)^{-1}. \tag{27.4}$$

Then we have

$$\begin{aligned}
&\Phi_2(x; \varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_3 v_3, \varepsilon_1\varepsilon_2 v_{12}, \varepsilon_2\varepsilon_3 v_{13}, \varepsilon_2\varepsilon_3 v_{23}) \\
&= (\varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_2\varepsilon_3 v_{23}) +_{\varepsilon_3 v_3} (\varepsilon_1 v_1 + \varepsilon_3 v_3 + \varepsilon_1\varepsilon_3 v_{13}) +_{\varepsilon_3 v_3} (\varepsilon_1\varepsilon_2 v_{12} + \varepsilon_3 v_3 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_3 v_{123}) \\
&= Q_{\varepsilon_3 v_3}(\varepsilon_2 \frac{v_2}{2} + \varepsilon_3 v_3 + \varepsilon_2\varepsilon_3 \frac{v_{23}}{2}) \circ Q_{\varepsilon_3 v_3}(\varepsilon_1 \frac{v_1}{2} + \varepsilon_3 v_3 + \varepsilon_1\varepsilon_3 \frac{v_{13}}{2}) \circ \\
&\qquad Q_{\varepsilon_3 v_3}(\varepsilon_1\varepsilon_2 \frac{v_{12}}{2} + \varepsilon_3 v_3 + \varepsilon_1\varepsilon_2\varepsilon_3 \frac{v_{123}}{2}).0_{\varepsilon_3 v_3} \\
&= Q_{\varepsilon_3 v_3}(Q(\varepsilon_2 \frac{v_2}{4}) Q(\varepsilon_3 \frac{v_3}{2}) Q(\varepsilon_2\varepsilon_3 \frac{v_{23}}{4}).0_x) \circ Q_{\varepsilon_3 v_3}(Q(\varepsilon_1 \frac{v_1}{4}) Q(\varepsilon_3 \frac{v_3}{2}) Q(\varepsilon_1\varepsilon_3 \frac{v_{13}}{4}).0_x) \circ \\
&\qquad Q_{\varepsilon_3 v_3}(Q(\varepsilon_1\varepsilon_2 \frac{v_{12}}{4}) Q(\varepsilon_3 \frac{v_3}{2}) Q(\varepsilon_1\varepsilon_2\varepsilon_3 \frac{v_{123}}{4}).0_x).0_{\varepsilon_3 v_3} \\
&= Q(\varepsilon_3 \frac{v_3}{2}) Q(Q(\varepsilon_2 \frac{v_2}{4}) Q(\varepsilon_2\varepsilon_3 \frac{v_{23}}{4}).0_x) \; Q(Q(\varepsilon_1 \frac{v_1}{4}) Q(\varepsilon_1\varepsilon_3 \frac{v_{13}}{4}).0_x) \\
&\qquad Q(Q(\varepsilon_1\varepsilon_2 \frac{v_{12}}{4}) Q(\varepsilon_1\varepsilon_2\varepsilon_3 \frac{v_{123}}{4}).0_x).0_x \\
&= Q(\varepsilon_3 \frac{v_3}{2}) Q(\varepsilon_2 \frac{v_2}{2}) Q(\varepsilon_2\varepsilon_3 \frac{v_{23}}{2}) Q(\varepsilon_1 \frac{v_1}{2}) Q(\varepsilon_1\varepsilon_3 \frac{v_{13}}{2}) Q(\varepsilon_1\varepsilon_2 \frac{v_{12}}{2}) Q(\varepsilon_1\varepsilon_2\varepsilon_3 \frac{v_{123}}{2}).0_x
\end{aligned}$$

where the simplifications in the last lines are due to the "fundamental formula" and the fact that Q-operators having an ε_i for argument in common commute. The formula for Φ_k with general k is obtained in a similar way; we state the result without proof: if, as in Theorem 24.8, κ denotes the order-reversing permutation $\kappa(j) = k + 1 - j$, then

$$\Phi_k(x; (v_\alpha)_{\alpha>0}) = (\prod_{\alpha>0}^{\uparrow} Q(\varepsilon^{\kappa\cdot\alpha} \frac{v_{\kappa\cdot\alpha}}{2})).0_x \tag{27.5}$$

where $\prod_\alpha^\uparrow A_\alpha = A_{0\ldots01} \circ \ldots \circ A_{1\ldots1}$ is the composition of operators A_α in the order given by the lexicographic order of the index set.

Theorem 27.6. *The curvature of the canonical connection of a symmetric space agrees with its Lie triple product (up to a sign): for $u, v, w \in T_oM$,*

$$R_o(u,v)w = -[u,v,w].$$

Proof. We are going to use the characterization of the curvature tensor by Theorem 18.3. (See [Lo69] and [Be00, I.2.5] for different arguments in the real finite-dimensional case, rather using covariant derivatives.) Let τ be the transposition (23). The formula for $\tau \cdot L_2$ and Φ_s^τ is obtained by exchanging in the formula for Φ_2: $\varepsilon_2 \leftrightarrow \varepsilon_3$, $v_1 \leftrightarrow v_3$, $v_{12} \leftrightarrow v_{13}$. Using the commutation rules, one sees that as a result one gets a formula of the same type, beginning with $A \circ B := Q(\varepsilon_2 \frac{v_2}{2}) Q(\varepsilon_2 \frac{v_3}{3})$ instead of $B \circ A = Q(\varepsilon_3 \frac{v_3}{2}) Q(\varepsilon_2 \frac{v_2}{2})$. Using the operator formula $AB = [A,B]BA$, where $[A,B]$ is described by Theorem 27.2 (the transposition denoted there by κ is now τ), we see that the difference between these terms (in any linearization) belongs to the axis $\varepsilon_1\varepsilon_2\varepsilon_3 V$ and is given by

$$-[Q(\frac{\varepsilon_1 v_1}{2}), Q(\frac{\varepsilon_2 v_2}{2})].\varepsilon_3 v_3.$$

On the one hand, by Theorem 18.3, this term is equal to $\varepsilon_1\varepsilon_2\varepsilon_3 R(v_1,v_2)v_3$; on the other hand, by Theorem 27.2, it is equal to $\varepsilon_1\varepsilon_2\varepsilon_3 [v_1,v_2,v_3]$, whence the claim. ∎

27.7. *On the exponential jet.* We say that a symmetric space M *has an exponential map* if, for every $x \in M$ and $v \in T_xM$, there exists a unique homomorphism of symmetric spaces $\gamma_v : \mathbb{K} \to M$ such that $\gamma(0) = x$, $\gamma_v'(0) = v$, and such that the map $\mathrm{Exp}_x : T_xM \to M$, $v \mapsto \gamma_v(1)$ is smooth (cf. [BeNe04, Ch. 2]). It is known that every real finite-dimensional symmetric space has an exponential map, and moreover this exponential map agrees with the exponential map of the canonical connection (see [Lo69]). In our general setting, not every symmetric space will have an exponential map, but, following the theory of Lie groups, one can (if the integers are invertible in $\mathbb{K}$) construct a canonical "exponential jet"

$$\mathrm{Exp}_x^{(k)} : T_0((T^kM)_x) \to (T^kM)_x$$

having the property that, if M has an exponential map, then $\mathrm{Exp}_x^{(k)} = T^k(\mathrm{Exp}_x)_0$. There are (at least) three different ways to develop the corresponding theory:

(1) One may use the *standard imbedding* of a Lie triple system in order to imbed the exponential map of the symmetric space into the exponential map of a polynomial group (Theorem PG.2). More precisely, if $(T^k\mathfrak{m})_0$ is the Lts of T^kM at the base point 0_o and $\mathfrak{g} := (T^k\mathfrak{m})_0 \oplus [(T^k\mathfrak{m})_0, (T^k\mathfrak{m})_0]$ its standard imbedding (associated Lie algebra with involution, cf. [Lo69] or [Be00]), then $\mathfrak{g}$ is a nilpotent Lie algebra. The main task is then to show that the transvection group of the fiber $(T^kM)_x$ has a natural structure of a polynomial group, corresponding to the Lie algebra $\mathfrak{g}$. Then, with the exponential map of this polynommial group, one has to establish an analog of the following diagram which is known to commute in the real finite-dimensional case:

$$\begin{array}{ccc} M & \overset{x \mapsto Q(x)}{\rightarrow} & G(M) \\ \mathrm{Exp}_M \uparrow & & \uparrow \exp_G \\ \mathfrak{m} & \overset{X \mapsto 2X}{\rightarrow} & \mathfrak{g}(\mathfrak{m}) \end{array}$$

Then $\exp_{(T^kM)_o}$ can be defined as the map induced by $\exp_{(T^kG)_e}$.

(2) A more conceptual strategy would be to develop from scratch a theory of *polynomial symmetric spaces*, similarly to what we have done for polynomial groups (Chapter PG), and to give a direct algebraic proof for the existence of an exponential map. It is to be expected that such a theory would give much insight into the higher order structure of symmetric spaces and would even contribute to a better understanding of the group case.

(3) Finally, the most general approach would be to develop a general "formal" theory of the exponential jet of an arbitrary connection on TM and then to apply it to the special case of the canonical connection of a symmetric space: it should be possible to define, for an arbitrary connection L on the tangent bundle of a manifold, an "exponential jet" $\mathrm{Exp}_x^{(k)} : T_0((T^kM)_x) \to (T^kM)_x$, such that $\mathrm{Exp}_x^{(k)} = T_0^k(\mathrm{Exp}_x)$ if the connection L has an exponential map Exp in the usual sense. Cf. Chapter 20 for some first steps in this direction, and for the case $k = 3$, see the next section and also the work [El67] by H. Eliasson.

27.8. *The structure of* J^3M. One can show that, for any torsionfree connection L on a the tangent bundle of a manifold M, the symmetrized linear structure $\mathrm{Sym}(DL)$ from Corollary 20.3 is the totally symmetric linear structure on T^3M corresponding to the (suitably defined) exponential jet $\mathrm{Exp}^{(3)}$. In the particular situation of the canonical connection of a symmetric space, this means that $\mathrm{Exp}^{(3)} = \mathrm{Sym}(\Phi_2)$ with Φ_2 calculated in Section 27.5. Put differently, the restriction of Φ_2 to the space of Σ_3-invariants,

$$\Phi_2 : \delta T_oM \oplus \delta^{(2)}T_oM \oplus \delta^{(3)}T_oM \to (J^3M)_o,$$

agrees with the restriction of the exponential jet $\mathrm{Exp}_o^{(3)}$ to the space of Σ_3-invariants. Using this canonical trivialization, we can describe explicitly the structure of the symmetric space $(J^3M)_o$. The result is given by the following formulae:

$$\mu(\delta v_1 + \delta^{(2)}v_2 + \delta^{(3)}v_3, \delta w_1 + \delta^{(2)}w_2 + \delta^{(3)}w_3)) = \delta(2v_1 - w_1) + \delta^{(2)}(2v_2 - w_2) + \delta^{(3)}(2v_3 - w_3 - ([v_1, w_1, v_1] + [w_1, v_1, w_1])).$$

In terms of the quadratic map, this reads

$$Q_o(\delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3, \delta w_1 + \delta^{(2)} w_2 + \delta^{(3)} w_3)) = \delta(2v_1 + w_1) + \delta^{(2)}(2v_2 + w_2) + \\ \delta^{(3)}(2v_3 + w_3 + ([v_1, w_1, v_1] - [w_1, v_1, w_1])).$$

In particular,

$$Q(\delta v) = \tau_v + \delta^{(3)}([v, \cdot, v] - [\cdot, v, \cdot])$$

is ordinary translation by v with respect to the linear structure plus an extra term which is the sum of a linear and a homogeneous quadratic map. Note that the Lie triple system of $(J^3 M, 0_o)$ is $(J^3 \mathfrak{m})_0 = \delta\mathfrak{m} \oplus \delta^2\mathfrak{m} \oplus \delta^3\mathfrak{m}$ with product

$$[\delta v_1 + \delta^{(2)} v_2 + \delta^{(3)} v_3, \delta u_1 + \delta^{(2)} u_2 + \delta^{(3)} u_3, \delta w_1 + \delta^{(2)} w_2 + \delta^{(3)} w_3] = (0, 0, \delta^3 [v_1, u_1, w_1]) \\ = (0, 0, 6\, \delta^{(3)} [v_1, u_1, w_1])$$

where $\mathfrak{m}$ is the Lie triple system of (M, o). It is nilpotent, but in general non-abelian, and it contains (if 6 is invertible in $\mathbb{K}$) all information of the original Lts $\mathfrak{m}$. Thus the space $(J^3 M)_o$ may be considered as a "faithful model" of the original symmetric space M in the sense that it contains all infinitesimal information on M.

References

[B57] Berger, M., *Les espaces symétriques non-compacts*, Ann. Sci. Ec. Norm. Sup. (3) **74** (1957), 85 – 177.

[Be00] Bertram, W., "The Geometry of Jordan and Lie Structures", Springer LNM **1754**, Berlin 2000.

[BGN04] Bertram, W., H. Glöckner and K.-H. Neeb, *Differential Calculus, manifolds and Lie groups over arbitrary infinite fields*, Expos. Math. **22** (2004), 213 – 282 arXiv: math.GM/0303300.

[BeNe04] Bertram, W. and K.-H. Neeb, *Projective completions of Jordan pairs. Part II: Manifold structures and symmetric spaces*, Preprint Darmstadt/Nancy 2004, arXiv: math.GR/0401236.

[Bou67] Bourbaki, N. "Variétés différentielles et analytiques – Fascicule de résultats", Hermann, Paris 1967 – 1971.

[Bou72] Bourbaki, N. "Groupes et algèbres de Lie. Chapitres 1 – 3.", Hermann, Paris 1972.

[D73] Dieudonné, J., "Introduction to the Theory of Formal Groups", Marcel Dekker, New York 1973.

[El67] Eliasson, H., *Geometry of manifolds of maps*, J. Diff. Geo. **1** (1967), 169–194.

[Gl03] Glöckner, H., *Every smooth p-adic Lie group admits a compatible analytic structure*, to appear in Forum Math. (also TU Darmstadt Preprint 2307, December 2003; arXiv: math.GR/0312113).

[Gl04] Glöckner, H., *Lie groups over non-discrete topological fields*, TU Darmstadt Preprint 2356, July 2004; also arXiv:math.GR/0408008.

[Hel78] Helgason, S., "Differential Geometry and Symmetric Spaces", Academic Press 1978.

[Haz78] Hazewinkel, M., "Formal Groups and Applications", Academic Press, New York 1978.

[KMS93] Kolář, I., Michor P. W. and J. Slovák, "Natural Operations in Differential Geometry", Springer-Verlag, Berlin 1993.

[Lo67] Loos, O., *Spiegelungsräume und homogene symmetrische Räume*, Math. Z. **99** (1967), 141 – 170.

[Lo69] Loos, O., “Symmetric Spaces I”, Benjamin, New York 1969.

[Ne02a] Neeb, K.-H., *A Cartan-Hadamard Theorem for Banach-Finsler Manifolds*, Geom. Dedicata **95** (2002), 115 – 156.

[Ne02b] Neeb, K.-H., “Nancy lectures on infinite dimensional Lie groups”, Lecture Notes, Nancy 2002.

[P62] Pohl, W.F., *Differential Geometry of Higher Order*, Topology **1** (1962), 169–211.

[Pos01] Postnikov, M.M., “Riemannian Geometry”, vol. **91** of the Encyclopedia of Mathematical Sciences, Springer-Verlag, New York 2001.

[Se65] Serre, J.-P., “Lie Algebras and Lie Groups”, Benjamin, New York 1965.

[Sh97] Sharpe, R.W. “Differential Geometry – Cartan’s Generalization of Klein’s Erlangen Program”, Springer, New York 1997.

[T04] Tu, L.W., *Une courte démonstration de la formule de Campbell-Hausdorff*, J. of Lie Theory **14**, no. 2 (2004), 501–508.