Differential Geometry over General Base Fields and Rings.
Part V: The exponential jet

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Abstract. Generalizing results of Kainz and Michor ([KM87]), we prove that the space $\mathfrak{X}^k(M)$ of sections of the higher order tangent bundle T^kM of a manifold over a general base field or -ring $\mathbb{K}$, has a natural group structure which we characterize in various ways. These groups can be seen as "derived groups" of the group $G := \mathrm{Diff}_{\mathbb{K}}(M)$ of diffeomorphisms of M, even if G is not a Lie group. If the integers are invertible in $\mathbb{K}$, then there is an exponential map associated to the group $\mathfrak{X}^k(M)$. This permits to define a k-th order analog of the flow of a vector field and – in presence of a linear connection on TM – of the jet of an exponential mapping (whereas the flows and the exponential mapping themselves do in general not exist) .

Contents. We start with a revised version of Chapter 14 (cf. Part I of this work) concerning the Lie-bracket of vector fields which is now put into the context of the group structure on the space of sections of T^2M.

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Introduction

This is the fifth and definitely last part of the series "Differential Geometry over General Base Fields and Rings" (prépublications de l'Institut Elie Cartan 2003/47, 2004/39, 2004/40 and 2004/41). It completes parts of the program of "formal differential geometry" set in Part II (Higher Order Geometry) and illustrates, for the example of flows of vector fields and the exponential jet, that it is indeed possible to work out a differential geometry "at arbitrary order" without ever using any result on solutions of differential equations.

1. *The group* $\mathrm{Diff}(M)$. If M is a manifold, then the group $G := \mathrm{Diff}(M)$ of all diffeomorphisms of M is in general not a Lie group. However, quite often it is useful to think of G as a Lie group with Lie algebra $\mathfrak{X}(M)$, the Lie algebra of vector fields on M, and exponential map associating to a vector field X the flow $\mathrm{Fl}_t^X \in G$ at time $t = 1$. Then, according to the approach used in Parts II and III of this work, we would like to analyze the group G by studying the higher order tangent group $T^k G$ and its fiber $(T^k G)_e$ over the origin $e \in G$. It turns out that this approach makes perfectly sense for any manifold M over a general base field or -ring $\mathbb{K}$: one can define a family of groups $G^k(M)$ taking the rôle of $T^k G$; more precisely, there is an exact sequence of groups

$$1 \to \mathfrak{X}^k(M) \to G^k(M) \to \mathrm{Diff}(M) \to 1 \tag{S}$$

such that the normal subgroup $\mathfrak{X}^k(M)$ of $G^k(M)$ is a polynomial group in the sense of Chapter PG (Part IV), whence behaves in all respects like a honest Lie group. The group $\mathfrak{X}^k(M)$ is (isomorphic to) the space of sections of the iterated tangent bundle $T^k M$ over M. For $\mathbb{K} = \mathbb{R}$ and M finite-dimensional, it is known that this space carries a natural group structure (this result is due to Kainz and Michor, [KM87, Theorem 4.6], see also [KMS93, Theorem 37.7]); we generalize this fact to the case of manifolds over general base fields and -rings, and we characterize the groups $\mathfrak{X}^k(M)$ and $G^k(M)$ as diffeomorphism groups of $T^k M$: they are precisely the groups of diffeomorphisms of $T^k M$ that are smooth over the ring $T^k\mathbb{K} = \mathbb{K}[\varepsilon_1, \ldots, \varepsilon_k]$ and fix (resp. permute) fibers over M. Then a splitting of the exact sequence (S) is simply given by $\mathrm{Diff}(M) \to G^k(M)$, $g \mapsto T^k g$, and thus $G^k(M)$ is a semidirect product of $\mathrm{Diff}(M)$ and $\mathfrak{X}^k(M)$. In particular, $G^1(M) \cong \mathfrak{X}(M) \rtimes \mathrm{Diff}(M)$ plays the rôle of a "tangent group" TG of $G = \mathrm{Diff}(M)$. In a similar way, all other features of Lie theory as developed in Part III of this work carry over the situation $G = \mathrm{Diff}(M)$, $T^k G \cong G^k(M)$: there are left- and right trivializations, and (if the integers are invertible in $\mathbb{K}$) there is an exponential map in the fiber over e; these three maps are bijections between the space of sections of the *axes-bundle* $A^k M$ (cf. Part II) and the space of sections of $T^k M$. Every vector field $X : M \to TM$ defines a symmetric section $\delta X : M \to A^k M$, and its exponential image is a section $\exp(\delta X) : M \to T^k M$, hence gives rise to a diffeomorphism $\Phi : T^k M \to T^k M$. If X admits a flow $\mathrm{Fl}_t^X : M \to M$ in the classical sense (e.g., if X is a complete vector field on a finite-dimensional real manifold), then the map Φ is closely related to the higher order tangent map of the flow, $T^k(\mathrm{Fl}_1^X) : T^k M \to T^k M$ (see Section 29.6 for the precise conditions). Summing up, we have constructed an algebraic substitute of the "missing exponential map" of the group $\mathrm{Diff}(M)$.

2. *The exponential jet of a connection.* Assume M is a real finite-dimensional (or Banach) manifold equipped with an affine connection on the tangent bundle TM. Then, for every $x \in M$, the exponential map $\mathrm{Exp}_x : U_x \to M$ of the connection is a local diffeomorphism from an open neighborhood of the origin in T_xM onto its image in M. Applying the k-th order tangent functor, we get $T^k \operatorname{Exp}_x : T^k(U_x) \to T^k M$, inducing bijections of fibers

$$(T^k \operatorname{Exp}_x)_{0_x} : \quad (A^k M)_x := (T^k(T_xM))_{0_x} \to (T^k M)_x .$$

The bundle $A^k M$ over M is just a direct sum of a certain numbers of copies of the tangent bundle TM, thus is a vector bundle, called the *(k-th order) axes-bundle of M*, and the bijections of fibers fit together to a smooth bundle isomophism $T^k \operatorname{Exp} : A^k M \to T^k M$, which we call the *exponential jet of the connection.* This isomorphism is of basic interest because it permits to translate, in a most canonical way, the complicated, non-linear and filtered geometry of the bundle $T^k M$, to the linear and graded geometry of the bundle $A^k M$.

For a general base field or ring $\mathbb{K}$, there is no exponential map Exp_x, but still it is possible to construct a bundle isomorphism $A^k M \to T^k M$ having all the good properties of the exponential jet and coinciding with $T^k \operatorname{Exp}$ in the real finite-dimensional or Banach case. In Chapters 30 and 31, we construct this isomorphism first for Lie groups and symmetric spaces, and then give a sketch of how defining it for a general (torsionfree) connection.

3. *Problems and further topics.* In Part I of this work, a final section "Problems and further topics" was anounced. However, for the reader who followed the text up to this point, it will be obvious that the range of problems and further topics is too vast to be summarized on a few pages. A compiled version of all parts together will, hopefully, soon be available on my homepage and on the arXiv; but even then the text is not meant to be the definitive version of a theory but rather a first step towards it.

14. The Lie bracket revisited

14.1. *Subgroups of the group of diffeomorphisms of TTM.* The group $\Gamma := \mathrm{Diff}(TTM)$ of diffeomorphisms (over $\mathbb{K}$) of TTM contains several subgroups which are defined via the behaviour with respect to the projections $p_i : TTM \to TM$, $i = 1,2$ and $p : TTM \to M$ (cf. also Section BG.1):

(1) Let Γ^+ be the subgroup preserving fibers of the projection $p : TTM \to M$ and permuting the fibers of p_1 and those of p_2. In a chart, $f \in \Gamma^+$ is represented by $f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) = (x + \varepsilon_1 f_1(x, v_1) + \varepsilon_2 f_2(x, v_2) + \varepsilon_1\varepsilon_2 f_{12}(x, \varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12}))$.

(2) Let Γ^1 be the subgroup of Γ^+ preserving all fibers of the projection p_1. In a chart: $f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) = x + \varepsilon_1 v_1 + \varepsilon_2 f_2(x, v_2) + \varepsilon_1\varepsilon_2 f_{12}(x, v_1, v_2, v_{12}))$

(3) Let Γ^2 be the subgroup of Γ^+ preserving fibers of p_2 and permuting those of p_1: in a chart, $f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) = (x + \varepsilon_1 f_1(x, v_1) + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12} f_{12}(x, \varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12}))$

(4) Let $\Gamma^{12} = \Gamma^1 \cap \Gamma^2$ be the subgroup preserving all vertical spaces; in a chart, $f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) = (x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 f_{12}(x, \varepsilon_1 v_1, \varepsilon_2 v_2, \varepsilon_1\varepsilon_2 v_{12})$.

It is clear that Γ^1 and Γ^2 are normal subgroups in Γ^+. Therefore, if $g \in \Gamma^1$ and $h \in \Gamma^2$, it follows that the group commutator $[g,h] = (ghg^{-1})h^{-1} = g(hg^{-1}h^{-1})$ belongs both to Γ^1 and to Γ^2 and hence belongs to Γ^{12}. In Theorem 14.4 we will express the Lie bracket of vector fields in terms of this commutator.

14.2. *Sections of the second order tangent bundle.* We denote by $\mathfrak{X}^2(M) := \Gamma(M, T^2M)$ the space of smooth sections of the bundle $T^2M \to M$. In a chart with chart domain $U \subset M$, we write $T^2U \cong U \times \varepsilon_1 V \times \varepsilon_2 V \times \varepsilon_1\varepsilon_2 V$, and a section $X : U \to T^2U$ is written in the form

$$X(p) = p + \varepsilon_1 X_1(p) + \varepsilon_2 X_2(p) + \varepsilon_1\varepsilon_2 X_{12}(p) \tag{14.1}$$

with (chart-dependent) vector fields $X_\alpha : U \to V$. Moreover, X is a section of $J^2M \to M$ if, and only if, in any chart representation we have $X_1(p) = X_2(p)$ for all $p \in U$. There are three canonical injections of the space of vector fields $\mathfrak{X}(M)$ into $\mathfrak{X}^2(M)$ which simply correspond to the three canonical inclusions of axes $\iota_\alpha : TM \to TTM$, $\alpha = 01, 10, 11$, by letting $\varepsilon^\alpha Y := \iota_\alpha \circ Y : M \to TTM$ for a vector field $Y : M \to TM$. In the notation (14.1), these are the sections X with $X_\alpha = Y$ and $X_\beta = 0$ for $\beta \neq \alpha$.

Theorem 14.3.

(1) *There is a natural group structure on the space $\mathfrak{X}^2(M)$, given in a chart representation as above, by the formula*

$$\begin{aligned}(X \cdot Y)(x) = x &+ \varepsilon_1(X_1(x) + Y_1(x)) + \varepsilon_2(X_2(x) + Y_2(x)) + \\ &\varepsilon_1\varepsilon_2(X_{12}(x) + Y_{12}(x) + dX_1(x)Y_2(x) + dX_2(x)Y_1(x)).\end{aligned}$$

The space $\Gamma(M, J^2M)$ of sections of J^2M is a subgroup of $\mathfrak{X}^2(M)$, and the three canonical injections $\mathfrak{X}(M) \to \mathfrak{X}^2(M)$ are group homomorphisms (where $\mathfrak{X}(M)$ is equipped with addition of vector fields).

(2) *Every* $X \in \mathfrak{X}^2(M)$ *gives rise, in a natural way, to a diffeomorphism* $\widetilde{X} : T^2M \to T^2M$*, which, in a chart representation as above, is described by*

$$\widetilde{X}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) = x + \varepsilon_1(v_1 + X_1(x)) + \varepsilon_2(v_2 + X_2(x)) + \varepsilon_1\varepsilon_2(v_{12} + X_{12}(x) + dX_1(x)v_2 + dX_2(x)v_1).$$

(3) *The map* $\mathfrak{X}^2(M) \to \mathrm{Diff}(TTM)$, $X \mapsto \widetilde{X}$ *is an injective group homomorphism.*

Proof. (2) We fix a point $p \in M$ and define $\widetilde{X}$ on a chart neighborhood T^2U of p by the formula given in the claim. We then have to show that this definition is independent of the chosen chart. This, in turn, amounts to proving that the map $X \mapsto \widetilde{X}$ is *natural* in the following sense: for all local diffeomorphisms f defined on a neibhborhood of x, we have $f_*\widetilde{X} = \widetilde{(f_*X)}$, where f_* is the natural action of f on $\mathrm{Diff}(TTM)$, resp. on $\mathfrak{X}^2(M)$ by $f_*h = T^2f \circ h \circ (T^2f)^{-1}$, resp. by $f_*X = T^2f \circ X \circ f^{-1}$. In order to get rid of the inverse, we show, more generally, that, if $X, Y \in \mathfrak{X}^2(M)$ are f*-related* (i.e. $T^2f \circ X = Y \circ f$) for a smooth map $f : U \to M$, then $T^2f \circ \widetilde{X} = \widetilde{Y} \circ T^2f$. Now,

$$\begin{aligned}
&T^2f\,(\widetilde{X}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12})) = \\
&\quad f(x) + \varepsilon_1(df(x)(v_1 + X_1(x))) + \varepsilon_2(df(x)(v_2 + X_2(x))) + \varepsilon_1\varepsilon_2 \cdot \\
&\quad (df(x)(v_{12} + X_{12}(x) + dX_1(x)v_2 + dX_2(x)v_1) + d^2f(x)(v_1 + X_1(x), v_2 + X_2(x))), \\
&\widetilde{Y}(T^2f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12})) = \\
&\quad f(x) + \varepsilon_1(df(x)v_1 + Y_1(f(x))) + \varepsilon_2(df(x)v_2 + Y_2(f(x))) + \varepsilon_1\varepsilon_2 \cdot \\
&\quad (d^2f(x)(v_1, v_2) + df(x)v_{12} + dY_1(f(x))df(x)v_2 + dY_2(f(x))df(x)v_1)
\end{aligned}$$

and both expressions are equal if X and Y are f-related. Thus the map $\widetilde{X} : TTM \to TTM$ is well-defined and natural in the sense explained above. It is bijective: in fact, a direct check shows that its inverse is $\widetilde{Z}$ with a section $Z : M \to TTM$ defined by

$$Z(x) = x - \varepsilon_1 X_1(x) - \varepsilon_2 X_2(x) - \varepsilon_1\varepsilon_2(X_{12}(x) - (dX_1(x)X_2(x) + dX_2(x)X_1(x))). \tag{14.2}$$

Proof of (1) and (3). By comparing the formulas from parts (1) and (2) of the claim, we see that $(X \cdot Y)(x) = \widetilde{X}(Y(x))$. Letting $v := Z(x)$, it follows that the relations $\widetilde{XY}(v) = \widetilde{X} \circ \widetilde{Y}(v)$ and $((X \cdot Y) \cdot Z)(x) = (X \cdot (Y \cdot Z))(x)$ (associativity) are equivalent. One may check one or the other of these relations by a straightforward direct calculation. Moreover, as remarked above, $\widetilde{X}^{-1} = \widetilde{Z}$ with Z as in (14.2). Summing up, $\mathfrak{X}^2(M)$ is a group (with neutral element the zero section), and $X \mapsto \widetilde{X}$ defines an action of this group on TTM. The action is faithful since, if $\widetilde{X}(x) = x$ for all $x \in U$, then $X_1(x) = 0 = X_2(x)$ and thus also $X_{12}(x) = 0$ for all $x \in U$. Moreover, it is clear from the explicit formula that, if X, Y are sections of J^2M, i.e., $X_1 = X_2$ and $Y_1 = Y_2$, then $X \cdot Y$ is again a section of J^2M over U, and similarly for the inverse, hence the sections of J^2M form a subgroup of $\mathfrak{X}^2(M)$. Finally, the explicit formula also shows that, if X and Y are vector fields, seen as elements of $\mathfrak{X}^2(M)$ in one of the three canonical ways, then $X \cdot Y$ simply corresponds to the sum $X + Y$ of vector fields. ∎

For a conceptual, chart-independent definition of the group structure on the space $\Gamma^\infty(M, T^kM)$ of sections of T^kM over M, see Theorem 28.2.

Theorem 14.4. *The group commutator in the group $\mathfrak{X}^2(M)$ and the Lie bracket in $\mathfrak{X}(M)$ are related via*

$$[\varepsilon_1 X, \varepsilon_2 Y]_{\mathfrak{X}^2(M)} = \varepsilon_1\varepsilon_2[X,Y]$$

for all $X, Y \in \mathfrak{X}(M)$. Equivalently, the group commutator in the group $\mathrm{Diff}(TTM)$ *and the Lie bracket in $\mathfrak{X}(M)$ are related via*

$$[\widetilde{\varepsilon_1 X}, \widetilde{\varepsilon_2 Y}] = \widetilde{\varepsilon_1\varepsilon_2[X,Y]}.$$

Proof. We fix $x \in M$. Let $X : M \to TM$ be a vector field. It gives rise, via the three imbeddings $\mathfrak{X}(M) \to X^2(M)$, to three diffeomorphisms of TTM. Specialising the chart formula from Theorem 14.2 (2), these three diffeomorphisms are described by

$$\begin{aligned}
\widetilde{\varepsilon_1 X}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) &= x + \varepsilon_1(v_1 + X(x)) + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2(v_{12} + dX(x)v_2),\\
\widetilde{\varepsilon_2 X}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) &= x + \varepsilon_1 v_1 + \varepsilon_2(v_2 + X(x)) + \varepsilon_1\varepsilon_2(v_{12} + dX(x)v_1)\\
\widetilde{\varepsilon_1\varepsilon_2 X}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 v_{12}) &= x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2(v_{12} + X(x)).
\end{aligned} \tag{14.3}$$

We observe that these diffeomorphisms belong to the groups Γ^2, Γ^1 resp. Γ^{12} defined in Section 14.1. (Without using Theorem 14.3, these diffeomorphisms may be defined as follows: the first two maps may also be seen as the tangent maps $T_{\varepsilon_i}\widetilde{X}$ of the infinitesimal automorphism $\widetilde{X} : TM \to TM$, and the third map is defined by translation in direction of the vertical bundle $VM \subset TTM$, and such translations are canonical in the bilinear space $(TTM)_x$, cf. Chapter BA.) Thus, if X and Y are vector fields, both diffeomorphisms $\widetilde{\varepsilon_1 X}$ and $\widetilde{\varepsilon_2 Y}$ preserve the fiber $(TTM)_x$ and are represented in a chart $V \times V \times V$ of this fiber by bijections g and h of the fiber given by

$$g(v, x', v') = (v + a, x', v' + \alpha(x')), \quad h(v, x', v') = (v, x' + b, v' + \beta(v))$$

with $a = X(x)$, $b = Y(x)$, $\alpha = dX(x)$, $\beta = dY(x)$. Then we obtain for the commutator

$$\begin{aligned}
[g,h](v, x', v') &= gh(v - a, x' - b, v' - \alpha(x' - b) - \beta(v))\\
&= g(v - a, x', v' - \alpha(x' - b)\beta(v) + \beta(v - a))\\
&= g(v, x', v' - \alpha(x') + \alpha(b) - \beta(a) + \alpha(x'))\\
&= (v, x', v' + \alpha(b) - \beta(a)).
\end{aligned}$$

The value at 0_x is therefore given by

$$\begin{aligned}
[g,h](0,0,0) &= (0, 0, \alpha(b) - \beta(a))\\
&= (0, 0, dX(x)Y(x) - dY(x)X(x)) = (0, 0, [X,Y](x)) = \varepsilon_1\varepsilon_2[X,Y](x)
\end{aligned}$$

by definition of the Lie bracket in Theorem 4.2. ■

14.5. *Remark on definition of the Lie bracket and the Jacobi identity.* The definition of the Lie bracket via Theorem 4.2 is not very conceptual. Theorem 14.4 offers a more conceptual way of defining it: *for two vector fields $X, Y \in \mathfrak{X}(M)$, there exists a unique vector field $[X,Y] \in \mathfrak{X}(M)$ such that the group commutator $[\widetilde{\varepsilon_1 X}, \widetilde{\varepsilon_2 Y}]$ agrees with $\widetilde{\varepsilon_1\varepsilon_2[X,Y]}$.* It is then easily seen that $[X,Y]$ depends $\mathbb{K}$-bilinearly on X and Y and that $[X,X] = 0$. One would like, then, to prove the Jacobi identity by intrinsic arguments not involving chart computations. For this, it is necessary to invoke the third order tangent bundle T^3M and the natural group structure on the space $\mathfrak{X}^3(M)$ of its sections (Theorem 28.2). Then the Jacobi identity can be proved in the same way as is done for the Lie algebra of a Lie group in Section 24.4.

Essentially, this strategy of defining the Lie bracket and proving the Jacobi identity is used in synthetic differential geometry, and it also corresponds to the definition of the Lie algebra of a *formal group* (see [Se65]). In the framework of formal groups, the Jacobi identity is obtained by a third order computation from Hall's identity for iterated group commutators,

$$[[u,v],w^u][[w,u],v^w][[v,w],u^v] = 1,$$

(here, $x^y = yxy^{-1}$) which is valid in any group (cf. [Se65, p. LG 4.18]). Compared to our framework, this proof rather corresponds to using the bundle J^3M which is more complicated than T^3M since it does not have "axes". Similar arguments are used for the proof of the Jacobi identity in synthetic differential geometry, cf. [Lav87, p. 74 ff] and [MR91, p. 187/88].

28. Group structure on the space of sections of T^kM

28.1. *Groups of diffeomorphisms.* We denote by $G^k(M) := \mathrm{Diff}_{T^k\mathbb{K}}(T^kM;M)$ the group of diffeomorphisms $f : T^kM \to T^kM$ which are smooth over the ring $T^k\mathbb{K}$ and preserve fibers of the bundle $T^kM \to M$. In particular, $G^0(M) = \mathrm{Diff}_{\mathbb{K}}(M)$ is the usual group of diffeomorphisms of M. There are natural projections and injections

$$G^k(M) \to \mathrm{Diff}_{\mathbb{K}}(M),\ F \mapsto f := \pi \circ F \circ z,$$
$$\mathrm{Diff}_{\mathbb{K}}(M) \to G^k(M),\ f \mapsto T^kf,$$

where $\pi : T^kM \to M$ and $z : M \to T^kM$ are the canonical maps.

28.2. *The space of sections of* T^kM. We denote by $\mathfrak{X}^k(M) := \Gamma(M, T^kM)$ the space of smooth sections of the bundle $T^kM \to M$. A chart $\varphi : U \to V$ induces a bundle chart $T^k\varphi : T^kU \to T^kV$, and with respect to such a chart, a section $X : M \to T^kM$ over U is represented by

$$X(x) = x + \sum_{\alpha>0} \varepsilon^\alpha X_\alpha(x) \tag{28.1}$$

with (chart-dependent) vector fields $X_\alpha : U \to V$. The inclusions of axes $\iota_\alpha : TM \to T^kM$ induce inclusions

$$\mathfrak{X}(M) \to \mathfrak{X}^k(M), \quad X \mapsto \iota_\alpha \circ X = \varepsilon^\alpha X.$$

We call such sections *(purely) vectorial.*

Theorem 28.3.

(1) *Every section $X : M \to T^kM$ admits a unique extension to a diffeomorphism $\widetilde{X} : T^kM \to T^kM$ which is smooth over the ring $T^k\mathbb{K}$. Here, the term "extension" means that $\widetilde{X} \circ z = X$, where $z : M \to T^kM$ is the zero section. In a chart representation as above, this extension is given by*

$$\widetilde{X}(x + \sum_\alpha \varepsilon^\alpha v_\alpha) = x + \sum_\alpha \varepsilon^\alpha (v_\alpha + X_\alpha(x) + \sum_{l=2}^{|\alpha|} \sum_{\Lambda\in\mathcal{P}_l(\alpha)} \sum_{j=1,\dots,l} (d^{l-1}X_{\Lambda^j})(x)(v_{\Lambda^1}, \dots, \widehat{v}_{\Lambda^j}, \dots, v_{\Lambda^l}))$$

(where "$\widehat{v}$" means that the corresponding term is to be omitted). If X is a section of J^kM over M, then the diffeomorphism $\widetilde{X}$ preserves J^kM.

(2) *There is a natural group structure on $\mathfrak{X}^k(M)$, defined by the formula*

$$X \cdot Y := \widetilde{X} \circ Y.$$

In a chart representation, the group structure of $\mathfrak{X}^k(M)$ is given by

$$(X \cdot Y)(p) = p + \sum_\alpha \varepsilon^\alpha (X_\alpha(p) + Y_\alpha(p) + \sum_{l=2}^{|\alpha|} \sum_{\Lambda\in\mathcal{P}_l(\alpha)} \sum_{j=1}^{l} (d^{l-1}X_{\Lambda^j})(p)(Y_{\Lambda^1}(p), \dots, \widehat{Y_{\Lambda^j}(p)}, \dots, Y_{\Lambda^l}(p))).$$

The sections of J^kM *form a subgroup of* $\mathfrak{X}^k(M)$, *and the imbeddings* $\mathfrak{X}(M) \to \mathfrak{X}^k(M)$, $X \mapsto \varepsilon^\alpha X$ *are group homomorphisms.*

(3) *The map* $X \mapsto \widetilde{X}$ *is an injective group homomorphism* $\mathfrak{X}^k(M) \to G^k(M)$, *and we have a splitting exact sequence of groups*

$$0 \to \mathfrak{X}^k(M) \to G^k(M) \to \mathrm{Diff}_{\mathbb{K}}(M) \to 1,$$

i.e., $G^k(M)$ *is a semidirect product of* $\mathrm{Diff}_{\mathbb{K}}(M)$ *and* $\mathfrak{X}^k(M)$.

Proof. (1) We prove uniqueness and existence of the extension of $X : M \to T^kM$ by using a chart representation $X : U \to T^kV$ of X. The statement to be proved is:

(A) *Let* V *and* W *be* $\mathbb{K}$*-modules and* $f : U \to T^kW$ *a smooth map (over* $\mathbb{K}$*) defined on an open set* $U \subset V$. *Then there exists a unique extension of* f *to a map* $\widetilde{f} : T^kU \to T^kW$ *which is smooth over the ring* $T^k\mathbb{K}$.

We first prove uniqueness. For $k = 1$, using that f is smooth over $T\mathbb{K}$, we get from the second order Taylor formula (cf. 1.5 (4)) for $x \in U$, $v \in V$,

$$\widetilde{f}(x + \varepsilon v) = \widetilde{f}(x) + \varepsilon d\widetilde{f}(x)v + \varepsilon^2 R_2(x, v, \varepsilon) = f(x) + \varepsilon df(x)v$$

since $\widetilde{f}(x) = f(x)$ and $d\widetilde{f}(x)v = df(x)v$. Thus $\widetilde{f}$ is uniquely determined by f. Similarly, for $k = 2$, using that $d\widetilde{f}$ and $d^2\widetilde{f}$ are smooth over $TT\mathbb{K}$ if so is $\widetilde{f}$,

$$\begin{aligned}\widetilde{f}(x + \varepsilon_2 v_2 + \varepsilon_1(v_1 + \varepsilon_2 v_{12}) &= \widetilde{f}(x + \varepsilon_2 v_2) + \varepsilon_1 d\widetilde{f}(x + \varepsilon_2 v_2)(v_1 + \varepsilon_2 v_{12}) \\ &= f(x) + \varepsilon_2 df(x)v_2 + \varepsilon_1 df(x)v_1 + \varepsilon_1\varepsilon_2(df(x)v_{12} + d^2f(x)(v_1, v_2)),\end{aligned}$$

and again $\widetilde{f}$ is uniquely determined by f. For general k, the same arguments lead to

$$\widetilde{f}(x + \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha) = f(x) + \sum_{\alpha > 0} \varepsilon^\alpha \Big(\sum_{l=1}^{|\alpha|} \sum_{\Lambda \in \mathcal{P}_l(\alpha)} d^l f(x)(v_{\Lambda^1}, \ldots, v_{\Lambda^l})\Big), \tag{28.2}$$

which is of course the chart formula (7.19) for T^kf, showing that $\widetilde{f}$ is determined by f. This proves uniqueness.

Next we prove the existence part of (A). By Theorem 7.2, $T^kf : T^kV \to T^k(T^kW)$ is smooth over the ring $T^k\mathbb{K}$. We will construct a canonical smooth map $\mu_k : T^k(T^kW) \to T^kW$ such that $\widetilde{f} := \mu_k \circ T^kf$ has the desired properties. Recall that $T^k\mathbb{K}$ is free over $\mathbb{K}$ with basis ε^α, $\alpha \in I_k$. The algebraic tensor product $T^k\mathbb{K} \otimes_{\mathbb{K}} T^l\mathbb{K}$ is again a commutative algebra over $\mathbb{K}$, isomorphic to $T^{k+l}\mathbb{K}$ after choice of a partition $\mathbb{N}_{k+l} = \mathbb{N}_k \dot{\cup} \mathbb{N}_l$. For any commutative algebra A, the product map $A \otimes A \to A$ is an algebra homomorphism, so we have, since $T^k\mathbb{K}$ is commutative, a homomorphism of $\mathbb{K}$-algebras

$$T^km : T^{2k}\mathbb{K} \cong T^k\mathbb{K} \otimes_{\mathbb{K}} T^k\mathbb{K} \to T^k\mathbb{K}$$

Explicitly, with respect to the $\mathbb{K}$-basis $\varepsilon^\alpha \otimes \nu^\beta$, $\alpha, \beta \in I_k$ (where the ν_i obey the same relations as the ε_i), it is given by

$$T^km\Big(\sum_{\alpha,\beta \in I_k} t_{\alpha,\beta}\, \varepsilon^\alpha \otimes \nu^\beta\Big) = \sum_{\gamma \in I_k} \varepsilon^\gamma \sum_{\substack{\alpha,\beta \in I_k \\ \alpha + \beta = \gamma}} t_{\alpha,\beta}.$$

For any $\mathbb{K}$-module V, the ring homomorphism $T^k m : T^{2k}\mathbb{K} \to T^k\mathbb{K}$ induces a homomorphism of modules

$$\mu_k : T^{2k}V = V \otimes_{\mathbb{K}} T^{2k}\mathbb{K} \to T^k V = V \otimes_{\mathbb{K}} T^k\mathbb{K},$$
$$x + \sum_{\substack{\alpha,\beta\in I_k \\ (\alpha,\beta)\neq 0}} \varepsilon^\alpha \nu^\beta v_{\alpha,\beta} \mapsto x + \sum_{\gamma\in I_k} \varepsilon^\gamma \sum_{\substack{\alpha,\beta\in I_k \\ \alpha+\beta=\gamma}} v_{\alpha,\beta}.$$

This map is $T^{2k}\mathbb{K}$-linear (recall that any ring homomorphism $R \to S$ induces an R-linear map $V_R \to V_S$ of scalar extensions; here $R = T^{2k}\mathbb{K}$, $S = T^k\mathbb{K}$). It is of class C^0 since it is a finite composition of certain sums and projections which are all C^0, and hence μ_k is of class C^∞ over $T^{2k}\mathbb{K}$. There are two imbeddings of $T^k\mathbb{K}$ as a subring of $T^{2k}\mathbb{K} \cong T^k\mathbb{K} \otimes T^k\mathbb{K}$ having image $1 \otimes T^k\mathbb{K}$, resp. $T^k\mathbb{K} \otimes 1$; thus μ_k is also smooth over these rings. Therefore, if we let

$$\widetilde{f} := \mu_k \circ T^k f : T^k U \to T^{2k} W \to T^k W,$$

$\widetilde{f}$ is smooth over $T^k\mathbb{K}$ (which we may identify with $T^k\mathbb{K} \otimes 1 \subset T^{2k}\mathbb{K}$). Let us prove that it is an extension: for all $x \in U$,

$$\widetilde{f}(x) = \mu_k(T^k f(x)) = \mu_k(f(x)) = f(x)$$

where the second equality holds since $T^k f(x) = T^k f(x+0) = f(x)$ and the third equality holds since $\mu_k(x + \sum_\alpha \varepsilon^\alpha v_{\alpha,0}) = x + \sum_\alpha \varepsilon^\alpha v_{\alpha,0}$, i.e. $\mu_k \circ \iota = \mathrm{id}_{T^k M}$, where $\iota : T^k V \to T^{2k} V$ is the imbedding induced by $T^k\mathbb{K} \to T^{2k}\mathbb{K}$, $r \mapsto r \otimes 1$. Thus claim (A) is completely proved.

Now we prove part (1) of the theorem. The extension $\widetilde{X}$ of X is defined, locally, in a chart representation; by the uniqueness statement of (A), this extension does not depend on the chart and hence is uniquely and globally defined. (See also Remark 28.4 for variants of this proof.) The chart formula given in the claim is simply Formula (28.2), where $f = X = \mathrm{pr}_1 + \sum_\alpha \varepsilon^\alpha X_\alpha$ is decomposed into its (chart-dependent) components according to Formula (28.1). All that remains to be proved is that $\widetilde{X}$ is bijective with smooth inverse. This will be done in the context of the proof of part (2).

(2) Clearly, $X \cdot Y$ is again a smooth section of $T^k M$, and hence the product is well-defined. The uniqueness part of (1) shows that $\widetilde{X \cdot Y} = \widetilde{X} \circ \widetilde{Y}$ since both sides are $T^k\mathbb{K}$-smooth extensions of $\widetilde{X} \circ Y$. This property implies associativity:

$$(X \cdot Y) \cdot Z = \widetilde{X \cdot Y} \circ Z = \widetilde{X} \circ \widetilde{Y} \circ Z = \widetilde{X} \circ (Y \cdot Z) = X \cdot (Y \cdot Z).$$

Moreover, for the zero section $z : M \to T^k M$ we have $\widetilde{z} = \mathrm{id}_{T^k M}$ and hence z is a neutral element.

All that remains to be proved is that X has an inverse in $\mathfrak{X}^k(M)$. Equivalently, we have to show that $\widetilde{X}$ is bijective and has an inverse of the form $\widetilde{Z}$ for some $Z \in \mathfrak{X}^k(M)$. This is proved by induction on k. In case $k = 1$, $\widetilde{X}$ is fiberwise translation in tangent spaces and hence is bijective with $(\widetilde{X})^{-1} = \widetilde{-X}$. For the case $k = 2$, see the explicit calculation of $(\widetilde{X})^{-1}$ in the proof of Theorem 14.3. For the proof of the induction step for general k, we give two slightly different versions:

(a) First version. Note that the chart formula for $X \cdot Y$ from the claim is a simple consequence of the chart formula for $\widetilde{X}(v)$ from Part (1) (take $v = Y(x)$). This formula shows that left multiplication by X, $l_X : \mathfrak{X}^k(M) \to \mathfrak{X}^k(M)$, $Y \mapsto X \cdot Y$, is an affine-multilinear self-map of the multilinear space $\mathfrak{X}^k(M)$ in the sense to be explained in the appendix to this chapter (Section 28.5, below). Moreover, l_X is unipotent, hence regular, and by the affine-multilinear version of Theorem MA.6 it is thus bijective. Now let $Z := (l_X)^{-1}(0)$ where $0 := z$ is the zero section. Then $X \cdot Z = l_X \circ (l_X)^{-1}(0) = 0$. Similarly, $Y \cdot X = 0$, where $Y := (r_X)^{-1}(0)$ is gotten from the bijective right multiplication by X. It follows that X is invertible and $X^{-1} = Z = Y$.

(b) Second version. Note first that, if $\varepsilon^\alpha X$, $\varepsilon^\alpha Y$ with $X, Y \in \mathfrak{X}(M)$ are purely vectorial, then $\varepsilon^\alpha X \cdot \varepsilon^\alpha Y = \varepsilon^\alpha (X + Y)$, as follows from the explicit formula in Part (2). Thus $\varepsilon^\alpha X$ is invertible with inverse $\varepsilon^\alpha(-X)$. One proves by induction that every element $X \in \mathfrak{X}^k(M)$ can be written as a product of purely vectorial sections (Theorem 29.2): $X = \prod_\alpha^\uparrow \varepsilon^\alpha X_\alpha$, and hence X is invertible.

(3) We have already remarked that $X \mapsto \widetilde{X}$ is a group homomorphism, and injectivity follows from the extension property. Let us prove that the imbedded image $\mathfrak{X}^k(M)$ is the kernel of the natural projection $G^k(M) \to \mathrm{Diff}_{\mathbb{K}}(M)$, $F \mapsto f$. In fact, $X := F \circ z : M \to T^k M$ is a section of $T^k M$ iff $f = \mathrm{id}_M$. But then, by the uniqueness statement in Part (1), $F = \widetilde{X}$; conversely, for any $X \in \mathfrak{X}^k(M)$, $\widetilde{X}$ clearly belongs to the kernel of the projection. Finally, we have already remarked in Section 28.1 that $f \mapsto T^k f$ is a splitting of the projection $G^k(M) \to G^0(M)$. ∎

28.4. *Comments on Theorem 28.3 and its proof.* For the finite-dimensional case over $\mathbb{K} = \mathbb{R}$ of Theorem 28.3 (2), see [KM87, Theorem 4.6] and [KMS93, Theorem 37.7], where the proof is carried out in the "dual" picture, corresponding to function-algebras. This proof has the advantage that the problem of bijectivity (i.e., the existence of inverses) is easily settled by using the Neumann series in a nilpotent associative algebra, but it does not carry over to our general situation. Moreover, in a purely real theory it is less visible why the extension of a section X to a diffeomorphism $\widetilde{X}$ is so natural.

Comparing with the proof of [KMS93, Theorem 37.7], the proof of the existence part in (1) may be reformulated as follows. We define $\widetilde{X} = \mu_{2k,k} \circ T^k X$ where $\mu_{2k,k} : T^{2k} M \to T^k M$ is the natural map which, in a chart representation, is given by $\mu_k : T^k(T^k V) \to T^k V$. Note that, for $k = 1$, $\mu_{2,1} : TTM \to TM$ can directly be defined by

$$\mu_{2,1} = a \circ (p_1 \times_M p_2) : TTM \to TM \times_M TM \to TM,$$

where $p_i : TTM \to TM$, $i = 1, 2$, are the canonical projections and $a : TM \times_M TM \to TM$ the addition map of the vector bundle $TM \to M$. In a chart,

$$\mu_{2,1}(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1 \varepsilon_2 v_{12}) = x + \varepsilon(v_1 + v_2)$$

which corresponds to the formula for $\mu_1 : TTV \to TV$. For general k, $\mu_{2k,k}$ may be obtained by deriving $\mu_{2,1} : T^2 M \to TM$:

$$\mu_{2k,k} := T^{2k-2}\mu_{2,1} \circ T^{2k-3}\mu_{2,1} \circ \ldots \circ T^{k-1}\mu_{2,1} : T^{2k} M \to T^k M.$$

Finally, one may note that any "natural" map $\nu : T^{2k} M \to T^k M$ leads to a product $X \cdot Y := \nu \circ TX \circ Y$, where the associativity of the product corresponds to the naturality of ν. For instance, "first" and "second" projection $T^{2k} M \to T^k M$ lead to the associative (but useless) products $X \cdot Y = X$, resp. $X \cdot Y = Y$.

28.5. Appendix to Chapter 28: the affine-multilinear group. Assume that $E = \oplus_{\substack{\alpha \in I_k \\ \alpha > 0}} V_\alpha$ is a multilinear space over a ring $\mathbb{K}$ (Chapter MA), and that the following data are given: for every $\alpha \in I_k$ and $\beta \subseteq \alpha$ and $\Lambda \in \mathcal{P}(\beta)$, assume given a multilinear map

$$C^{\Lambda;\alpha} : V_{\Lambda^1} \times \ldots \times V_{\Lambda^\ell} \to V_\alpha .$$

We say that $f : E \to E$ is *affine-multilinear* if it is of the form

$$f(\sum_\alpha v_\alpha) = \sum_\alpha \sum_{\beta \subset \alpha} \sum_{\Lambda \in \mathcal{P}(\beta)} C^{\Lambda;\alpha}(v_{\Lambda^1}, \ldots, v_{\Lambda^\ell})$$

for some family $(C^{\Lambda;\alpha})$. Note that for $\beta = \emptyset$, $C^{\Lambda;\alpha}$ is a constant belonging to V_α; in particular, for $k = 1$, we get the usual affine group of $E = V_1$. For $\beta = \alpha$, $C^{\Lambda;\alpha} =: b^\Lambda$ contributes to the "multilinear part" of f which is defined by

$$F : E \to E, \quad \sum_\alpha v_\alpha \mapsto \sum_\alpha \sum_{\Lambda \in \mathcal{P}(\alpha)} b^\Lambda(v_{\Lambda^1}, \ldots, v_{\Lambda^\ell}).$$

Clearly, F is a multilinear map in the sense of Section MA.5. We say that f is *regular* (resp. *unipotent*) if so is F (cf. Def. MA.5). Then the following analog of Theorem MA.6 (2) holds: *An affine-multilinear map f is invertible iff it is regular, and then the inverse of f is again affine-multilinear.* (The proof is left to the reader.) Thus we have defined groups $\mathrm{Am}^{0,k}(E)$, resp. $\mathrm{Am}^{1,k}(E)$, of regular (resp. unipotent) affine-multilinear maps. The structure of these groups will be investigated elsewhere.

29. The exponential jet for vector fields

29.1. *The "Lie algebra of $\mathfrak{X}^k(M)$".* As explained in the introduction, heuristically we may think of $G := \mathrm{Diff}_{\mathbb{K}}(M)$ as a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{X}(M)$. Then the group $G^k(M)$ takes the rôle of T^kG and the group $\mathfrak{X}^k(M)$ the one of $(T^kG)_e$. The Lie algebra of T^kG is then $T^k\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{K}} T^k\mathbb{K}$ and therefore the one of $\mathfrak{X}^k(M)$ has to be $(T^k\mathfrak{g})_0 = \oplus_{\alpha>0}\varepsilon^\alpha\mathfrak{g}$. It is the space of sections of the *axes-bundle* $A^kM = \bigoplus_{\alpha\in I_k,\alpha>0}\varepsilon^\alpha TM$. The space of sections of A^kM will be denoted by

$$\mathcal{A}^k(M) := \Gamma(M, A^kM) = \bigoplus_{\alpha\in I_k,\alpha>0} \varepsilon^\alpha\mathfrak{X}(M). \tag{29.1}$$

We will construct canonical bijections $\mathcal{A}^k(M) \to \mathfrak{X}^k(M)$, taking the rôle of left-trivialization, resp. of an exponential map.

Theorem 29.2. *There is a bijective "left trivialization" of the group $\mathfrak{X}^k(M)$, given by*

$$\mathcal{A}^k(M) \to \mathfrak{X}^k(M), \quad (X_\alpha)_\alpha \mapsto \prod^{\uparrow}_{\substack{\alpha\in I_k\\ \alpha>0}} \varepsilon^\alpha X_\alpha,$$

where the product is taken in the group $\mathfrak{X}^k(M)$, with order corresponding to the lexicographic order of the index set I_k. Then the product in $\mathfrak{X}^k(M)$ is given by

$$\prod^{\uparrow}_{\alpha}\varepsilon^\alpha X_\alpha \cdot \prod^{\uparrow}_{\beta}\varepsilon^\beta X_\beta = \prod^{\uparrow}_{\gamma}\varepsilon^\gamma Z_\gamma$$

with Z_γ given by the "left product formula" from Theorem 24.7.

Proof. First of all, note that Theorem 14.4 implies the following commutation relations for purely vectorial elements in the group $\mathfrak{X}^k(M)$: for all vector fields X_α, $Y_\beta \in \mathfrak{X}(M)$,

$$[\varepsilon^\alpha X_\alpha, \varepsilon^\beta Y_\beta] = \varepsilon^{\alpha+\beta}[X_\alpha, Y_\beta], \tag{29.2}$$

where the last bracket is taken in the Lie algebra $\mathfrak{X}(M)$. Now we define the map $\Psi : \mathcal{A}^k(M) \to \mathfrak{X}^k(M)$, $(X_\alpha)_\alpha \mapsto \prod^{\uparrow}_{\substack{\alpha\in I_k\\ \alpha>0}} \varepsilon^\alpha X_\alpha$ as in the claim and show that it is a bijection. We claim that the image of Ψ is a subgroup of $\mathfrak{X}^k(M)$. In fact, this follows from the relations (29.2): re-ordering the product of two ordered products is carried out by the procedure described in the proof of Theorem 24.7 and leads to the left-product formula mentioned in the claim. Thus the image of Ψ is the subgroup of $\mathfrak{X}^k(M)$ generated by the purely vectorial sections, with group structure as in the claim. But this group is in fact the whole group $\mathfrak{X}^k(M)$: using the explicit chart formula for the product in $\mathfrak{X}^k(M)$ (Theorem 28.3 (2)), it is seen that Ψ is a unipotent multilinear map between multilinear spaces over $\mathbb{K}$, whence is bijective (Theorem MA.6). ■

In the same way as in the theorem, a "right-trivialization map" $\mathcal{A}^k(M) \to \mathfrak{X}^k(M)$ is defined. Next we prove the analog of Theorem 25.2 on the exponential map:

Theorem 29.3. *Assume that the integers are invertible in* $\mathbb{K}$. *Then there is a unique map* $\exp_k : \mathcal{A}^k(M) \to \mathfrak{X}^k(M)$ *such that:*

(1) *In every chart representation,* $\exp_k$ *is a polynomial map (of degree at most* k*); equivalently,* $\exp_k^L := \Psi^{-1} \circ \exp_k : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$, *with* Ψ *as in the preceding theorem, is a polynomial map,*

(2) *for all* $n \in \mathbb{Z}$ *and* $X \in \Gamma(M, A^k M)$, $\exp(nX) = (\exp X)^n$,

(3) *on the axes,* $\exp_k$ *agrees with the inclusion maps, i.e., if* $X : M \to TM$ *is a vector field, then* $\exp(\varepsilon^\alpha X) = \varepsilon^\alpha X$ *is the purely vectorial section corresponding to* $\varepsilon^\alpha X$.

Proof. As for Theorem 25.2, this in an immediate consequence of Theorem PG.6, applied to the polynomial group $\mathfrak{X}^k(M)$. ■

Theorem 29.4. *The map* $\exp_k$ *from the preceding theorem commutes with the canonical action of the symmetric group* Σ_k *and hence restricts to a bijection*

$$\exp_k : (\mathcal{A}^k(M))^{\Sigma_k} = \oplus_{j=1}^k \delta^{(j)}\mathfrak{X}(M) = \Gamma^\infty(\oplus_{j=1}^k \delta^{(j)} TM) \to (\mathfrak{X}^k(M))^{\Sigma_k} = \Gamma^\infty(J^k M).$$

In particular, we get an injection

$$\mathfrak{X}(M) \to G^k(M), \quad X \mapsto \widetilde{\exp_k(\delta X)}$$

such that $\widetilde{\exp_k(\delta n X)} = (\widetilde{\exp_k(\delta X)})^n$ *for all* $n \in \mathbb{Z}$.

Proof. This is the analog of Theorem 25.4, and it is proved in the same way. ■

29.5. *Smooth actions of Lie groups.* Assume G is a Lie group over $\mathbb{K}$ and $a : G \times M \to M$ a smooth action of G on M. Deriving, we get an action $T^k a : T^k G \times T^k M \to T^k M$ of $T^k G$ on $T^k M$. We claim that $T^k a(g, \cdot)$ belongs to the group $G^k(M)$. Indeed, since $T^k a$ is smooth over $T^k\mathbb{K}$, it follows that $T^k G$ acts by $T^k\mathbb{K}$-diffeomorphisms of $T^k M$, and since $T^k a$ is a bundle homomorphism with a being the corresponding map on the base, we see that $(T^k G)_e$ preserves fibers over M. Therefore we may consider $T^k a$ and $(T^k a)_e$ as group homomorphisms

$$T^k a : T^k G \to G^k(M), \quad \text{resp.} \quad (T^k a)_e : (T^k G)_e \to \mathfrak{X}^k(M).$$

In particular, for $k = 1$ we get a homomorphism of abelian groups $\mathfrak{g} = T_e G \to \mathfrak{X}(M)$, associating to $X \in \mathfrak{g}$ the "induced vector field on M", classically often denoted by $X_* \in \mathfrak{X}(M)$. This map is a Lie algebra homomorphism since $T^2 a$ is a group homomorphism and the Lie bracket on $\mathfrak{g}$, resp. on $\mathfrak{X}(M)$, can be defined in similar ways via group commutators (Theorem 14.5 and Theorem 23.2). Taking a direct sum of such maps, we get a Lie algebra homomorphism $(T^k\mathfrak{g})_0 \to \mathcal{A}^k(M)$, and from the uniqueness property of the exponential map we see that the following diagram commutes:

$$\begin{array}{ccc} (T^k G)_e & \stackrel{T^k a}{\to} & \mathfrak{X}^k(M) \\ \exp \uparrow & & \uparrow \exp_k \\ (T^k \mathfrak{g})_0 & \to & \mathcal{A}^k(M) \end{array}$$

Summing up, everything behaves as if $a : G \to \mathrm{Diff}(M)$ were a honest Lie group homomorphism.

29.6. *Relation with the flow of a vector field.* The diffeomorphism $\widetilde{\exp_k(\delta X)}$ of $T^k M$ (Theorem 29.4) can be related to the k-jet of the flow of X, if flows exist. Let us explain this. We assume that $\mathbb{K} = \mathbb{R}$ and M is finite-dimensional (or a Banach manifold). Then any vector field $X \in \mathfrak{X}(M)$ admits a flow map $(t,x) \mapsto \mathrm{Fl}_t^X(x)$, defined on some neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ (cf. [La99, Ch. IV]). In general, Fl_1^X is not defined as a diffeomorphism on the whole of M, and this obstacle prevents us from defining the exponential map of the group $G = \mathrm{Diff}(M)$ by $\exp(X) := \mathrm{Fl}_1^X$. Nevertheless, this is what one would like to do, and one may give some sense to this definition by fixing some point $p \in M$ and by considering the subalgebra

$$\mathfrak{X}(M)_p := \{X \in \mathfrak{X}(M) |\, X(p) = 0, \forall Y \in \mathfrak{X}(M) : [X,Y]_p = 0\}$$

of vector fields vanishing at p of order at least 2. Then it is easily seen from the proof of the existence and uniqueness theorem for local flows that, for all $X \in \mathfrak{X}(M)_p$, $\exp(X) := \mathrm{Fl}_1^X$ is defined on some open neighborhood of p, and moreover that $T_p(\exp(X)) = \mathrm{id}_{T_pM}$ (cf., e.g., [Be96, Appendix (A2)]). Thus the group $\mathrm{Diff}(M)_p$ of germs of local diffeomorphisms that are defined at p and have trivial first order jet there, admits an exponential map given by $\exp = \mathrm{Fl}_1 : \mathfrak{X}(M)_p \to \mathrm{Diff}(M)_p$. We claim that, under these assumptions, the k-jet of Fl_1^X at p is given by the map from Theorem 29.4:

$$(\widetilde{\exp_k(\delta X)})_p = (T^k(\mathrm{Fl}_1^X))_p,$$

or, in other words, that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Diff}(M)_p & \overset{T^k}{\to} & G^k(M) & \overset{\mathrm{ev}_p}{\to} & \mathrm{Am}^{k,1}(T^kM)_p \\
\mathrm{Fl}_1 \uparrow & & & & \uparrow \mathrm{ev}_p \\
\mathfrak{X}(M)_p & \overset{X \mapsto \delta X}{\to} & \mathcal{A}^k(M) & \overset{\exp_k}{\to} & G^k(M)
\end{array}$$

where $\mathrm{Am}^{k,1}(T^kM)_p$ denotes the affine-multilinear group of the fiber $(T^kM)_p$. The proof is by using the uniqueness of the map exp from Theorem PG.6: we contend that the map $X \mapsto (T^k(\mathrm{Fl}_1^X))_p$ also has the properties (1) and (2) from Theorem PG.6 and hence coincides with the exponential map constructed above by using Theorem PG.6. Now, Property (1) follows by observing that

$$T^k(\mathrm{Fl}_1^{nX}) = T^k((\mathrm{Fl}_1^X)^n) = (T^k(\mathrm{Fl}_1^X))^n,$$

and Property (2) follows from the fact that $T_p(\mathrm{Fl}_1^X) = \mathrm{id}_{T_pM}$ (here we need the fact that X vanishes at p of order at least two, cf. [Be96, Appendix (A.2)]) and hence our candidate for the exponential mapping acts trivially on axes, as it should. Summing up, for any $X \in \mathfrak{X}(M)_p$ the k-jet of $\exp(X)$ at p may be constructed in a purely "formal way", i.e., without using the integration of differential equations. If X does not belong to $\mathfrak{X}(M)_p$, then there still is a relation, though less canonical, with the k-jet of the flow of X, see the next subsection.

29.7. *Relation with "Chapoton's formula".* If X is a vector field on $V = \mathbb{R}^n$, identified with the corresponding map $V \to V$, then the local flow of X can be written

$$\mathrm{Fl}_t^X(v) = v + \sum_{j \geq 1} \frac{t^j}{j!} X^{\star j}(v)$$

where

$$(X \star Y)(x) = (\nabla_X Y)(x) = dY(x) \cdot X(x)$$

denotes the product on $\mathfrak{X}(V)$ given by the canonical flat connection of V, and $X^{\star j+1} = X \star X^{\star j}$ are the powers with respect to this product – see [Ch01, Prop. 4]. Thus, letting $t=1$, the exponential map associated to the Lie algebra of vector fields on V can be written as a formal series

$$\exp(X) = \mathrm{id}_V + \sum_{j\geq 1} \frac{t^j}{j!} X^{\star j}.$$

This formal series may be seen as the projective limit of our $\exp_k(\delta X) : V \to J^k V$ for $k \to \infty$. In fact, according to Theorem PG.6, we have the explicit expression of the exponential map of the polynomial group $\mathfrak{X}^k(V)$ by $\exp(\delta X) = \sum_{j=1}^k \frac{1}{j!}\psi_{j,j}(\delta X)$ where $\psi_{j,j}(X)$ is gotten from the homogeneous term of degree j in the iterated multiplication map $m^{(j)}$. The chart formula in the global chart V for the product (Theorem 28.3 (2)) permits to calculate these terms explicitly, leading to the formula

$$\exp_k(\delta X) = \mathrm{id}_V + \sum_{j\geq 1} \frac{\delta^{(j)}}{j!} X^{\star j}.$$

A variant of these arguments is implicitly contained in [KMS93, Prop. 13.2] where the exponential mapping of the "jet groups" is determined: for a vector field X vanishing at the origin, $\exp(j_0^k X) = j_0^k \mathrm{Fl}_1^X$.

30. The exponential jet of a symmetric space

30.1. As an application of the preceding results, we will construct a canonical isomorphism $\mathrm{Exp}_k : A^k M \to T^k M$ (called the *exponential jet*, cf. Introduction) for symmetric spaces over $\mathbb{K}$. In order to motivate our construction, we start by considering the group case, where such a construction has already been given in Chapter 25 (cf. also Section 29.5). We relate those results to the context explained in the preceding chapters.

30.2. *Case of a Lie group.* We continue to assume that the integers are invertible in $\mathbb{K}$. We construct the exponential jet $\mathrm{Exp}_k : A^k G \to T^k G$ of a Lie group G over $\mathbb{K}$ as follows: for $X \in \mathfrak{g}$, let $X^R \in \mathfrak{X}(G)$ be the right-invariant vector field such that $X^R(e) = X$. The "extension map"

$$l_R : \mathfrak{g} \to \mathfrak{X}(G), \quad X \mapsto l_R(X) := X^R$$

is linear and injective. Denote by

$$(T^k l_R)_0 : (T^k \mathfrak{g})_0 \to (T^k \mathfrak{X}(G))_0 = \mathcal{A}^k(G)$$

its scalar extension (simply a direct sum of copies of l_R). Then the exponential map $\mathrm{Exp} = \mathrm{Exp}_k : (T^k \mathfrak{g})_0 \to (T^k G)_e$ of the polynomial group $(T^k G)_e$ (Theorem 25.2) is recovered from these data via

$$\mathrm{Exp}_k(X) = (\exp((T^k l_R)X))(e) : \qquad \begin{array}{ccc} (T^k G)_e & \overset{\mathrm{ev}_e}{\leftarrow} & \mathfrak{X}^k(G) \\ \mathrm{Exp} \uparrow & & \uparrow \exp_k \\ (T^k \mathfrak{g})_0 & \overset{(T^k l_R)_0}{\to} & \mathcal{A}^k(G) \end{array}$$

In fact, this is seen by the arguments given in Section 29.5, applied to the action of G on itself by left translations.

In the case of finite dimension over $\mathbb{K} = \mathbb{R}$, or more generally for real Banach Lie-groups, the group G does admit an exponential map $\mathrm{Exp} : \mathfrak{g} \to G$ in the sense explained in Section 25.9, and it is well-known that then the exponential map of G is given by $\exp(X) = \mathrm{Fl}_1^{X^R}(e)$, i.e., by the commutative diagram

$$\begin{array}{ccc} G & \overset{\mathrm{ev}_e}{\leftarrow} & \mathrm{Diff}(G) \\ \mathrm{Exp} \uparrow & & \uparrow \mathrm{Fl}_1 \\ \mathfrak{g} & \overset{l_R}{\to} & \mathfrak{X}(G)_c \end{array}$$

(where $\mathfrak{X}(G)_c$ is the set of complete vector fields on G). The preceding characterization of $\mathrm{Exp} : (T^k \mathfrak{g})_0 \to (T^k G)_e$ is obtained by applying, formally (i.e., as if $\mathrm{Diff}(G)$ were a Lie group), the functor $(T^k)_0$ to this diagram.

30.3. *Exponential jet of a symmetric space.* Assume M is a symmetric space over $\mathbb{K}$ with base point $o \in M$. Recall from the proof of Prop. 5.9 that every tangent vector $v \in T_o M$ admits a unique extension to a vector field $l(v)$ (denoted by $\widetilde{v}$ in Chapter 5) such that $l(v)$ is a derivation of M and $\sigma_o \cdot l(v) = -l(v)$. The "extension map" $l : T_o M \to \mathfrak{X}(M)$ is linear, and its higher order tangent map $(T^k l)_0 : (A^k M)_o \to \mathcal{A}^k(M)$ is simply a direct sum $\oplus_{\alpha > 0} \varepsilon^\alpha l$ of copies of this map.

Theorem 30.4. *Let (M,o) be a symmetric space over $\mathbb{K}$ and assume that the integers are invertible in $\mathbb{K}$. Then the map* $\mathrm{Exp} := \mathrm{Exp}_k : (A^k M)_o \to (T^k M)_o$ *defined by*

$$\mathrm{Exp}_k(v) = (\exp_k(T^k l(v)))(o) : \qquad \begin{array}{ccc} (T^k M)_o & \overset{\mathrm{ev}_o}{\leftarrow} & \mathfrak{X}^k(M) \\ \mathrm{Exp} \uparrow & & \uparrow \exp_k \\ (A^k M)_o & \overset{(T^k l)_0}{\rightarrow} & \mathcal{A}^k(M) \end{array}$$

is a unipotent multilinear isomorphism of fibers over $o \in M$ commuting with the action of the symmetric group Σ_k. Moreover, Exp_k satisfies the "one-parameter subspace property" from Section 27.7*: for all $u \in (A^k M)_o$, the map*

$$\gamma_u : \mathbb{K} \to (T^k M)_o, \quad t \mapsto \mathrm{Exp}_k(tu)$$

is a homomorphism of symmetric spaces such that $\gamma_u'(0) = u$. In particular, the property $\mathrm{Exp}(nu) = (\mathrm{Exp}(u))^n$ holds for all $n \in \mathbb{Z}$ (where the powers in the symmetric space $(T^k M)_o$ are defined as in Equation (5.5)*).*

Proof. By its definition, Exp_k is a composition of multilinear maps and hence is itself multilinear. In order to prove that Exp_k is bijective it suffices, according to Theorem MA.6, to show that Exp_k is unipotent, i.e., the restriction of Exp_k to axes is the identity map. But this is an immediate consequence of the corresponding property of the map $\exp_k$ (Theorem 29.3, (3)): let $v = \varepsilon^\alpha v_\alpha$ be an element of the axis $\varepsilon^\alpha V$; then $X := T^k l(\varepsilon^\alpha v_\alpha) = \varepsilon^\alpha l(v_\alpha)$ is a purely vectorial section, hence $\exp_k$ applied to this section is simply given by inclusion of axes. Evaluating at the base point, we get

$$(\exp_k(\varepsilon^\alpha l(v_\alpha)))(o) = \varepsilon^\alpha l(v_\alpha)(o) = \varepsilon^\alpha v_\alpha$$

since $\mathrm{ev}_o \circ l = \mathrm{id}$. The map Exp commutes with the Σ_k-action since so do all maps of which it is composed.

Finally, we prove the one-parameter subspace property. In a first step, we show that, for all $n \in \mathbb{Z}$, $\mathrm{Exp}(nu) = \mathrm{Exp}(u)^n$. We write $u = x + \sum_\alpha \varepsilon^\alpha v_\alpha$; then, using that $\exp_k(nX) = (\exp_k(X))^n$,

$$\begin{aligned} \mathrm{Exp}(nu) &= \exp_k(n \sum_\alpha \varepsilon^\alpha l(v_\alpha)).o = (\exp(\sum_\alpha \varepsilon^\alpha l(v_\alpha)))^n.o \\ &= (\exp(\sum_\alpha \varepsilon^\alpha l(v_\alpha)).o)^n = (\mathrm{Exp}\, u)^n, \end{aligned}$$

where the third equality is due to a general property of symmetric spaces: if (N,p) is a symmetric space with base point and $g \in \mathrm{Aut}(N)$ is such that $\sigma_p \circ g \circ \sigma_p = g^{-1}$, then $(g.p)^n = g^n.p$. (This is proved by an easy induction, using the definition of the powers in a symmetric space given in Equation (5.5).) This remark is applied to $(N,p) = ((T^k M)_o, 0_o)$ and $g = \exp_k(\sum_\alpha \varepsilon^\alpha l(v_\alpha))$.

Now let $\gamma(t) := \mathrm{Exp}(tu)$. Then, in any chart representation, $\gamma(t) = tu$ modulo higher order terms in t (since Exp is unipotent, as already proved), and hence $\gamma'(0) = u$. We have to show that $\gamma : \mathbb{K} \to T^k M$ is a homomorphism of symmetric spaces. As we have just seen, $\gamma(n) = (\gamma(1))^n$. By power associativity in symmetric spaces (cf. Equation (5.7)), and denoting by μ the binary product in the symmetric spaces $\mathbb{K}$, resp. $T^k M$, this implies $\gamma(\mu(n,m)) = \gamma(2n-m) = (\gamma(1))^{2n-m} = \mu(\gamma(m), \gamma(n)$, and thus the restriction of γ to $\mathbb{Z}$ is an "algebraic symmetric space homomorphism". But $\gamma : \mathbb{K} \to (T^k M)_o$ is a polynomial map between two symmetric spaces with polynomial multiplication maps, and hence by the "polynomial density argument" used several times in Section PG, it follows that also γ must be a homomorphism. ∎

As in Step 1 of the proof of Theorem PG.6, it is seen that γ_u is the only *polynomial* one-parameter subspace with $\gamma_u'(0) = u$. We do not know wether this already implies that γ_u is the only *smooth* one-parameter subspace with this property.

In the finite-dimensional case over $\mathbb{K} = \mathbb{R}$, it is known that the exponential map Exp_o of a symmetric space M with base point o can be defined by

$$\mathrm{Exp}_o(v) = \mathrm{Fl}_1^{l(v)}(o) = \mathrm{ev}_o \circ \mathrm{Fl}_1 \circ l : \qquad \begin{array}{ccc} M & \stackrel{\mathrm{ev}_o}{\leftarrow} & \mathrm{Diff}(M) \\ \mathrm{Exp}_o \uparrow & & \uparrow \mathrm{Fl}_1 \\ T_o M & \stackrel{l}{\rightarrow} & \mathfrak{X}(M)_c \end{array}$$

(see [Lo69] or [Be00, I.5.7]). As in the case of Lie groups, the definition of Exp from the preceding theorem is formally obtained by applying the functor $(T^k)_0$ to this diagram.

31. Remarks on the exponential jet of a general connection

31.1. *The exponential jet.* As explained in the Introduction, for real finite dimensional or Banach manifolds M equipped with a (torsionfree) linear connection on TM, we can define a canonically associated bundle isomorphism $T^k \operatorname{Exp} : A^k M \to T^k M$, called the *exponential jet (of order k).* For a general base field or -ring $\mathbb{K}$, there is no exponential map in the classical sense, but still it is possible to construct a bundle isomorphism $A^k M \to T^k M$ having all the good properties of the exponential jet and coinciding with $T^k \operatorname{Exp}$ in the real finite-dimensional or Banach case. In the following, we explain the basic ideas of two different approaches to such a construction; details will be given in later work.

31.2. *Vector field extensions.* The definition of the exponential jet for Lie groups and symmetric spaces in the preceding chapter suggests the following approach: in the real finite-dimensional case, every connection L gives rise to a, locally, on a star-shaped neighborhood U of x, defined "vector field extension map" $l := l_x : T_x M \to \mathfrak{X}(U)$, assigning to $v \in T_x M$ the *adapted vector field* $l(v)$ which is defined by taking as value $(l(v))(p) \in T_p M$ the parallel transport of v along the unique geodesic segment joining x and p in U. Then clearly l is linear, and $l(v)(x) = v$. Moreover, one may check that the covariant derivative of L is recovered by $(\nabla_X Y)(x) = [l_x(X(x)), Y](x)$. The exponential map Exp_x is given, as in the preceding chapter, by $\operatorname{Exp}_x(v) = (\operatorname{Fl}_1^{l(v)})(x)$.

For general base fields and rings $\mathbb{K}$, one may formalize the properties of such "vector field extension maps". One does not really need the whole vector field $l(v) \in \mathfrak{X}(U)$, but only its k-jet at x. Then one may try to define the exponential jet in a "synthetic way" by using the exponential maps of the polynomial groups $\operatorname{Gm}^{1,k}(T^k M)_x$ and $\operatorname{Am}^{1,k}(T^k M)_x$ acting on the fiber over x.

31.3. *The geodesic flow.* Another "synthetic" approach to the exponential jet is motivated as follows: in the classical case, if the connection is complete, the geodesic flow $\Phi_t : TM \to TM$ is the flow of a vector field $S : TM \to T(TM)$ which is called the *spray of the connection* and which is equivalent to the (torsionfree part of) the connection itself (cf. Chapter 11). Then the exponential map $\operatorname{Exp}_x : T_x M \to M$ at a point $x \in M$ is recovered from the flow Fl_1^S via $\operatorname{Exp}_x(v) = \pi(\operatorname{Fl}_1^S(v))$, where $\pi : TM \to M$ is the canonical projection:

$$\begin{array}{ccc} TM & \overset{\operatorname{Fl}_1^S}{\to} & TM \\ \uparrow & & \downarrow \\ T_x M & \overset{\operatorname{Exp}_x}{\to} & M \end{array}$$

Applying the k-th order tangent functor T^k to this diagram and restricting to the fiber over 0_x, we express the exponential jet via the k-jet of Fl_1^S and canonical maps:

$$\begin{array}{ccc} T^{k+1}M & \overset{T^k \operatorname{Fl}_1^S}{\to} & T^{k+1}M \\ \uparrow & & \downarrow \\ T^k(T_x M) & \overset{T^k(\operatorname{Exp}_x)}{\to} & T^k M \end{array}$$

As explained in Chapter 29, under some conditions, the map $T^k \operatorname{Fl}_1^S$ can be expressed in our general framework via $\exp(\delta S)$, and thus we should get an expression of the exponential jet in terms of maps that all can be defined in a "synthetic" way. However, the problem in the preceding diagram is that we cannot simply restrict everything to the fiber over $x \in M$, for Exp_x does not take values in some "fiber over x". One may try to overcome this problem by applying the tangent functor once more, or, in other words, by introducing one more scalar extension, thus imbedding $A^k M$ into $T^{k+2}M$ and not into $T^{k+1}M$ and defining

$$\mathrm{Exp}_k := \mathrm{pr} \circ \widetilde{\exp(\delta S)} \circ \iota : \qquad \begin{array}{ccc} T^{k+1}(TM) & \overset{\widetilde{\exp(\delta S)}}{\to} & T^{k+1}(TM) \\ \iota \uparrow & & \downarrow \mathrm{pr} \\ A^k M & \overset{\mathrm{Exp}_k}{\to} & T^k M \end{array}$$

where the imbedding ι and the projection pr are defined by suitable choices of k infinitesimal units among the $k+2$ infinitesimal units $\varepsilon_0, \ldots, \varepsilon_{k+1}$, and $\widetilde{\exp(\delta S)}$: $T^{k+2}M \to T^{k+2}M$ is associated to the spray $S : T_{\varepsilon_0}M \to T_{\varepsilon_1}T_{\varepsilon_0}M$ the spray of the connection L in the way described by Theorem 29.4.

31.4. *Jet of the holonomy group.* A major interest of the construction of the exponential jet, either in the way as indicated in 31.2 or as in 31.3, should be the possibility to give a "synthetic" theory of the (restricted) *holonomy group* of (M, L) at x, i.e., to define intrinsically its k-jet for all $k \in \mathbb{N}$, by using only "infinitesimal parallel transport" which may be expanded into its "Taylor series" at arbitrary order. We hope to come back to this topic in future work.

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