

Differential Geometry over General Base Fields and Rings. Part II: Higher Order Geometry

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Abstract. In the general framework introduced in Part I of this work, we investigate the structure of higher order tangent bundles $T^k F$ of a vector bundle F over M and in particular the structure of $T^{k+1} M$: we describe these bundles as *multilinear bundles over M* , and we define *multilinear connections* on these bundles to be certain vector bundle structures L on $T^k F$ over the base M . Such connections can be constructed by deriving linear connections on F , and *curvature* measures the non-linearity of the action of the symmetric group Σ_k on the vector bundle $(T^k F, L)$.

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bundle, curvature, differential operator, jet, linear connection, multilinear connection

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Apologie

L'entreprise d'écrire un autre traité de géométrie différentielle nécessite peut-être une justification, d'autant plus que la longueur du texte dépasse déjà largement les bornes prévues au moment de la prépublication de la première partie de ce texte (prépublication de l'Institut Elie Cartan 2003/47) et qu'il est toujours incomplet. Je ne sais pas si un jour il sera vraiment complet, et pour cette raison je profite de la possibilité de prépublication comme un moyen de communication plutôt "interne", destiné à informer des collègues et à susciter des discussions, remarques ou questions. J'ai divisé le texte en plusieurs parties : la partie IV est purement algébrique et indépendante du reste, la partie III est en grande partie indépendante de la partie II ; c'est la partie II qui est la plus incomplète. J'espère pouvoir mettre bientôt une rédaction de l'ensemble du texte à la disposition du lecteur sur ma page web.

Pour justifier mon entreprise, j'aurais envie de dire qu'il s'agit d'un essai d'ajouter aux "Eléments de Mathématique" de N. Bourbaki un tome qui y manque toujours, à savoir un tome intitulé "géométrie différentielle". Pourquoi Bourbaki n'a-t-il pas attaqué ce sujet ? Certainement une raison en est que, du point de vue bourbachique, la géométrie différentielle n'était pas (encore) suffisamment "formalisée" ou "algébrisée". Dans la théorie telle qu'elle est présentée habituellement, il faut invoquer, tôt ou tard, des théorèmes sur l'existence et unicité de solutions d'équations différentielles, limitant par là la théorie aux cas de dimension finie (ou banachiques) sur les nombres réels ou complexes ; souvent, les résultats sont de la forme "il existe un voisinage tel que...". Par contre, on attend d'une théorie "algébrisée" qu'elle s'applique dans un cadre très général, sans limiter la dimension ou trop gêner le choix du corps ou anneau de base, et que ses résultats ne soient pas de la forme "il existe $\varepsilon > 0$ tel que...". Appellons "géométrie différentielle formelle" les parties de la géométrie différentielle usuelle qui répondent à ces attentes. Une telle théorie existe ; c'est ce que je cherche à montrer par cette série de travaux. L'idée est simple : il faut formaliser le concept (si cher aux français et difficilement traduisable dans d'autres langues) de "développement limité", d'abord en un calcul différentiel général, puis l'utiliser systématiquement dans le cadre des variétés, sans jamais intégrer quoi que ce soit. J'essaierai d'expliquer dans ce travail pourquoi, à mon avis, l'utilisation systématique des fibrés tangents itérés TTM, T^3M, \dots, T^kM , est la méthode idéale pour unifier ce point de vue rigoureusement formel avec la tradition classique de la géométrie différentielle. En effet, les fibrés T^kM (ou leur limite projective, si l'on veut) remplacent bien les séries formelles, et ils posent moins de problèmes en caractéristique positive. Bourbaki n'a pas vu ces aspects purement formels et algébriques du calcul différentiel et s'est donc forcé de les retrouver plutôt dans le contexte des variétés et groupes de Lie analytiques. Or, au bout de compte, ce choix n'est pas naturel car la structure différentielle est une structure beaucoup plus élémentaire et générale que la structure analytique. Heureusement, dans le volume "Groupes et Algèbres de Lie" [Bou72], l'aspect formel domine tellement sur l'aspect analytique que l'on peut facilement généraliser une bonne partie de la théorie au cadre différentiel (voir la partie III de ce travail); ceci est beaucoup moins vrai pour le fascicule de résultats "Variétés différentielles et analytiques" [Bou67], et enfin pour la géométrie différentielle proprement dite il n'existe même pas un fascicule de résultats. La partie II du présente texte n'est qu'une mise en place d'un cadre de "géométrie différentielle formelle", et je suis parfaitement conscient du fait que le vrai travail reste encore à faire.

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W.B.

Introduction to Part II

In Part I of this work we defined manifolds M over very general topological base fields or rings \mathbb{K} , modelled on arbitrary topological \mathbb{K} -modules, and we identified the tangent and jet functors, T^k , resp. J^k , with functors of scalar extension by jet rings $T^k\mathbb{K}$, resp. $J^k\mathbb{K}$; we started to investigate the structure of bundles like TTM over M , called *bilinear bundles*, and introduced vector bundle structures over M on such bundles, which correspond to *linear connections*.

In the subsequent Part II, *Higher order geometry*, we extend the bilinear theory of TTM and of TF to a multilinear theory of higher order tangent bundles $T^{k+1}M$, resp. T^kF . The purely algebraic ingredients of the theory are extracted into an appendix which gained independence and now forms Part IV, *Geometric multilinear algebra*. The theory of Lie groups and symmetric spaces occupies Part III, *Lie theory*, which, although inspired by Parts II and IV, is almost independent of the other parts.

1. Formal Differential Geometry. The present Part II is at the same time the central and most incomplete part of the whole work: it is an attempt to formalize in a most conceptual way the idea of a “limited expansion”¹ of finite order k . The most naive idea might be to use the higher order tangent functors T^k to take the role of the “ k -th order Taylor expansion functor”. However, one immediately realizes that the “differential information” is stored, in the higher order tangent maps $T^k f$ on the one hand and in a k -th order Taylor polynomial on the other hand, in two completely different ways: polynomials form a linear, graded space, whereas the tangent bundles $T^k M$ are non-linear and filtered, but not graded; moreover, the permutation group Σ_k acts in a non-trivial way on $T^k M$, but not on the space of polynomials in one argument. Therefore, if one wants to turn our naive idea into a serious theory, one has to study the relation between these two viewpoints: can one introduce linear structures on the fibers of $T^k M$, defining at the same time a graded structure such that the maps $T^k f$ essentially become polynomials? Giving a (positive) answer to this question occupies the major part of the present work. Linear and graded structures on $T^k M$ in the above sense are called *multilinear connections*; they are important additional structures on a manifold.

Trying to work out this program is the beginning of what I would like to call *formal differential geometry*. The mere notion of the tangent bundle is already a “formal” concept, introducing auxiliary variables and a space of twice the dimension of the “geometric” space. The power of this formal concept becomes visible if one iterates the construction (whereas from a “classical” point of view, TTM and its relatives are very unpleasant objects which one prefers to avoid), so that one could define: *formal differential geometry is simply the theory of the higher order tangent functors T^k* . It is essentially an algebraic theory and excludes all “transcendental” aspects of classical differential geometry which arise by using various sorts of “integration procedures” (solution of differential equations, convergence, measures...) which have been invented in order to transfer formal results from $T^\infty M$ onto the space M itself.

2. Duality. Of course, a formal theory should not exclude the cotangent-“functor” T^* , which in some sense plays a more important rôle in most modern theories than the tangent functor T . In fact, the duality between T and T^* corresponds to a duality between *curves* $\gamma : \mathbb{K} \rightarrow M$ and *functions* $f : M \rightarrow \mathbb{K}$. If the theory is rather based on curves, then the tangent functor will play a dominant role since $\gamma'(t)$ is a tangent vector, and if the theory is rather based on functions, then T^* will play a dominant role since $df(x)$ is a cotangent vector. Under the influence of contemporary algebraic geometry, most modern theories adopt the second point of view. By choosing the first one, we do not want to take a decision on “correctness” of one point of view or the other, but, being interested in the “covariant theory”, we just simplify our life and avoid technical problems such as double dualization and topologies on dual spaces which inevitably arise if one tries to develop a theory of the tangent functor in an approach based on

¹ translated from the french “développement limité”

functions. This allows to drop almost all restrictions on the model space in infinite dimension and opens the theory for the use of base fields of positive characteristic and even of base rings (giving the whole strength to the point of view of scalar extensions).

Although we do not define, e.g., tangent vectors as differential operators acting on functions, they may be *represented* by such operators, and similarly, connections may be represented by covariant derivatives of sections, and so on. In other words, one can develop a *representation theory*, representing geometric objects from the “covariant setting”, in a contravariant way, by operators on functions and sections – some sketchy remarks on the outline of such a theory are given in Chapter 20, the theory itself remains to be worked out.

3. Related literature. We do not pretend that the present work were the first attempt to formalize structures of differential geometry – on the contrary, there exist quite a few approaches, and we hope to better exploit relations with these approaches in subsequent work. However, it seems that most of them are “contravariant” in the sense explained above, and thus the treatment of scalar extensions and higher order tangent functors is less natural than here. A foundational paper of these approaches is the text [W53] by A. Weil (intended as a draft for Bourbaki, but refused by the group), which later inspired the developpement of “synthetic differential geometry” (see [Ko81], [Lav87], [MR91], already discussed in Part I of this work) and found an echo in the russian school of A. M. Vinogradov (cf. [ALV91], [Nes03]). A very useful reference on Weil’s construction is [KMS93, Chapter VIII]. Another related concept (still in finite dimension over the reals) is “formal geometry” (see [CFT01]). Our basic reference for differential geometry in infinite dimension, modelled an real Banach spaces, is [La99], where a first step in direction of a complete formalization is made by avoiding double dualizations and rejecting the quite widespread confusion between the use of charts and the use of charts with bases (“coordinates”).

N. Notational remarks on Part I (version preprint 2003/no. 47)

On Chapter 7: *Higher order tangent rings*. The canonical flip $TTM \rightarrow TTM$ shall be denoted by κ (as in [KMS93]) rather than by τ .

On Chapter 10: *Connections (Direct sum of connections)*. In Section 10.10, a chart-depending construction of the direct sum of connections is given. (The notation used there is slightly misleading.) We add here a more conceptual construction which will be used later on:

Canonical isomorphisms of bundles. We describe two canonical isomorphisms of bundles. First of all, there is a canonical isomorphism

$$T(G \times_M H) \cong TG \times_{TM} TH \quad (\text{N.1})$$

for vector bundles G, H over M . In a chart, if G is modelled on W_1 and H on W_2 , this amounts to the canonical identification

$$T(U \times W_1 \times W_2) \cong TU \times TW_1 \times TW_2$$

which is a special case of the general isomorphism $T(M \times N) \cong TM \times TN$ for manifolds M, N . Next, assume F, G, H are vector bundles over M . Then $F \times_M G$ is a vector bundle over M , but it may also be considered as a vector bundle over G : the fiber over $g \in G_x$ with $x \in M$ is $\{(f, g) \mid f \in F_x\} \subset F_x \times G_x$. Then

$$(F \times_M G) \times_G (H \times_M G)$$

is a vector bundle over G , whereas over M , *a priori*, it is just a fiber bundle. But

$$F \times_M G \times_M H \rightarrow (F \times_M G) \times_G (H \times_M G), \quad (x; f, g, h) \mapsto (x, g; f, h) \quad (\text{N.2})$$

is a bijection of bundles over M – in a chart, if F, G, H are modelled on W_1, W_2, W_3 , respectively, then this amounts to the canonical identification

$$U \times (W_1 \times W_2 \times W_3) \cong (U \times W_2) \times W_1 \times W_3.$$

Now, the left hand side in (N.2) is a vector bundle *over* M , and hence, via the isomorphism (N.2), also the right hand side carries a canonical vector bundle structure over M .

Direct sum of connections. Assume F, G are vector bundles over M with connections L_i . Then, using the canonical isomorphisms described above, the following isomorphism defines a linear structure on $T(F \oplus_M G)$:

$$\begin{aligned} T(F \oplus_M G) &\cong TF \oplus_{TM} TG \\ &\cong (F \oplus_M TM \oplus \varepsilon TF) \oplus_{TM} (G \oplus_M TM \oplus \varepsilon TG) \\ &\cong F \oplus_M \varepsilon F \oplus TM \oplus G \oplus \varepsilon G. \end{aligned}$$

In a chart, this isomorphism is described, if b_x and a_x are the Christoffel tensors on F , resp. G ,

$$(x; \varepsilon v, u, w, \varepsilon u', \varepsilon w') \mapsto (x; \varepsilon v, u, w, \varepsilon(u' + b_x(v, u)), \varepsilon(w' + a_x(v, w))),$$

i.e. by the Christoffel tensor $c_x(v, (u, w)) = b_x(v, w) + a_x(v, u)$.

Chapters on Lie groups and symmetric spaces. See the new version in Part III of this work.

Third and higher order differential geometry

Having linearized the bundle TF over M by means of a linear connection, our plan of attack is to linearize in a similar way all higher tangent bundles $T^k F$ over M . It will turn out that there is an essentially canonical procedure to do so, starting with a connection on F and one on TM . The procedure is canonical up to a permutation acting on $T^k F$; in particular, for $k = 2$, we get two versions of this procedure. Their difference can be interpreted as the curvature of L .

15. The structure of $T^k F$: Multilinear bundles

15.1. Higher order models of F : projection and injection cube. For a vector bundle $p : F \rightarrow M$, we consider the fiber bundle $T^k F \rightarrow M$ as a “ k -th order model of F .” The case $F = TM$ deserves special attention: $T^k F = T^{k+1} M$ may be seen as the model of order $k + 1$ of M . The bundle $T^k F$ comes with a natural “ $k + 1$ -dimensional cube of projections”: e.g., for TTF , resp. for $T^3 M$, this is the commutative diagram of projections

$$\begin{array}{ccc}
 & & TTF \\
 & \swarrow & \downarrow & \searrow \\
 TTM & & \varepsilon_1 TF & & \varepsilon_2 TF \\
 \swarrow & & \downarrow & & \downarrow \\
 \varepsilon_1 TM & & \varepsilon_2 TM & & F \\
 \swarrow & & \downarrow & & \downarrow \\
 & & M & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & \varepsilon_0 \varepsilon_1 \varepsilon_2 T^3 M \\
 & \swarrow & \downarrow & \searrow \\
 \varepsilon_1 \varepsilon_2 T^2 M & & \varepsilon_0 \varepsilon_1 T^2 M & & \varepsilon_0 \varepsilon_2 T^2 M \\
 \swarrow & & \downarrow & & \downarrow \\
 \varepsilon_1 TM & & \varepsilon_2 TM & & \varepsilon_0 TM \\
 \swarrow & & \downarrow & & \downarrow \\
 & & M & &
 \end{array}
 \tag{15.1}$$

where the symbol ε_i in $\varepsilon_i TM$ and $\varepsilon_i TF$ means simply that we regard TM , resp. TF as scalar extension with new infinitesimal unit ε_i , and in case $F = TM$ we write $F = \varepsilon_0 TM$. Each vertex of the cube is the projection of a vector bundle onto its base, but the composition of more than one of such projections is just the projection of a fiber bundle onto its base. In particular, $T^k F$ is a fiber bundle over M . The fiber over $x \in M$ will be denoted by $(T^k F)_x$. For general k , the cube for $T^{k+1} M$ will have vertices $\varepsilon^\alpha T^{|\alpha|} M$ (notation for multi-indices $\alpha \in \{0, 1\}^{k+1}$ being as in Chapter 7, resp. MA.1), and the cube for $T^k F$ will have vertices $\varepsilon^\alpha T^{|\alpha|} M$ and $\varepsilon^\alpha T^{|\alpha|} F$ with $\alpha \in \{0, 1\}^k$; it contains the k -dimensional cube for $T^k M$ as a sub-cube. Diagram (15.1) can also be interpreted as a diagram of zero sections, just by reading the vertices as upward arrows. In particular, inclusion of spaces of the second row from below in the top space defines $k + 1$ injections

$$\varepsilon_i TM \rightarrow T^k F \quad (i = 1, \dots, k), \quad F \rightarrow T^k F \tag{15.2}$$

which, for $x \in M$ fixed, define $k + 1$ “basic axes” in $(T^k F)_x$ which will be denoted by $\varepsilon_i T_x M$, F_x . Altogether there are $2^{k+1} - 1$ “axes in $(T^k F)_x$ ” whose definition will be given below.

15.2. The axes-bundle. For a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k) \in I_{k+1} = \{0, 1\}^{k+1}$, we denote by

$$F^\alpha := \begin{cases} \varepsilon^\alpha F & \text{if } \alpha_0 = 1 \\ \varepsilon^\alpha TM & \text{if } \alpha_0 = 0 \end{cases} \tag{15.3}$$

a copy of F , resp. of TM (for the moment, ε^α is just a label to distinguish the various copies of F and of TM) and we define the *axes-bundle* to be the direct sum (over M) of all of these copies of F and of TM :

$$A^k F := \bigoplus_{\substack{\alpha \in I_{k+1} \\ \alpha_0 > 0}} F^\alpha \tag{15.4}$$

We let also

$$A^{k+1}M := A^k(TF) = \bigoplus_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} \varepsilon^\alpha TM. \quad (15.5)$$

These are direct sums of vector bundles over M and hence are itself a vector bundles over M , in contrast to T^kF and to $T^{k+1}M$ which do not carry a canonical vector bundle structure.

15.3. Isomorphism $T^kF \cong A^kF$ over a chart domain. Assume $\varphi_i : M \supset U_i \rightarrow V$ is a chart such that $\varphi_i(x) = 0$. Then $T^k\varphi_i : T^kM \supset T^kU_i \rightarrow T^kV$ is a bundle chart of T^kM defining a bijection of fibers

$$(\Phi_i)_x := (T^k\varphi_i)_x : (T^kM)_x \rightarrow (T^kV)_0.$$

Taking another chart φ_j with $\varphi_j(x) = 0$, we have the following commutative diagram of bijections, describing the transition functions of the atlas of T^kM induced from the one of M :

$$\begin{array}{ccc} & (T^kM)_x & \\ (T^k\varphi_j)_x \swarrow & & \searrow (T^k\varphi_i)_x \\ (T^kV)_0 & \xrightarrow{(T^k\varphi_{ij})_0} & (T^kV)_0 \end{array} \quad (15.6)$$

The transition functions of T^kM ,

$$(\Phi_{ij})_0 := (T^k\varphi_{ij})_0 : (T^kV)_0 = \bigoplus_{\alpha > 0} \varepsilon^\alpha V_\alpha \rightarrow \bigoplus_{\alpha > 0} \varepsilon^\alpha V_\alpha$$

are not linear: they are given by the Tangent Map Formula (7.19), with $f = \varphi_{ij}$,

$$T^k f(0) \left(\sum_{\alpha} v_{\alpha} \right) = \sum_{\alpha} \sum_{l=1}^k \sum_{\lambda \in \mathcal{P}_l(\alpha)} b^{\lambda}(v_{\lambda^1}, \dots, v_{\lambda^l})$$

where, for a partition λ of α of length l , we have written b^{λ} for the multilinear map $d^l f(0)$. This shows that $(\Phi_{ij})_0$ is a *multilinear map* in the sense of Section MA.5. Thus transition functions of T^kM are multilinear maps, and T^kM is a *multilinear bundle* in the following sense:

15.4. Multilinear bundles. A *multilinear bundle* (with base M , of degree k) is a fiber bundle E over M together with a bundle atlas such that, in a bundle chart around $x \in M$, the fiber E_x carries a natural structure of a multilinear space (given by a k -dimensional cube of \mathbb{K} -modules) over \mathbb{K} , and change of charts, restricted to fibers, is by isomorphisms of multilinear spaces. For $k = 1$, a multilinear bundle is simply a vector bundle, and for $k = 2$ it is bilinear bundle in the sense of Section 10. .

Theorem 15.5. *If F is a vector bundle over M , then the iterated tangent bundle T^kF carries a canonical structure of a multilinear bundle over M . In particular, the general multilinear group $\text{Gm}^{1,k+1}((T^kF)_x)$ of the fiber is intrinsically defined, and there are canonical inclusions of axes $F^\alpha \subset T^kF$.*

Proof. The arguments are essentially the same as the ones used above in the special case $F = TM$: bundle charts of F are of the form $\tilde{g}_i : F \supset \tilde{U}_i \rightarrow V \times W$, with transition functions $\tilde{g}_{ij}(v, w) = (\varphi_{ij}(v), g_{ij}(v)w)$, where $g_{ij}(v) : W \rightarrow W$ is a linear map. We assume $\varphi_i(x) = \varphi_j(x) = 0$. Bundle charts of T^kF are of the form $T^k\tilde{g}_{ij}$, and as an analog of (15.6) we have the following commutative diagram of bijections:

$$\begin{array}{ccc} & (T^kF)_x & \\ (T^k\tilde{g}_j)_x \swarrow & & \searrow (T^k\tilde{g}_i)_x \\ (T^kV)_0 \times T^kW & \xrightarrow{(T^k\tilde{g}_{ij})_0} & (T^kV)_0 \times T^kW \end{array} \quad (15.7)$$

For $k = 1$, the chart expression has been calculated in Section 9.1. In order to give the general chart expression, let $\tilde{U} \subset F$ be a bundle chart domain with base $U \subset M$ and let W be the

model space for the fibers of F . Then $T^k(\tilde{U}) \subset T^kF$ is a bundle chart domain for T^kF , and elements of $T^k(\tilde{U})$ are represented in the form

$$u = z + \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha \quad z \in U, \varepsilon^\alpha v_\alpha \in V_\alpha = \begin{cases} \varepsilon^\alpha W & \text{if } \alpha_0 = 1 \\ \varepsilon^\alpha V & \text{if } \alpha_0 = 0. \end{cases} \quad (15.8)$$

For $k = 2$, we may also use the following two equivalent notations (similar as in Section 7.4, Eqn. (7.17)):

$$\begin{aligned} u &= z + \varepsilon^{001} w_{001} + \varepsilon^{010} v_{010} + \varepsilon^{011} w_{011} + \varepsilon^{100} v_{100} + \varepsilon^{101} w_{101} + \varepsilon^{110} v_{110} + \varepsilon^{111} w_{111} \\ &= z + \varepsilon_0 w_0 + \varepsilon_1 v_1 + \varepsilon_0 \varepsilon_1 w_{01} + \varepsilon_2 v_2 + \varepsilon_0 \varepsilon_2 w_{02} + \varepsilon_1 \varepsilon_2 v_{12} + \varepsilon_0 \varepsilon_1 \varepsilon_2 w_{012} \end{aligned} \quad (15.9)$$

where we replaced v by w whenever the argument belongs to W . Of course, addition is commutative, so we may re-order the sum in any order we like. For fixed $z \in U$, these formulas describe the fiber $(T^kF)_z$; in particular, the fiber $(T^kF)_z$ carries a chart-dependent structure of a \mathbb{K} -module, isomorphic to the direct sum of axes $\bigoplus_{\alpha > 0} V_\alpha$ with the axes V_α as in (15.8). For $\tilde{g}_{ij}(x, w) = (\varphi_{ij}(x), g_{ij}(x)w)$, a transition map of F , the transition map $T\tilde{g}_{ij}$ of TF has been calculated in Section 9.1. The general formula for the transition functions is gotten from Equation (7.19): we let

$$\begin{aligned} \varepsilon^\alpha v &:= \begin{cases} \varepsilon^\alpha d\varphi_{ij}(x)v & \text{if } \alpha_0 = 1 \\ \varepsilon^\alpha g_{ij}(x)v & \text{if } \alpha_0 = 0 \end{cases} \\ c_x^j &:= d^j \varphi_{ij}(x), \\ b_x^{j+1}(w, v_1, \dots, v_j) &:= \dots := b_x^{j+1}(v_1, \dots, v_j, w) := \partial_{v_1} \cdots \partial_{v_j} g_{ij}(x, w). \end{aligned}$$

Then

$$\begin{aligned} T^k \tilde{g}_{ij}(x + \sum_{\alpha=1} \varepsilon^\alpha w_\alpha + \sum_{\alpha=0} \varepsilon^\alpha v_\alpha) &= \varphi_{ij}(x) + \sum_{\alpha=1} \varepsilon^\alpha \bar{w}_\alpha + \sum_{\alpha=0} \varepsilon^\alpha \bar{v}_\alpha + \\ &\sum_{j=2}^{k+1} \left(\sum_{\substack{\alpha_0=1 \\ |\alpha|=j}} \varepsilon^\alpha \sum_{l=2}^j \sum_{\substack{\beta^1+\dots+\beta^l=\alpha \\ \beta^1 < \dots < \beta^l}} b_x^l(w_{\beta^0}, v_{\beta^1}, \dots, v_{\beta^l}) + \sum_{\substack{\alpha_0=0 \\ |\alpha|=j}} \varepsilon^\alpha \sum_{l=2}^j \sum_{\substack{\beta^1+\dots+\beta^l=\alpha \\ \beta^1 < \dots < \beta^l}} c_x^l(v_{\beta^1}, \dots, v_{\beta^l}) \right) \end{aligned} \quad (15.10)$$

(For $j = k + 1$, only terms of the first sum over α actually appear.) If $V = TM$, then this is the expression for $T^{k+1}\varphi_{ij}$, with ε_0 being an additional infinitesimal unit. We want to describe the linear structure induced on the fiber by this chart. First of all, the linear part

$$h(x + \sum_{\alpha=1} \varepsilon^\alpha w_\alpha + \sum_{\alpha=0} \varepsilon^\alpha v_\alpha) = \varphi_{ij}(x) + \sum_{\alpha=1} \varepsilon^\alpha \bar{w}_\alpha + \sum_{\alpha=0} \varepsilon^\alpha \bar{v}_\alpha$$

does not affect the linear structure on the fiber; hence we may replace $T^k \tilde{g}_{ij}$ by $T^k \tilde{g}_{ij} \circ h^{-1}$. Next, using the notation for *partitions* of $\alpha \in I$ (cf. Section MA.3), Formula (15.10) can be rewritten: if $\lambda \in \mathcal{P}_l(\alpha)$ is a partition of α of length l , we let

$$b_x^\lambda(v_1, \dots, v_l) := \begin{cases} b_x(h^{-1}v_1, \dots, h^{-1}v_l) & \text{if } \alpha_0 = 1 \\ c_x(h^{-1}v_1, \dots, h^{-1}v_l) & \text{if } \alpha_0 = 0 \end{cases}$$

Then (15.10) can be written

$$\begin{aligned} T^k \tilde{g}_{ij} \circ h^{-1}(\varphi_{ij}(x) + \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha) &= \varphi_{ij}(x) + \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha + \sum_{\substack{\alpha > 0 \\ |\alpha| \geq 2}} \varepsilon^\alpha \left(\sum_{l=2}^{|\alpha|} \sum_{\lambda \in \mathcal{P}_l(\alpha)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^l}) \right) \\ &=: \varphi_{ij}(x) + f_b \left(\sum_{\alpha > 0} \varepsilon^\alpha v_\alpha \right). \end{aligned} \quad (15.11)$$

The map f_b defined in the second line belongs to the *general multilinear group* $\text{Gm}^{1,k}(E)$ of the fiber $E := (T^kF)_x$, as had to be shown. \blacksquare

Theorem 15.5 is the basic result for applying results of (multi-)linear algebra to the analysis of the structure of higher order tangent bundles.

16. The structure of $T^k F$: Multilinear connections

16.1. Definition of multilinear connections and their homomorphisms. Recall that a linear structure L on $T^k F$ over M is simply given by smooth structure maps $T^k F \times_M T^k F \rightarrow T^k F$, $\mathbb{K} \times T^k F \rightarrow T^k F$ defining a \mathbb{K} -module structure on each fiber of $T^k F$ over M . A *multilinear connection on $T^k F$* is a linear structure L that is *multilinearly related* (i.e. conjugate under $\text{Gm}^{0,k}(T^k F)_x$) to all linear structures induced from charts. (As in case $k = 2$ it is clear from Theorem 15.4 that this notation is well-defined.) Assume K is a multilinear connection on $T^k F$ and L a multilinear connection on $T^k G$ and $f : F \rightarrow G$ a vector bundle homomorphism. We say that f is a *K - L homomorphism* if $T^k f : T^k F \rightarrow T^k G$ is fiberwise linear (with respect to K_x and $L_{f(x)}$). Clearly, the usual functorial rules hold, and thus vector bundles with multilinear connections (of a fixed order k) form a category. For $k = 0$ this is just the category of vector bundles, and for $k = 1$ it is the category of vector bundles with linear connection.

Theorem 16.2. (The generalized Dombrowski Splitting Theorem.) *For any vector bundle F over M , the following objects are in bijection to each other:*

- (1) *multilinear connections L on $T^k F$*
- (2) *isomorphisms $\Phi : A^k F \rightarrow T^k F$ of multilinear bundles over M such that the restriction of Φ to each axis F^α is the identity map on this axis (here and in the sequel, the corresponding axes F^α of $A^k F$ and of $T^k F$ are identified with each other).*

The bijection is described as follows: given Φ , the linear structure $L = L^\Phi$ is just the push-forward of the canonical linear structure L_0 on $A^k F$ by Φ . Given L , we let

$$\Phi := \Phi^L : A^k F = \bigoplus_{\alpha > 0} F^\alpha \rightarrow T^k F, \quad (x; v_\alpha)_{\alpha > 0} \mapsto \sum_{\alpha} v_\alpha.$$

In this formula, the sum on the right hand side is taken in the fiber $(T^k F)_x$ with respect to the linear structure L_x .

Proof. Assume first that $\Phi : A^k F \rightarrow T^k F$ is any isomorphism of multilinear bundles over M . By definition of an isomorphism, the push-forward L^Φ of L_0 is then a multilinear connection on $T^k F$.

Conversely, assume a multilinear connection L on $T^k F$ be given and let $\Phi = \Phi^L$. In order to prove that Φ is a diffeomorphism, we need the chart representation of multilinearly related linear structures: with respect to a fixed chart, both the fiber $E := (T^k F)_x$ and the fiber $(A^k F)_x$ are represented in the form $E = \bigoplus_{\alpha > 0} V_\alpha$ with V_α as in Equation (15.8), V being the model space for the manifold M and W the one for the fiber F_x . Since $A^k F$ is a vector bundle, the linear structure of the fiber agrees with the one induced from the chart, and the linear structure induced by L is multilinearly related to this linear structure: it is obtained by push-forward via a map f_b which is the chart representation of Φ , where f_b is defined as in Section MA.5: $b = (b_\lambda)_{\lambda \in \text{Part}(I)}$ is a family of multilinear maps

$$b^\lambda := b_x^\lambda : V_{\lambda^{(1)}} \times \dots \times V_{\lambda^{(l(\lambda))}} \rightarrow V_\alpha, \quad (16.1)$$

λ is a partition of α , and $V_\beta = \varepsilon^\beta V$ if $\beta_0 = 0$, $V_\beta = \varepsilon^\beta W$ if $\beta_0 = 1$, and for $v = \sum_{\alpha > 0} v_\alpha$,

$$f_b(v) = v + \sum_{|\alpha| \geq 2} \sum_{\lambda \in \mathcal{P}(\alpha)} b^\lambda(v_{\lambda^{(1)}}, \dots, v_{\lambda^{(l(\lambda))}}). \quad (16.2)$$

(The map $f_b = f_{b_x}$, resp. their components $(b^\lambda) = (b_x^\lambda)$, will be called the *Christoffel symbols of L in the given chart*; for $k = 1$, there is just one component, and it agrees with the classical Christoffel symbols of a connection, cf. Section 10.4.) We claim that the linear structure L is

smooth if and only if, for all bundle charts, the Christoffel symbols b depend smoothly on x in the sense that

$$x \mapsto b_x^\lambda(v_{\lambda(1)}, \dots, v_{\lambda(l(\lambda))})$$

is smooth for all choices of the $v_{\lambda(i)}$. In fact, it is clear that the linear structure is smooth if it is related to chart structures via smooth f_b 's, and conversely, f_b and hence b can be recovered from the vector addition: in a chart, addition of elements from the axes is given by $\sum_{\alpha>0}^{(b)} v_\alpha = f_b(v)$, where the superscript (b) in the first sum means "addition with respect to the linear structure given by b_x ". Hence the map f_b defined by a smooth linear structure is also smooth. We have proved that Φ is smooth. It is bijective with smooth inverse: in fact, f_{b_x} belongs to the group $\text{Gm}^{1,k}(E)$ and hence is invertible (Theorem MA.6), and its inverse is of the form f_{c_x} with another multilinear map c_x which can be calculated by the algorithm used in the proof of Theorem MA.6. Since this algorithm consists of a sequence of compositions of multilinear maps and multiplications by -1 , $c_x(v)$ depends again smoothly on x , and hence Φ^{-1} is smooth. ■

The isomorphism $\Phi = \Phi^L$ is called the *linearization map of L* . In case $F = TM$ is the tangent bundle, we have $F^\alpha = \varepsilon^\alpha TM$ for all α , and the linearization map can be written

$$\Phi_k^L : \bigoplus_{\alpha \in I_k, \alpha > 0} \varepsilon^\alpha TM \rightarrow T^k M.$$

16.3. Multilinear differential calculus on manifolds with multilinear connection. We can define a "matrix calculus" for multilinear maps between higher order tangent bundles $T^k M$, $T^k N$ equipped with multilinear connections, simply by using fiberwise the purely algebraic "matrix calculus" from Section MA.12. Basically, the choice of a multilinear connection on $T^k M$ is the analog of the choice of a basis in a (say, finite-dimensional) vector space E ; the splitting map $\Phi^L : A^k F \rightarrow T^k F$ corresponds to the induced isomorphism $\mathbb{K}^n \rightarrow E$, the "matrix" is the induced homomorphism $\mathbb{K}^n \rightarrow \mathbb{K}^m$, individual "matrix coefficients" are defined by composing with the induced projections $\mathbb{K}^m \rightarrow \mathbb{K}$ and injections $\mathbb{K} \rightarrow \mathbb{K}^n$, and finally there is a "matrix multiplication rule" describing the matrix coefficients of the matrix of a composition.

Let us describe these concepts in more detail. Assume K is a multilinear connection on $T^k M$ and L a multilinear connection on $T^k N$ (the case of general vector bundles is treated similarly), and $f : M \rightarrow N$ is an arbitrary smooth map. We define its *multilinear higher differentials (with respect to K and L)* by $d_K^L f := (\Phi^L)^{-1} \circ T^k f \circ \Phi^K : A^k M \rightarrow A^k N$, i.e., by the following commuting diagram:

$$\begin{array}{ccc} T^k M & \xleftarrow{\Phi^K} & A^k M = \bigoplus_{\alpha>0} \varepsilon^\alpha TM \\ T^k f \downarrow & & \downarrow d_K^L f \\ T^k N & \xleftarrow{\Phi^L} & A^k N = \bigoplus_{\alpha>0} \varepsilon^\alpha TN \end{array} \quad (16.3)$$

As for matrices of linear maps, we have the functorial rules

$$d_{L_2}^{L_3} g \circ d_{L_1}^{L_2} f = d_{L_1}^{L_3} (g \circ f), \quad d_L^L(\text{id}) = \text{id} \quad (16.4)$$

which are immediate from the definitions. In each fiber, the maps

$$(d_K^L)_x f : (A^k M)_x \rightarrow (A^k N)_{f(x)}$$

are homomorphisms of multilinear spaces (in the sense of MA.5) – in fact, with respect to chart structures, this is true by Formula (7.19), and hence it is true with respect to any other structure that is multilinearly related to chart structures. It follows that the "matrix coefficients", indexed by partitions Λ and Ω ,

$$\begin{aligned} d_x^\Lambda f &:= (d_K^L f)_x^\Lambda := \text{pr}_\alpha \circ T^k f \circ \iota_\Lambda : \bigoplus_{i=1}^{l(\Lambda)} \varepsilon^{\Lambda^i} TM \rightarrow \varepsilon^\alpha TM \\ (d_K^L f)_x^{\Omega|\Lambda} &:= \text{pr}_\Omega \circ D_{K^L} f \circ \iota_\Lambda : \bigoplus_{i=1}^{l(\Lambda)} \varepsilon^{\Lambda^i} TM \rightarrow \bigoplus_{i=1}^{l(\Omega)} \varepsilon^{\Omega^i} TN \end{aligned} \quad (16.5)$$

are multilinear maps in the usual sense; here pr and ι denote the canonical projections and injections in the direct sum bundle $A^k N$ defined by

$$\text{pr}_\Omega : A^k N \rightarrow \bigoplus_{i=1}^{l(\Omega)} \varepsilon^{\Omega^i} TN, \quad \iota_\lambda : \bigoplus_{i=1}^{l(\lambda)} \varepsilon^{\lambda^i} TM \rightarrow T^k M.$$

The whole map is recovered from the matrix entries via

$$T^k f(x + \sum_\alpha \varepsilon^\alpha v_\alpha) = f(x) + \sum_\alpha \varepsilon^\alpha \sum_{\lambda \in \mathcal{P}(\alpha)} (d^\lambda f)_x(v_{\lambda^1}, \dots, v_{\lambda^l}). \quad (16.6)$$

Homomorphisms in the sense of 16.1 are characterized by the property that the linearization maps commute with the natural action $f_* : TM \rightarrow TN$, $\sum_\alpha v_\alpha \mapsto \sum_\alpha Tf(v_\alpha)$:

$$\begin{array}{ccc} T^k M & \xleftarrow{\Phi^K} & A^k M = \bigoplus_{\alpha > 0} \varepsilon^\alpha TM \\ T^k f \downarrow & & \downarrow f_* \\ T^k N & \xleftarrow{\Phi^L} & A^k N = \bigoplus_{\alpha > 0} \varepsilon^\alpha TN \end{array}. \quad (16.7)$$

This, in turn, means that the matrix $d_K^L f$ is a “diagonal matrix”, i.e. all components $(d_K^L f)_x^\Lambda$ for $l(\Lambda) > 1$ vanish, and for $l(\Lambda) = 1$ they agree with Tf .

16.4. Composition rules and “Matrix multiplication”. Let us return to Equation (16.5). The “matrix coefficients” of $f \circ g$ with respect to L^3 and L^1 can be calculated from those of f with respect to L^3 , L^2 and those of g with respect to L^2 , L^1 by using the composition rule for multilinear maps (MA.19), resp. Prop. MA.11:

$$(d_{L_3}^{L_1}(f \circ g))^\Lambda = \sum_{\Omega \leq \Lambda} (d_{L_2}^{L_1} f)^\Omega \circ (d_{L_3}^{L_2} g)^{\Omega|\Lambda}. \quad (16.8)$$

This notation generalizes the usual differential d since, taking $M = V$ with the flat connection of a linear space, the matrix coefficient $d^\Lambda f$ as defined here coincides with the usual higher differential $d^l f$ with l being the length of Λ , cf. Equation (7.19).

16.5. On existence of multilinear connections. For $k > 1$, the space of all multilinear connections on $T^k F$ is no longer an affine space over \mathbb{K} in any natural way (since $\text{Gm}^{1,k}(E)$ is no longer a vector group), and hence it is more difficult to construct multilinear connections than to construct linear ones. They may be defined in charts by brute force by choosing arbitrary “Christoffel symbols” b^Λ which then have to behave under chart changes according to the composition rule in the general multilinear group (16.8). In the sequel we will see that there are canonical procedures to construct multilinear connections out of linear ones.

17. Construction of multilinear connections

17.1. Deriving linear structures. Assume $L = (a, m)$ is a linear structure on a fiber bundle E over M , with structural maps $a : E \times_M E \rightarrow E$, $m : \mathbb{K} \times E \rightarrow E$. Then $TL = (Ta, Tm)$ defines a linear structure on TE over TM , called the *derived linear structure* (this is even a linear structure with respect to $T\mathbb{K}$, but we will only consider it as a linear structure with respect to \mathbb{K}). However, in general there is no canonical way to see the derived linear structure TL as a linear structure on the bundle TE over M .

17.2. Deriving multilinear connections. Assume we are given the following data: a multilinear connection L on $T^k F$ with corresponding isomorphism of linear bundles $\Phi : A^k F = \bigoplus_{\alpha > 0} F^\alpha \rightarrow T^k F$, as well as linear connections L' on TM and L'' on F . Since $A^k F$ is a direct sum of certain copies of F and of TM , we can equip $A^k F$ with the direct sum of the connections L' (on the copies of TM) and L'' (on the copies of F) – cf. Section 10.10 and the remarks in Section N at the beginning of this text concerning notation. This gives us a linear structure on $T(A^k F)$ and an isomorphism of linear bundles $T(A^k F) \cong A^{k+1} F$. On the other hand,

$$T\Phi : T\left(\bigoplus_{\alpha > 0} F^\alpha\right) = T(A^k F) \rightarrow T(T^k F) = T^{k+1} F$$

is a bundle-isomorphism, so that we end up with a bundle isomorphism $T^{k+1} F \cong A^{k+1} F$ which, by transport of structure, can be used to define a linear structure on $T^{k+1} F$. Let us make this isomorphism more explicit: in the following equalities, we use first the isomorphism $T\Phi$, then the canonical isomorphism (N.1) $T(A \oplus_M B) \cong TA \oplus_{TM} TB$, the isomorphisms Φ_1 corresponding to L' and to L'' and finally the canonical isomorphism (N.2) $(A \oplus_M TM) \oplus_{TM} (B \oplus_M TM) \cong A \oplus_M TM \oplus_M B$. By \bigoplus^{TM} we denote the direct sum of vector bundles over TM and by $\bigoplus = \bigoplus_M$ the direct sum of vector bundles over M , and e_1, \dots, e_k is the canonical basis of $I_k = \{0, 1\}^k$:

$$\begin{aligned} T^{k+1} F &\stackrel{T\Phi}{\cong} T\left(\bigoplus_{\substack{\alpha \in I_k \\ \alpha > 0}} F^\alpha\right) \stackrel{(N.1)}{\cong} \bigoplus_{\substack{\alpha \in I_k \\ \alpha > 0}}^{TM} T F^\alpha \\ &\stackrel{\Phi_1}{\cong} \bigoplus_{\substack{\alpha \in I_k \\ \alpha > 0}}^{TM} (F^\alpha \oplus \varepsilon_{k+1} TM \oplus \varepsilon_{k+1} F^\alpha) \\ &\stackrel{(N.2)}{\cong} \varepsilon_{k+1} TM \oplus \bigoplus_{\substack{\alpha \in I_k \\ \alpha > 0}} F^\alpha \oplus \bigoplus_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon_{k+1} F^\alpha \stackrel{\text{def.}}{\cong} \bigoplus_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} F^\alpha. \end{aligned} \tag{17.1}$$

Let us write $D\Phi$ for the isomorphism (17.1) of bundles over M . Suppressing the canonical isomorphisms (N.1) and (N.2), we may write

$$D\Phi = T\Phi \circ (\times_{\alpha \in I_k} \Phi_1^{e_{k+1}, \alpha}) : \bigoplus_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} F^\alpha \rightarrow T^{k+1} F, \tag{17.2}$$

where $\Phi_1^{e_{k+1}, \alpha}$ is the splitting isomorphism Φ_1 for $T F^\alpha$. By transport of structure, $D\Phi$ defines on $T^{k+1} F$ the structure of a linear bundle over M which we denote by DL .

Theorem 17.3. *The linear structure DL defined on $T^{k+1} F$ by $D\Phi$ is a multilinear connection on $T^{k+1} F$.*

Proof. As seen in the proof of the Generalized Dombrowski Splitting Theorem 16.2, in a chart, Φ is given by

$$\Phi(x + v) = x + \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^\alpha \sum_{\lambda \in \mathcal{P}(\alpha)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^{l(\lambda)}})$$

with multilinear maps b_x^λ , depending smoothly on x , and $b_x^\lambda(u) = u$ if $l(\lambda) = 1$. The tangent map $T\Phi$ is calculated by taking ε_{k+1} as new infinitesimal unit:

$$\begin{aligned} T\Phi(x + \sum_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} \varepsilon^\alpha v_\alpha) &= \Phi(x + \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^\alpha v_\alpha) + \varepsilon^{k+1} (d\Phi(x + \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^\alpha v_\alpha) \cdot \sum_{\alpha \in I_k} \varepsilon^\alpha v_{\alpha+e_{k+1}}) \\ &= x + v + \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^\alpha \sum_{\lambda \in \mathcal{P}(\alpha)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^{l(\lambda)}}) + \\ &\quad \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^{\alpha+e_{k+1}} \sum_{\lambda \in \mathcal{P}(\alpha)} \left(\sum_{i=1}^{l(\lambda)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^i+e_{k+1}}, \dots, v_{\lambda^{l(\lambda)}}) + \partial_{v_{e_{k+1}}} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^{l(\lambda)}}) \right) \end{aligned}$$

(the partial derivative in the last term is with respect to x ; the dependence on the other variables is linear and hence deriving amounts to replace the argument v_{λ^i} by the direction vector $v_{\lambda^i+e_{k+1}}$). Next, we have to insert the chart representation of the isomorphisms Φ_1 : we have to replace, for all $\beta \in I_k$, $v_{\beta+e_{k+1}}$ by $v_{\beta+e_{k+1}} + d_x(v_\beta, v_{e_{k+1}})$ where, if $\beta_0 = 0$, d_x is the Christoffel tensor of the connection L' on TM , and, if $\beta_0 = 1$, d_x the Christoffel tensor of the connection L'' on F . The resulting formula for $D\Phi$ is:

$$\begin{aligned} D\Phi\left(\sum_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} \varepsilon^\alpha v_\alpha\right) &= x + \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^\alpha \sum_{\lambda \in \mathcal{P}(\alpha)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^{l(\lambda)}}) + \\ &\quad \sum_{\substack{\alpha \in I_k \\ \alpha > 0}} \varepsilon^{\alpha+e_{k+1}} \sum_{\lambda \in \mathcal{P}(\alpha)} \left(\sum_{i=1}^{l(\lambda)} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^i+e_{k+1}}, \dots, v_{\lambda^{l(\lambda)}}) + \right. \\ &\quad \left. b_x^\lambda(v_{\lambda^1}, \dots, d_x(v_{\lambda^i}, v_{e_{k+1}}), \dots, v_{\lambda^{l(\lambda)}}) + \partial_{v_{e_{k+1}}} b_x^\lambda(v_{\lambda^1}, \dots, v_{\lambda^{l(\lambda)}}) \right) \end{aligned} \quad (17.3)$$

This expression is again a sum of multilinear terms and hence defines a new multilinear map $f_B = f_{B_x}$ where B_x depends smoothly on x . Thus DL is a multilinear connection on $T^{k+1}F$. ■

For later use, let us spell out the formula in case $k = 2$, writing $b_x(w, v)$ for the Christoffel tensor of L and $c_x(u, v)$ for the one of L' in the chart, and using the more traditional notation $v = \varepsilon_0 w_0 + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_0 \varepsilon_1 w_{01} + \varepsilon_0 \varepsilon_2 w_{02} + \varepsilon_1 \varepsilon_2 v_{12} + \varepsilon_0 \varepsilon_1 \varepsilon_2 w_{012}$ (see Eqn. (15.9)) for elements in the fiber $(TTF)_x$:

$$\begin{aligned} D\Phi_1(x, v) &= v + \varepsilon_0 \varepsilon_1 b_x(w_0, v_1) + \varepsilon_0 \varepsilon_2 b_x(w_0, v_2) + \varepsilon_1 \varepsilon_2 c_x(v_1, v_2) + \\ &\quad \varepsilon_0 \varepsilon_1 \varepsilon_2 (b_x(w_0, v_{12}) + b_x(w_{01}, v_2) + b_x(w_{02}, v_1) + \\ &\quad b_x(w_0, c_x(v_1, v_2)) + b_x(b_x(w_0, v_2), v_1) + \partial_{v_2} b_x(w_0, v_1)). \end{aligned} \quad (17.4)$$

If $F = TM$, then this formula describes a linear structure on T^3M , depending on two linear connections L, L' on TM . In this case, unless otherwise stated, we will choose $L = L'$, i.e. $b_x = c_x$.

17.4. The preceding construction has the property that the zero section $z : T^k F \rightarrow T^{k+1} F = T_{\varepsilon_{k+1}} T^k F$ is a L - DL -linear homomorphism of linear bundles over M ; more precisely, by construction L and DL agree on the image of the zero section:

$$\begin{array}{ccc} T^k F & \xrightarrow{z} & T^{k+1} F \\ \Phi \uparrow & & \uparrow D\Phi \\ A^k F & \subset & A^{k+1} F \end{array}$$

(this may also be seen from the chart representation (17.4), where the image of the zero section corresponds to points v with $v_{\beta+e_{k+1}} = 0$ for all β .)

17.5. *The sequence of derived linear structures.* Assume L is a linear connection on the vector bundle $p : F \rightarrow M$ and L' a linear connection on TM . Then, using the preceding construction,

we can derive L and construct a connection $L_2 := DL$ on TTM over M , and so on: we get a sequence of derived linear structures $L_k = D^{k-1}L$ on $T^k F$ over M , $k = 2, 3, \dots$ with corresponding isomorphisms $\Phi_k = D^{k-1}\Phi$ of linear bundles over M . In case of tangent bundles, $F = TM$, then (unless otherwise stated) we will take $L = L'$ in this construction, and the construction is functorial in the following sense: if $f : M \rightarrow N$ is affine for given connections L on TM , L' on TN , then $T^k f : T^k M \rightarrow T^k N$ is an $L_k - L'_k$ homomorphism in the sense of 16.1 (and conversely). This is seen by an easy induction using that the construction involves only natural isomorphisms and the tangent functor. For instance, $\gamma : \mathbb{K} \rightarrow M$ is a geodesic if and only if, for some $k > 1$, $T^k \gamma : T^k \mathbb{K} \rightarrow T^k M$ is a $L_k - L'_k$ homomorphism, where L_k is the natural connection on \mathbb{K} given by the canonical identification $T^k \mathbb{K} = A^k \mathbb{K}$.

However, the ‘‘operator D of covariant derivative’’ has not the property of the usual derivative d that higher derivatives are independent of the order of the ε_i : in general, Φ_k and L_k will depend on the order $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ of the subsequent scalar extensions leading from F to $T^k F$, and we denote by L_k^σ , $\sigma \in \Sigma_k$, the linear structure obtained with respect to the order $\varepsilon_{\sigma(1)}, \dots, \varepsilon_{\sigma(k)}$. Then the push-forward of L_k by the canonical action of σ on $T^k F$ is given by

$$\sigma \cdot L_k = L_k^\sigma. \quad (17.5)$$

These remarks make it necessary to study in more detail the behavior of the linear structures L_k with respect to the action of the symmetric group (see next chapter).

17.6. Example: trivialisable tangent bundles. We will see later that the tangent bundle of a Lie group is always trivialisable, and thus the following example can be applied to the case of an arbitrary Lie group: assume that the tangent bundle TM is trivialisable over M , i.e. that there is a diffeomorphism $\Phi_0 : M \times V \rightarrow TM$ which is fiber-preserving and linear in fibers over M . By the following diagram

$$\begin{array}{ccc} M \times V \times V \times V & \xrightarrow{\Psi_1} & TTM \\ \downarrow & & \uparrow T\Phi_0 \\ M \times V \times TV & \xrightarrow{\Phi_1} & TM \times TV \end{array}$$

we define a diffeomorphism $\Psi_1 : (M \times V) \times (V \times V) \rightarrow TTM$ which is a trivialization of the bundle TTM over M . We can continue in this way and trivialisate all higher tangent bundles $T^k M$ over M via

$$\Psi_{k+1} := T^k \Phi_0 \circ (\Psi_k \times \text{id}) = T\Psi_k \circ (\Phi_0 \times \text{id}) : M \times \bigoplus_{\substack{\alpha \in I_{k+1} \\ \alpha > 0}} \varepsilon^\alpha V \rightarrow T^{k+1} M.$$

Via push-forward, this defines linear structures L_k on all higher order tangent bundles $T^k M$. By our assumption on Φ_0 , L_1 is just the vector bundle structure on TM .

Lemma 17.7. *The linear structure L_2 is a connection on the tangent bundle TM , and $L_k = D^{k-2}L_2$ for $k > 2$, i.e., the linear structures L_k agree with the derived connections of L_2 defined in the preceding section.*

Proof. In order to prove that L_2 is a connection, in a bundle chart, Φ_0 is represented by $\Phi_0(x, w) = (x, g_x(w))$; then

$$T\Phi_0((x, w) + \varepsilon(x', w')) = (x, g_x(w)) + \varepsilon(g_x(w'), \partial_x g_x(w)x')$$

(cf. Section 9.1) where the last term depends bilinearly on (w, x') , proving that L_2 is a connection. Explicitly, the Christoffel symbols of this connection are calculated as in Section 9.1:

$$b_x(w, x') = \frac{\partial}{\partial y} \Big|_{y=x} (g_y((g_x)^{-1}(w)))x'.$$

(Mind that the bilinear map b_x is in general not symmetric.) Next, it follows that $L_3 = DL_2$ since both L_3 and DL_2 are constructed by deriving L_2 and composing with Ψ_2 , resp. with Φ_2 , and Ψ_2 and Φ_2 induce the same linear structure L_2 on TTM . By induction, this argument shows that $L_k = D^{k-1}L_2$. ■

18. Curvature

18.1. Totally symmetric multilinear connections. The symmetric group Σ_k acts on $T^k F$; in the notation (15.8) this action is by permuting the last k coordinates of $\{0, 1\}^{k+1}$. If $F = TM$, then $T^k F = T^{k+1} M$, and ε_0 may be considered as another infinitesimal unit, and the permutation group Σ_{k+1} acts on all $k+1$ coordinates. We say that a multilinear connection L is *totally* or *very symmetric* if it is invariant under the action of Σ_k , resp. of Σ_{k+1} . Equivalently, this means that Σ_k acts linearly with respect to L , or that L is fiberwise conjugate to all chart-structures under the action of *very symmetric group* $\text{Vsm}^{1,k}((T^k F)_x) = (\text{Gm}^{1,k}(T^k F))^{\Sigma_k}$ defined in Section SA.2. In terms of the Dombrowski Splitting Map $\Phi : A^k F \rightarrow T^k F$, the condition is equivalent to saying that Φ is equivariant with respect to the natural Σ_k -actions on both sides.

18.2. Curvature operators and curvature forms. Curvature is the obstruction for a multilinear connection L to be very symmetric, or, in other words, it measures the difference between L and the linear structure $L^\sigma = \sigma.L$ obtained by push-forward via $\sigma \in \Sigma_k$. In order to formalize this idea, let us denote the canonical linear structure on the axes-bundle $A^k F$ by L^0 ; it is Σ_k -invariant. The linear structure L is obtained as push-forward of L^0 via Φ , i.e. $L = \Phi.L^0$. It follows that $\sigma.L = \sigma.\Phi.L^0 = \sigma\Phi\sigma^{-1}.L^0$, and hence the linear structure L^σ is given by push-forward via $\Phi^\sigma = \sigma \circ \Phi \circ \sigma^{-1}$. It is proved in Prop. SA.3 that Φ^σ is again a multilinear map, and thus we see that $\sigma.L$ is again a multilinear connection. Now we define the *curvature operator* $R := R^\sigma := R^{\sigma,L}$ by

$$R^{\sigma,L} := \Phi^\sigma \circ \Phi^{-1} = \sigma \circ \Phi \circ \sigma^{-1} \circ \Phi^{-1} : T^k F \rightarrow T^k F, \quad (18.1)$$

respectively its action on the axes-bundle $\tilde{R} := \tilde{R}^\sigma := \tilde{R}^{\sigma,L}$ given by

$$\tilde{R}^{\sigma,L} := \Phi^{-1} \circ R \circ \Phi = \Phi^{-1} \circ \sigma \circ \Phi \circ \sigma^{-1} = \Phi^{-1} \circ \Phi^{\sigma^{-1}} : A^k F \rightarrow A^k F \quad (18.2)$$

Both operators are compositions of multilinear maps and hence are multilinear automorphisms of $T^k F$, resp. of $A^k F$. The curvature operator R can be characterized as the unique element of $\text{Gm}^{1,k}(T^k F)$ relating the linear structures L and L^σ :

$$R \cdot L = L^\sigma. \quad (18.3)$$

In fact, $R \cdot L = \sigma \circ \Phi \circ \sigma^{-1} \circ \Phi^{-1} \cdot L = \sigma \circ \Phi \circ \sigma^{-1} \circ L_0 = \sigma \circ \Phi \cdot L_0 = \sigma \cdot L$. Immediately from the definition we get the following ‘‘cocycle relations’’: for fixed L and for all $\sigma, \tau \in \Sigma_k$,

$$R^{\text{id}} = \text{id}, \quad R^{\sigma\tau} = \sigma R^\tau \sigma^{-1} \circ R^\sigma. \quad (18.4)$$

In fact, it is also possible to interpret the curvature operators as higher differentials (‘‘matrices’’) in the sense of Section 16.3, and then the cocycle relations are seen as a special case of the matrix multiplication rule (16.5): from (18.2) it follows that

$$\begin{array}{ccc} T^k F & \xleftarrow{\Phi^{\sigma^{-1}}} & A^k F \\ T^k \text{id}_F \downarrow & & \downarrow \tilde{R} \\ T^k F & \xleftarrow{\Phi} & A^k F \end{array}$$

commutes, which shows that

$$\tilde{R}^\sigma = d_{\sigma^{-1}.L}^L(\text{id}_F) \quad (18.5)$$

is a higher differential in the sense of 16.3. The matrix coefficients of the curvature operators are called the *curvature forms*,

$$\tilde{R}^{\Omega|\Lambda} := \text{pr}_\Omega \circ \tilde{R} \circ \iota_\Lambda : F_{\Lambda^1} \oplus_M \dots \oplus_M F_{\Lambda^l} \rightarrow F_{\Omega^1} \oplus_M \dots \oplus_M F_{\Omega^r}. \quad (18.6)$$

Of particular interest are the *elementary curvature forms*

$$\tilde{R}^\Lambda := \tilde{R}^{\Lambda|\Lambda} : F_{\Lambda^1} \oplus_M \dots \oplus_M F_{\Lambda^l} \rightarrow F_\Lambda$$

which are ordinary tensor fields, i.e. fiberwise multilinear maps in the usual sense. For the longest partition $\Lambda = (e_1, \dots, e_k)$ we get the *top curvature form*. In particular, for $k = 3$ one finds the classically known curvature tensor as a top curvature form:

Theorem 18.3. *Assume L is a linear connection on the vector bundle F over M and L' a torsionfree connection on TM . Let $k = 2$ and $R = R^{\kappa, DL}$ where $\kappa : T^2F \rightarrow T^2F$ is the canonical flip and DL is the connection derived from L as described in Theorem 17.3. Then R does not depend on the choice of L' . Moreover, R belongs to the subgroup $\text{Gm}^{2,3}(TTF)$, i.e. \tilde{R}_x has only one non-trivial component, namely the top curvature form*

$$\tilde{R}^\Lambda : \varepsilon_1 TM \times \varepsilon_2 TM \times \varepsilon_0 F \rightarrow \varepsilon_0 \varepsilon_1 \varepsilon_2 \cdot F$$

where $\Lambda = (e_1, e_2, e_3)$. In other words, for $v = (v_\alpha) \in (A^2F)_x$,

$$\tilde{R}_x(v) = v + \tilde{R}_x^\Lambda(\varepsilon_1 v_1, \varepsilon_2 v_2) \varepsilon_0 v_0.$$

In a chart, with Christoffel tensor b_x of L , the top curvature form \tilde{R}_x^Λ is given by

$$\tilde{R}_x^\Lambda(v_1, v_2)w_0 = (b_x(b_x(w_0, v_1), v_2) - b_x(b_x(w_0, v_2), v_1) + \partial_{v_2} b_x(w_0, v_1) - \partial_{v_1} b_x(w_0, v_2)).$$

Proof. We use the chart representation (17.4) of the map $\Phi_2 = D\Phi$ for $L_2 = DL$: the chart formula for L_2^κ is obtained from this formula by exchanging $\varepsilon_1 \leftrightarrow \varepsilon_2$, $v_1 \leftrightarrow v_2$, $w_{01} \leftrightarrow w_{02}$. Note that the resulting formula differs from the one of Φ_2 only in the $\varepsilon_0 \varepsilon_1 \varepsilon_2$ -component and in the $\varepsilon_1 \varepsilon_2$ -component, and the difference in the $\varepsilon_1 \varepsilon_2$ -component vanishes if $c(v_1, v_2) = c(v_2, v_1)$, i.e. if L' is torsionfree. Let us assume that this is the case. Then from Formula (17.4) we get the following chart formula for the difference between L_2 and L_2^κ :

$$\begin{aligned} \Phi_2(v) - \Phi_2^\kappa(v) &= \varepsilon_0 \varepsilon_1 \varepsilon_2 \cdot \\ &(b_x(b_x(w_0, v_1), v_2) - b_x(b_x(w_0, v_2), v_1) + \partial_{v_2} b_x(w_0, v_1) - \partial_{v_1} b_x(w_0, v_2)) \end{aligned} \quad (18.7)$$

since c_x is assumed to be symmetric. This formula proves all claims of the theorem: if we denote by $A_x(v_1, v_2)w_0$ the right-hand side of (18.7) and by f_A the multilinear map given by $f_A(v) = v + A(v_1, v_2)w_0$, then from the composition rule of multilinear maps it follows that $f_A \cdot L_2 = L_2^\kappa$; thus (18.3) is satisfied, and by uniqueness in (18.3) we get $R = f_A$, and A is the only non-trivial component of the curvature maps R , resp. \tilde{R} . Formula (18.7) shows that it is indeed independent of c and hence of L' . ■

The chart representation (18.7) of the top curvature form coincides with the usual expression of the curvature tensor of a connection in a chart by its Christoffel tensor (cf. e.g. [La99, p. 232]), thus showing that our interpretation of the curvature agrees with all other known concepts.

18.4. Case of the tangent bundle. Assume we are in the case of the tangent bundle: $F = TM$; then we usually choose $L = L'$. If L has torsion, then also L' has torsion, and the assumptions of Theorem 18.3 are not satisfied. However, Formula (17.4) still can be used to establish a relation between the classical curvature tensor and the curvature forms of L ; we will not go into the details and leave the topic of connections with torsion. Let us just add remark that in this case the Σ_3 -orbit of the linear structure L_3 on T^3M has 6 elements which we may represent by a hexagon. None of the 6 structures is invariant under a transposition, thus we may represent the three transpositions by symmetries with respect to three axes that do not pass through the vertices.

18.5. Bianchi's identity. Now assume that $F = TM$ is tangent bundle and that L is torsionfree. Then, by torsionfreeness, L and DL are invariant under the transposition (12), and the usual curvature tensor is the top curvature form of $R = R^{(23)}$. Let us look at the Σ_3 -orbit of the linear structure $K := DL$ on T^3M . This orbit contains at most 3 elements since K is stabilized by the transposition (12). We denote these elements by $K^1 := K = K^{\text{id}} = K^{(12)}$, $K^2 := K^{(23)} = K^{(132)}$ and $K^3 := K^{(13)} = K^{(123)}$. From the ‘‘matrix multiplication rule’’ (16.5), we get

$$\text{id} = D_{K^1}^{K^1} \text{id} = D_{K^2}^{K^1} \text{id} \circ D_{K^3}^{K^2} \text{id} \circ D_{K^1}^{K^3} \text{id}.$$

Taking the top curvature forms, we get the *Bianchi identity* for the curvature form $R = R^{(23)}$

$$R + R^{(132)} + R^{(123)} = 0.$$

In a similar way, we get the usual skew-symmetry of the curvature form $R = R^{(23)}$. More generally, one can show that the top degree curvature forms $R^{(23), D^k L}$ for the derived linear structures L_{k+1} agree with the covariant derivatives of the curvature tensor in the usual sense, and under suitable assumptions one can prove algebraic relations for these.

The following result is included for the sake of completeness ; it relates the curvature as defined here to the usual definition in terms of covariant derivatives:

Theorem 18.6. *Assume ∇ is the covariant derivative associated to the linear connection L on the vector bundle F . Then, for all sections X, Y of TM and Z of F and all $x \in M$, the curvature is given by*

$$R_x(X(x), Y(x))Z(x) = ([\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z)(x).$$

In particular, the right hand side depends only on the values of X, Y, Z at x .

Proof. The right hand side has the following chart expression (the calculation given in [La99, p. 232] can be repeated verbatim):

$$b_x(b_x(Y, Z), X) - b_x(b_x(X, Z), Y) + \partial_Y b_x(X, Z) - \partial_X b_x(Y, Z) \quad (18.8)$$

where b_x is the Christoffel tensor of L . This coincides with the chart expression for the curvature R given by Eqn. (18.7). ■

18.7. Structure equations. Assume L, L' are two connections on F with covariant derivatives ∇, ∇' and curvature tensors R, R' ; let $A := L' - L = \nabla' - \nabla$ (tensor field of type $(2, 1)$). For $X, Y \in \mathfrak{X}(M)$, we write $A_X(Y) := A(X, Y)$. Then the “new” curvature is calculated from the “old” one via

$$\begin{aligned} R'(X, Y) &= [(\nabla + A)_X, (\nabla + A)_Y] - (\nabla + A)_{[X, Y]} \\ &= R(X, Y) + [A_X, \nabla_Y] - [A_Y, \nabla_X] - A_{[X, Y]} + [A_X, A_Y]. \end{aligned}$$

Thus R' depends on A in a quadratic way. In particular, the set of flat connections can be seen as the zero set of a quadratic map. If we assume that $F = TM$ and that ∇ is torsionfree, then we can write $[\nabla_X, A_Y] - A_{\nabla_X Y} = (\nabla A)(X, Y)$, and the preceding equation reads

$$R'(X, Y) = R(X, Y) + (\nabla A)((X, Y) - (Y, X)) + [A_X, A_Y]. \quad (18.9)$$

If, moreover L was the flat connection of a chart domain, then $R = 0$, and $A : TM \times W \rightarrow W$ can be interpreted as an $\text{End}(W)$ -valued one-form (where W is the model space for the fibers of F) whose exterior derivative is $dA(X, Y) = (\nabla A)((X, Y) - (Y, X))$, which gives the second term. The third term is symbolically written $\frac{1}{2}[A, A]$, and we get

$$R' = dA + \frac{1}{2}[A, A] \quad (18.10)$$

which is a version of the structure equations of E. Cartan. If also L' is flat, we get the condition $dA + \frac{1}{2}[A, A] = 0$ which is another version of the structure equations.

19. Linear structures on jet bundles

19.1. Jet-compatible multilinear connections. We say that a multilinear connection L on $T^k F$ is *weakly symmetric* or *jet-compatible* if the jet-bundle $J^k F = (T^k F)^{\Sigma_k}$ is a *linear* subbundle of the linear bundle $(T^k F, L)$. Equivalently, L is conjugate to all chart-structures under the *symmetric multilinear group* $\text{Sm}^{1,k}((T^k F)_x)$ defined in Section SA.2. In terms of the Dombrowski Splitting Map $\Phi : A^k F \rightarrow T^k F$, the condition is equivalent to saying that

$$\Phi((A^k F)^{\Sigma_k}) = (T^k F)^{\Sigma_k}.$$

A totally symmetric multilinear connection is weakly symmetric (since $\text{Vsm}(E) \subset \text{Sm}(E)$) but the converse is not true: for instance, every connection on TM is weakly symmetric since the chart representation of Φ is $\Phi(x; u, v, w) = (u, v, w + b_x(v, w))$, and this preserves the diagonal $\{(u, u, w) | u, w \in V\}$. On the other hand, Φ not always totally symmetric (it is totally symmetric if and only if b_x is a symmetric bilinear map). More generally, a connection may be weakly symmetric even if it has non-trivial curvature:

Theorem 19.2. *Assume L is a linear connection on F and L' a linear connection on L' . Then the derived multilinear connections $L_k = D^{k-1}L$ on $T^k F$ are weakly symmetric.*

Proof. The claim is proved by induction on k . The case $k = 1$ is trivial since $\Sigma_1 = \{\text{id}\}$.

The case $k = 2$ can be proved by using the explicit formula (17.4) for Φ_2 in a chart: the vector v is fixed under Σ_2 iff $v_1 = v_2$ and $w_{01} = w_{02}$; then (17.4) shows that also $\Phi(v)$ is fixed under Σ_2 .

Now assume that the claim holds for the linear structure L_k on $T^k F$: $\Phi_k((A^k F)^{\Sigma_k}) = (T^k F)^{\Sigma_k}$. In order to prove that $\Phi_{k+1}((A^{k+1} F)^{\Sigma_{k+1}}) \subset (T^{k+1} F)^{\Sigma_{k+1}}$, note first that the set $R = \Sigma_k \cup \{(k, k+1)\}$ generates Σ_{k+1} , where Σ_k is imbedded in Σ_{k+1} as the subgroup permuting $\varepsilon_1, \dots, \varepsilon_k$. Thus the fixed point spaces are

$$\begin{aligned} (A^{k+1} F)^{\Sigma_{k+1}} &= (A^{k+1} F)^{\Sigma_k} \cap (A^{k+1} F)^{(k+1,1)}, \\ (T^{k+1} F)^{\Sigma_{k+1}} &= (T^{k+1} F)^{\Sigma_k} \cap (T^{k+1} F)^{(k+1,1)}, \end{aligned}$$

and it suffices to show that

- (a) $\Phi_{k+1}((A^{k+1} F)^{\Sigma_k}) = (T^{k+1} F)^{\Sigma_k}$ and
- (b) $\Phi((A^{k+1} F)^{(k+1,k)}) = (T^{k+1} F)^{(k+1,k)}$.

In order to prove (a), recall from Formula (17.2) that Φ_{k+1} is defined as a composition of two maps: $\Phi_{k+1} = T\Phi_k \circ \oplus_{\alpha} \Phi_1^{\varepsilon_{k+1}, \alpha}$. We claim that both maps preserve the subspace of Σ_k -invariants, i.e., the following diagram commutes:

$$\begin{array}{ccccc} A^{k+1} F & \xrightarrow{\oplus_{\alpha} \Phi_1^{\varepsilon_{k+1}, \alpha}} & T(A^k F) & \xrightarrow{T\Phi_k} & T(T^k F) \\ \uparrow & & \uparrow & & \uparrow \\ (A^{k+1} F)^{\Sigma_k} & \xrightarrow{\oplus_{\alpha} \Phi_1^{\varepsilon_{k+1}, \alpha}} & (T(A^k F))^{\Sigma_k} & \xrightarrow{T\Phi_k} & T(T^k F)^{\Sigma_k} \end{array}$$

In fact, for the square on the right of the diagram this follows by applying the tangent functor to the induction hypothesis since the action of $\tau \in \Sigma_k$ on $T^{k+1} F$ is simply given by the tangent map $T\tau$ of the action on $T^k F$. The map on the left of the diagram commutes also with the Σ_k -action; this follows from the fact that Σ_k does not act on ε_{k+1} , hence simply interchanges the various copies of Φ_1 used in the construction, and it can also be directly seen from the chart representation

$$\oplus_{\alpha} \Phi_1^{\varepsilon_{k+1}, \alpha} : \sum_{\alpha \in I_{k+1}} \varepsilon^{\alpha} v_{\alpha} \mapsto \sum_{\alpha_{k+1}=0} \varepsilon^{\alpha} v_{\alpha} + \varepsilon_{k+1} v_{k+1} + \sum_{\alpha_{k+1}=0} \varepsilon^{\alpha + e_{k+1}} (v_{\alpha + e_{k+1}} + d_x(v_{\alpha}, v_{e_{k+1}})).$$

Next, we will show that the space of invariants of the transposition $(k, k+1)$ is preserved. In order to prove this, we will use another representation of the map Φ_{k+1} , introducing first a more detailed notation for (17.2): we write $\Phi_j^{\varepsilon_{i_j}, \dots, \varepsilon_{i_1}; \beta}$ for the map $\Phi_j : A^j F^\beta \rightarrow T^k F^\beta$ taken with respect to the sequence $\varepsilon_{i_1}, \dots, \varepsilon_{i_j}$ of scalar extensions of the bundle F^β . Then our recursion formula for $\Phi_{k+1} = \Phi_k^{\varepsilon_{k+1}, \dots, \varepsilon_1; \varepsilon_0}$ reads

$$\Phi_{k+1} = T\Phi_k^{\varepsilon_k, \dots, \varepsilon_1; \varepsilon_0} \circ \left(\times_{\substack{\alpha \in I_k \\ \alpha > 0}} \Phi_1^{\varepsilon_{k+1}, \alpha} \right). \quad (19.1)$$

Applying twice this recursion formula, we get another recursion formula for Φ_{k+1} , this time in terms of Φ_{k-1} and Φ_2 :

$$\begin{aligned} \Phi_{k+1} &= T\Phi_k^{\varepsilon_k, \dots, \varepsilon_1; \varepsilon_0} \circ \left(\times_{\substack{\alpha \in I_k \\ \alpha > 0}} \Phi_1^{\varepsilon_{k+1}, \alpha} \right) \\ &= T^2\Phi_{k-1}^{\varepsilon_{k-1}, \dots, \varepsilon_1; \varepsilon_0} \circ \left(\times_{\substack{\alpha \in I_{k-1} \\ \alpha > 0}} T\Phi_1^{\varepsilon_k; \alpha} \right) \circ \left(\times_{\substack{\alpha \in I_k \\ \alpha > 0}} \Phi_1^{\varepsilon_{k+1}, \alpha} \right) \\ &= T^2\Phi_{k-1}^{\varepsilon_{k-1}, \dots, \varepsilon_1; \varepsilon_0} \circ \left(\times_{\substack{\alpha \in I_{k-1} \\ \alpha > 0}} \Phi_2^{\varepsilon_{k+1}, \varepsilon_k; \alpha} \right). \end{aligned} \quad (19.2)$$

Now, $T^2\Phi_{k-1}$, being a second tangent map with respect to scalar extensions $\varepsilon_{k+1}, \varepsilon_k$, commutes with the transposition $(k, k+1)$. As seen for the case $k=2$ (beginning of our induction), $\Phi_2^{\varepsilon_{k+1}, \varepsilon_k; \alpha}$ preserves the subspace fixed under $(k, k+1)$. Altogether, it follows that also Φ_{k+1} preserves the subspace fixed under $(k, k+1)$, as had to be shown. ■

19.3. Multilinear differential calculus for jets. If L is a jet-compatible connection on $T^k F$, then structures such as the linearization map or the higher differentials from Chapter 16 can be restricted to the subbundle $J^k F$. We give a short description of the theory, where, for simplicity, in the following we only consider the case of the tangent bundle, $F = TM$, leaving to the reader the generalization to arbitrary vector bundles. The model for the jet bundle $J^k M$ is the subbundle of $A^k M$ fixed under Σ_k ,

$$(A^k M)^{\Sigma^k} = \left(\bigoplus_{\alpha} \varepsilon^\alpha TM \right)^{\Sigma_k} = \bigoplus_{j=1}^k \delta^{(j)} TM$$

with the $\delta^{(j)}$ as in (8.1). Note, however, that the axes of $A^k M$ are not invariant under Σ_k , and hence there are no natural inclusions of the factors $\delta^j TM$ in $J^k M$. If L is a jet-compatible connection on $T^k M$, then it induces a bundle isomorphism

$$J\Phi : \bigoplus_{j=1}^k \delta^{(j)} TM \rightarrow J^k M,$$

and $J\Phi$ is the same for all connections L^σ , $\sigma \in \Sigma_k$. In particular, if $L = L_k = D^{k-1}K$ is derived from some linear connection K on TM , then $J\Phi$ depends only on K and not on the order of the scalar extensions in the definition of the iterated tangent functor T^k .

Now let $f : M \rightarrow N$ be a smooth map between manifolds with linear connections K on TM and L on TN . Restricting the covariant derivative $d_K^L f$ from Section 16.3 to a map $J_K^L f : J^k M \rightarrow J^k N$,

$$\begin{array}{ccc} J^k M & \xrightarrow{\Phi^k} & \bigoplus_{j=1}^k \delta^{(j)} TM \\ J^k f \downarrow & & \downarrow J_K^L f \\ J^k N & \xrightarrow{\Phi^k} & \bigoplus_{j=1}^k \delta^{(j)} TN \end{array},$$

we get a jet version of the higher covariant derivatives that no longer depends on the standard order of scalar extensions but only on the connections themselves. Restricting the expression (16.6) of $T^k f$ by its matrix coefficients to spaces of Σ_k -invariants and recalling from Section MA.3 that $\lambda \sim \mu$ iff $l(\lambda) = l(\mu) =: l$ and $|\lambda^j| = |\mu^j| =: i_j$ for all j , we get

$$J^k f(x + \sum_j \delta^{(j)} v_j) = f(x) + \sum_{j=1}^k \delta^{(j)} \sum_{l=1}^j \sum_{\substack{i_1 \leq \dots \leq i_l \\ i_1 + \dots + i_l = j}} (d^{(i_1, \dots, i_l)} f)_x(v_{i_1}, \dots, v_{i_l})$$

with the “matrix coefficient”

$$(d^{(i_1, \dots, i_l)} f)_x(v_{i_1}, \dots, v_{i_l}) = \sum_{\substack{\lambda: \forall p=1, \dots, l: \\ |\lambda^p|=i_p}} (d^\lambda f)_x(v_{i_1}, \dots, v_{i_l}).$$

These matrix coefficients will be symmetric as multilinear maps if we use a totally symmetric connection, and if we use a shift-invariant connection (see next chapter), then there will be relations between the various matrix coefficients; if the connection is both very symmetric and shift-invariant, then the formula can be rewritten such that combinatorial constants similar to the ones in (8.14) appear. Details will be given elsewhere.

20. Shifts and symmetrization

20.1. Totally symmetrizable multilinear connections. We say that a multilinear connection L on $T^k F$ is (totally) *symmetrizable* if all its curvature operators $R^{\sigma, L}$ belong to the central vector group $\text{Gm}^{k-1, k}(T^k F)$ of $\text{Gm}^{1, k}(T^k F)$. Equivalently, all curvature forms \tilde{R}^Λ for $l(\Lambda) < k$ vanish, and only the top curvature form possibly survives.

Theorem 20.2. *Assume that L is a totally symmetrizable multilinear connection on $T^k F$ and that $2, \dots, k$ are invertible in \mathbb{K} . Then all connections $\sigma \cdot L$, $\sigma \in \Sigma_k$, belong to the orbit $\text{Gm}^{k-1, l}(E) \cdot L$, which is an affine space over \mathbb{K} , and their barycenter in this affine space,*

$$\text{Sym}(L) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma \cdot L,$$

is a very symmetric multilinear connection on $T^k F$. Moreover, if L is invariant under Σ_{k-1} , then

$$\text{Sym}(L) = \frac{1}{k} \sum_{j=1}^k (12 \dots k)^j \cdot L.$$

Proof. This follows by applying pointwise the corresponding purely algebraic result (Cor. SA.9). ■

Corollary 20.3. *Assume L is a torsionfree connection on TM . Then the linear structure DL on $T^3 M$ is symmetrizable.*

Proof. According to Theorem 18.3, the only non-vanishing curvature form of DL is the top curvature form, which means that DL is symmetrizable. ■

In the situation of the corollary, another proof of Bianchi's identity (cf. Section 17.5) is obtained by observing that the curvature of $\text{Sym}(L)$ vanishes – see Section SA.10 for the purely algebraic argument.

20.4. Shift invariance. Recall that, for any scalar $r \in \mathbb{K}$, we have the canonical action $r_{TM} : TM \rightarrow TM$ by multiplication with r in tangent spaces. Now, $T^k M$ is a manifold over the ring $T^k \mathbb{K}$, and hence, for all $\beta \in \{0, 1\}^k$, we have a canonical action of ε^β on $T(T^k M)$:

$$S_{k+1, \beta} := (\varepsilon^\beta)_{T(T^k M)} : T_{\varepsilon_{k+1}}(T^k M) \rightarrow T_{\varepsilon_{k+1}}(T^k M).$$

Note that, with respect to the lexicographic order, $\beta < e_{k+1}$; we then say that $S_{k+1, \alpha}$ is a *positive shift (in direction β with base e_{k+1})*. Of course, choosing another order we may define shifts $S_{i, \beta}$ for any β and i such that $\beta \perp e_i$. If $\beta = e_j$, i.e. $\varepsilon^\beta = \varepsilon_j$, we write $S_{ij} := S_{i, e_j}$, called an *elementary shift (in direction j with basis i)*; if $i < j$ it is a *negative shift*. In a chart, the elementary shift is given by

$$S_{ij} : T^k M \rightarrow T^k M, \quad (x; \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha) \mapsto (x; \sum_{\substack{\alpha_i=1 \\ \alpha_j=0}} \varepsilon^{\alpha+e_j} v_\alpha + \sum_{\alpha_i=0} \varepsilon^\alpha v_\alpha).$$

Every shift is a composition of elementary shifts, and the square of any shift is zero. For $k = 2$, the two elementary shifts are precisely the two almost dual structures on TTM (Section 4.8). Invariance under shifts is a natural condition: Formula (7.19) shows that, for any smooth map $f : M \rightarrow N$, the higher order tangent map $T^k f : T^k M \rightarrow T^k N$ commutes with all shift operators S_{ij} . Thus invariance under shifts and under permutations are necessary conditions on a multilinear map $F : T^k M \rightarrow T^k N$ to be *integrable*, i.e. to be of the form $F = T^k f$ for some $f : M \rightarrow N$. More generally, the shifts may also be defined for general vector bundles: for instance, on TF there is just one shift S (action of ε on the fiber of the projection $TF \rightarrow TM$).

We say that a multilinear connection L on $T^k F$ is (*positively*) *shift-invariant* if all (positive) shifts are linear operators with respect to L . Equivalently, L is G -related to all chart structures where $\text{Shi}(T^k M)$, resp. $\text{Shi}_+(T^k M)$ are the (positively) shift invariant groups defined by (SA.20). Still equivalently, the map $\Phi : A^k F \rightarrow T^k F$ is equivariant under the groups $\text{Shi}(T^k M)$, resp. $\text{Shi}_+(T^k M)$.

Theorem 20.5. *Assume L is a torsionfree connection on TM .*

- (1) *The linear structure DL on $T^3 M$ is invariant under all shifts.*
- (2) *The linear structure $D^k L$ on $T^{k+2} M$ is invariant under the shifts $S_{k+2, k+1}$ and $S_{k+1, k+2}$.*

Proof. (1) We use the chart representation (17.4) for the Christoffel tensor of DL :

$$D\Phi_1(x, v) = v + \varepsilon_0 \varepsilon_1 b_x(w_0, v_1) + \varepsilon_0 \varepsilon_2 b_x(w_0, v_2) + \varepsilon_1 \varepsilon_2 b_x(v_1, v_2) + \varepsilon_0 \varepsilon_1 \varepsilon_2 (b_x(w_0, v_{12}) + b_x(w_{01}, v_2) + b_x(w_{02}, v_1) + t(w_0, v_1, v_2)) \quad (20.1)$$

with trilinear component $t(w_0, v_1, v_2) = b_x(w_0, b_x(v_1, v_2)) + b_x(b_x(w_0, v_2), v_1) + \partial_{v_2} b_x(w_0, v_1)$. Clearly, all bilinear components of $D\Phi_1(x, \cdot)$ are conjugate to each other, and hence the Shift Invariance Condition (Prop. SA.13) is fulfilled.

(2) The claim is proved by induction. For $k = 1$ it follows from Part (1). Now assume the claim is proved for $k > 1$. We use the recursion formula (19.2) for Φ_{k+1} :

$$\Phi_{k+1} : A^{k+1} F \xrightarrow{\oplus_\alpha \Phi_2^{e_{k+1}, e_k; \alpha}} TT(A^{k-1} F) \xrightarrow{TT\Phi_{k-1}} TT(T^{k-1} F) = T^{k+1} F$$

The map $TT\Phi_{k-1}$, like any second tangent map, commutes with both shifts $S_{k+1, k}$ and $S_{k, k+1}$. By the case $k = 1$ of the induction, $\Phi_2^{e_{k+1}, e_k; \alpha}$ also commutes with both shifts $S_{k+1, k}$ and $S_{k, k+1}$, whence the claim. ■

As seen in the algebraic part (Theorem SA.14), the condition of invariance under *all* shifts is a rather strong condition: it implies in turn that DL is symmetrizable. – Part (1) of the preceding theorem does not carry over in this form to arbitrary k : using the recursion formula (17.3), one can see that $D^2 L$ is in general no longer invariant under the shifts S_{12} and S_{21} . On the other hand, the statement (2) is not the strongest possible result: combinatorial arguments, using again the recursion formula (17.3), show that $D^k L$ is invariant under all shifts of the form $S_{i, k+2}$, $i < k + 2$. One would like to understand this situation in a conceptual way.

Corollary 20.6. *Under the assumptions of the theorem, and if 3 is invertible in \mathbb{K} , then the linear structure $\text{Sym}(DL)$ on $T^3 M$ is totally symmetric and invariant under all shifts.*

Proof. In a chart, symmetrization amounts to replace in Equation (19.1) the trilinear map t by its symmetrized version $\frac{1}{3}(t + (123).t + (132).t)$, whereas the other components do not change (recall that b_x is already symmetric, by torsionfreeness). Thus $\text{Sym}(DL)$ still satisfies the shift invariance condition SA.13. ■

Theorem 20.7. *Assume L is a linear structure on TM without torsion and such that all curvature forms of DL vanish. Then all linear structures $D^k L$, $k \in \mathbb{N}$, are totally symmetric and invariant under all shifts.*

Proof. The proof is by induction on k : for $k = 1$, DL is totally symmetric by assumption and shift-invariant by Theorem 20.5. For the induction step, we use exactly the same arguments as in the proof of Theorem 19.2, replacing the property of preserving the space of Σ_k -invariants by full Σ_k -invariance. In this way it is seen that $D^k L$ also is Σ_{k+1} -invariant, i.e. all $D^k L$ are totally symmetric. By Theorem 20.5 they are invariant under one shift (namely $S_{k+1, k}$), but since they are totally symmetric and all other shifts are conjugate among each other under the permutation group, $D^k L$ is then invariant under all shifts. ■

For a general linear structure L on TM , if all integers are invertible in \mathbb{K} , one would like to define a sequence of totally symmetric and shift-invariant multilinear structures coming from the

linear structures $D^k L$, $k = 1, 2, \dots$. See also Section 27.7 for other aspects of this problem. Unfortunately, already $D(\text{Sym}(DL))$ is in general no longer totally symmetrizable in the sense of 20.1, and thus an appropriate symmetrization procedure has to be more subtle than the one used in Theorem 20.2. In the following chapter we recall a similar construction in a context which is “dual” to the one treated here, namely in the context of linear differential operators.

21. Remarks on differential operators and symbols

21.1. Overview over definitions of differential operators. In the literature several definitions of differential operators can be found. They agree for finite-dimensional real manifolds but may lead to quite different concepts in our much more general setting. All have in common that a *linear differential operator* between vector bundles $F \rightarrow M$ and $F' \rightarrow M$ should be a \mathbb{K} -linear map

$$D : \Gamma^\infty(F) \rightarrow \Gamma^\infty(F')$$

from sections of F to sections of F' satisfying some additional conditions such as:

- (1) D is *support-decreasing*. Since there is no hope to prove an analog of the classical Peetre theorem (cf., e.g., [Hel84, Th. II.1.4] or [KMS93, Ch. 19]) in our setting, this cannot serve as possible definition in the general case.
- (2) The value $Df(x)$ depends only on the k -jet of the section $f : M \rightarrow F$ at x . More precisely, let $J^k(M, F, \text{sec})$ be the space of k -jets of sections of F (if $F = M \times \mathbb{K}$ is the trivial bundle, then $J^1(M, F, \text{sec})$ essentially is the cotangent bundle T^*M). Following [KMS93, p. 143], we say that D is *local and of order k* if there is a map $D_k : J^k(M, F, \text{sec}) \rightarrow F'$ such that $Df(x) = D_k(J_x^k f)$. In case of the trivial line bundle, i.e., $F = F' = M \times \mathbb{K}$, a differential operator of order one is thus seen as a section of the dual bundle of T^*M , and not as a section of TM . Already this example shows that, when trying to generalize this definition, we inevitably run into difficulties related to double dualization.
- (3) Following the elegant algebraic definition of differential operators given e.g. in [ALV91, p. 85] or [Nes03, p. 125, p. 131], we may require that there exists $k \in \mathbb{N}$ such that, for all smooth functions $f_0, \dots, f_k \in \mathcal{C}^\infty(M, \mathbb{K})$,

$$[\dots [[D, f_0], f_1] \dots f_k] = 0,$$

where, for an operator A from sections to sections, $[A, f]X = A(fX) - fA(X)$. Moreover, one would add some smoothness condition. It is not hard to see that (2) implies (3), whereas already in the infinite-dimensional Banach case the converse fails in general (cf. remarks in Section 4.6). Hence it seems that this definition is best suited for the finite-dimensional case.

- (4) In the special case $F = F'$, we may assume that there exists $k \in \mathbb{N}$ such that D can (at least locally over a chart domain) be written as a finite product of operators $A \nabla_{X_1} \circ \dots \circ \nabla_{X_k}$, where ∇ runs over the covariant derivatives associated to connections on F and A is a field of endomorphisms (cf. [BGV92, p.64]). In case $F = M \times \mathbb{K}$ is the trivial bundle over a finite-dimensional manifold, this means that, in a chart, $D = g_0 + \sum_\alpha g_\alpha \partial^\alpha$ is a sum a partial derivatives multiplied by functions. This is the most classical notion; it implies (2) and hence (3).

All of these concepts are useful, and hence one should distinguish between different classes of differential operators. In any case, k is the *degree* or *order of D* , and the degree defines a natural filtration on the space of differential operators. Although the second approach is, in a sense, the most conceptual one, we will not follow it here because in our general context one cannot deal with features related to double duals; the approach outlined in the sequel is closest to (4). Things will be most clear if we start with differential operators on the trivial line bundle, i.e. with scalar differential operators.

21.2. Pure differential operators. A *pure scalar differential operator of order at most k* ($k \geq 1$) on a manifold M is a smooth section $D : M \rightarrow T^k M$ of the canonical projection $T^k M \rightarrow M$. If $f : M \rightarrow \mathbb{K}$ is a smooth function, then $T^k f \circ D : M \rightarrow T^k \mathbb{K}$ is smooth, and we let

$$Df := \text{pr}_k \circ T^k f \circ D : M \rightarrow \mathbb{K}, \tag{21.1}$$

where $\text{pr}_k : T^k \mathbb{K} \rightarrow \mathbb{K}$, $\sum_{\alpha \geq 0} a_\alpha \varepsilon^\alpha \mapsto a_{(1, \dots, 1)}$ is the projection onto the last factor. Thus D induces a map

$$D^{op} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \quad f \mapsto Df$$

which clearly is \mathbb{K} -linear. Let $\mathcal{D}_k(M)$ be the \mathbb{K} -span of all such operators in $\text{End}_{\mathbb{K}}(\mathcal{C}^\infty(M))$, called the space of *linear scalar differential operators of degree at most k* . In a chart, using Formula (7.19), the “constant” operator $D = \sum_{\alpha > 0} \varepsilon^\alpha v_\alpha$ acts by

$$\left(\sum_{\alpha > 0} \varepsilon^\alpha v_\alpha \cdot f \right)(x) = \text{pr}_k(T^k f(x + \sum_{\alpha} v_\alpha)) = \sum_{l=1}^k \sum_{\beta \in \mathcal{P}_l(1, \dots, 1)} d^l f(x)(v_{\beta^1}, \dots, v_{\beta^l}) \quad (21.2)$$

i.e.

$$\left(\sum_{\alpha > 0} \varepsilon^\alpha v_\alpha \right)^{op} = \sum_{l=1}^k \sum_{\beta \in \mathcal{P}_l(1, \dots, 1)} \partial_{v_{\beta^1}} \cdots \partial_{v_{\beta^l}}$$

is a scalar differential operator in the most classical sense. For instance, $\varepsilon_1 v_1 + \varepsilon_2 v_2$ acts as $\partial_{v_1} \partial_{v_2}$ and $\delta^{(2)} v$ as ∂_v^2 . The chart formula also shows that, for all $\sigma \in \Sigma_k$ acting on $T^k M$ in the canonical way, $\sigma \circ D$ and D give rise to the same operator; therefore, if $2, \dots, k$ are invertible in \mathbb{K} , then we could as well work with sections of $J^k M$. Also, if $T^k z : T^k M \rightarrow T^{k+1} M$ is the injection obtained by deriving some zero section $z = z_j : M \rightarrow T_{\varepsilon_j} M$, then D and $z \circ D$ give rise to the same operator on functions. Somewhat loosely, we thus may say that a differential operator of order k is also one of degree $k+1$. (Taking in the definition of Df instead of pr_k the projection onto the “augmentation ideal” of $T^k \mathbb{K}$, one gets another kind of operators satisfying an analog of the Leibniz rule, called *expansions* in [KMS93, Section 37.6].)

The *composition* of two pure differential operators $D_i : M \rightarrow T^{k_i} M$, $i = 1, 2$, is defined by

$$D_2 \cdot D_1 := T^{k_1} D_2 \circ D_1 : M \rightarrow T^{k_1} M \rightarrow T^{k_1+k_2} M. \quad (21.3)$$

It is easily checked that this composition is associative, and that $D \mapsto D^{op}$ is a “contravariant” representation:

$$\begin{aligned} (D_2 \cdot D_1)^{op} f &= \text{pr}_{k_1+k_2} \circ T^{k_1+k_2} f \circ (D_2 \cdot D_1) \\ &= \text{pr}_{k_1+k_2} \circ T^{k_1+k_2} f \circ T^{k_1} D_2 \circ D_1 \\ &= \text{pr}_{k_1+k_2} \circ T^{k_1} (T^{k_2} f \circ D_2) \circ D_1 \\ &= \text{pr}_{k_1} \circ T^{k_1} (\text{pr}_{k_2} \circ T^{k_2} f \circ D_2) \circ D_1 \\ &= (D_1^{op} \circ D_2^{op}) f. \end{aligned}$$

It follows that $\mathcal{D}^k(M) \cdot \mathcal{D}^l(M) \subset \mathcal{D}^{k+l}(M)$, and hence $\mathcal{D}^\infty(M) := \bigcup_{i=1}^\infty \mathcal{D}^i(M)$ is an associative algebra over \mathbb{K} .

21.3. *Principal symbol of a scalar differential operator.* Let $D : M \rightarrow T^k M$ be a section and

$$\text{pr} : T^k M \rightarrow \varepsilon_1 TM \times \dots \times \varepsilon_k TM$$

be the product of the canonical projections, and let

$$t : \varepsilon_1 TM \times \dots \times \varepsilon_k TM \rightarrow \varepsilon_1 TM \otimes \dots \otimes \varepsilon_k TM$$

be the tensor product map (where $\otimes = \otimes_M$ is the algebraic tensor product of vector bundles over M , i.e. given by tensor products in fibers, without considering a topology). We say that

$$\sigma^k(D) := t \circ \text{pr} \circ D : M \rightarrow \varepsilon_1 TM \otimes \dots \otimes \varepsilon_k TM$$

is the *principal symbol of D* . It is a section of the (algebraic) bundle $\otimes^k TM$ over M . Comparison with the chart representation (21.2) shows that the principal symbol of $\sum_{\alpha > 0} \varepsilon^\alpha v_\alpha$ is $v_1 \otimes \dots \otimes v_k$, coming from the “highest term” $\partial_{v_1} \cdots \partial_{v_k}$. If D happens to be a section of $J^k M$, then $\sigma^k(D)$

is a section of the algebraic bundle $S^k(TM)$ over M (k -th symmetric power of TM). If the “operator representation” $D \mapsto D^{op}$ is injective, then one can show that the principal symbol map factors through a \mathbb{K} -linear map

$$\tilde{\sigma}^k : \mathcal{D}^k(M) \rightarrow \Gamma(S^k TM)$$

(where $\Gamma(S^k TM)$ is the space of sections that locally are finite sums of t composed with smooth sections of $TM \times \dots \times TM$), i.e. $\sigma^k(D) = \tilde{\sigma}^k(D^{op})$, and that the sequence

$$0 \rightarrow \mathcal{D}^{k-1}(M) \rightarrow \mathcal{D}^k(M) \xrightarrow{\tilde{\sigma}^k} \Gamma(S^k(TM)) \rightarrow 0 \quad (21.4)$$

is exact. Details are left to the reader.

21.4. Correspondence between total symbols and differential operators (scalar case). For real finite dimensional manifolds it is known that, in presence of connection on TM , one can associate to a (scalar) differential operator a *total symbol* (which is a section of $S^k TM$) in such a way that this correspondence becomes a bijection between operators and their total symbols.¹ For the case of second order operators, the construction is essentially described in [Lo69] (see also [P62]); the general construction seems to be folklore – the few references we know about include [BBG98] and [Bo02] (where the construction is called *préscription d'ordre standard*). Let us briefly describe the construction: clearly, the problem is equivalent to find $C^\infty(M)$ -linear sections of the exact sequences (21.4) for all k . Using the covariant derivative ∇ of L , for a function f one defines $\nabla^k f : \times^k TM \rightarrow \mathbb{K}$, the “ k -th order Hessian of f ” (defined as usual in differential geometry by $\nabla^1 f := df$, $\nabla^{j+1} f := \nabla(\nabla^j f)$). Then to a section s of $S^k TM$ we associate a differential operator $D := D(s)$: if $s(x) = v_1 \otimes \dots \otimes v_k$ with all $v_j \in T_x M$, let

$$Df(x) = (\nabla^k f)(x; v_1, \dots, v_k).$$

One easily proves that $\tilde{\sigma}^k D(s) = s$, hence $s \mapsto D(s)$ gives the desired section of the exact sequence (21.4). This construction should be seen as an analog of the construction of linear structures on higher order tangent bundles (Chapter 17); it can be adapted to the case of differential operators acting on sections of (finite dimensional) vector bundles over finite dimensional real manifolds (cf. [Bo02]).

21.5. Differential operators on general vector bundles. First of all, differential operators on a trivial bundle $M \times W$ may be defined as in the scalar case. For general vector bundles, the problem arises that we need a good class of differential operators of degree zero (which can be avoided on trivial vector bundles). The reason for this is that, although vector bundles can be trivialized locally, the preceding definition of differential operator over chart domains would no longer be chart-independent. The smallest class of differential operators of degree zero which one could take here are fields of endomorphisms that arise precisely from chart changes. Taking a somewhat bigger class, one arrives at Definition (4) of a differential operator mentioned in Section 21.1.

Finally, in order to define differential operators acting from sections of one vector bundles F , to sections of *another* bundle F' , one needs to single out some subspace in $\text{Hom}_{\mathbb{K}}(F_x, F'_x)$ giving the differential operators of degree zero; but there is no natural candidate for such a space (in the general case).

¹ [ALV91, p.87]: “Obviously, to one and only one symbol there correspond many differential operators. However, it is a remarkable fact that in presence of a connection this correspondence can be made one-to-one.” On the same page, the authors add the remark: “This procedure of restoring a differential operator from its symbol is completely analogous to the basic procedure of quantum mechanics – quantization.”

22. The de Rham complex

22.1. Skewsymmetric derivatives. In the preceding chapters, we have focused on symmetric derivatives. They depend on an additional structure (connection) and thus may be considered as “not natural”. The skewsymmetric version of the derivative (the exterior derivative), in contrast, is completely natural, and therefore plays a dominant role in differential geometry¹. Since the exterior derivative is so natural, we wish to give also a definition which, in our context, is as natural as possible. Recall (Section 4.5) that we consider a differential k -form ω as a smooth map $TM \times_M \dots \times_M TM \rightarrow \mathbb{K}$ which is multilinear and alternating in fibers. One could proceed e.g. as in [La99, Ch. V], where the exterior derivative of a differential form ω is defined in a chart by

$$\begin{aligned} d\omega(x; v_1, \dots, v_{k+1}) &= \sum_{\sigma \in \Sigma_{k+1}} \operatorname{sgn}(\sigma) \partial_{v_{\sigma(1)}} \omega(\widehat{v_{\sigma(1)}}, v_{\sigma(2)}, \dots, v_{\sigma(k+1)}) \\ &= \sum_{i=1}^{k+1} (-1)^i \partial_{v_i} \omega(v_1, \dots, \widehat{v_i}, \dots, v_{k+1}). \end{aligned} \quad (22.1)$$

Then one shows that evaluation of $d\omega$ on vector fields X_1, \dots, X_k is given by the formula

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^i X_i \omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) + \\ &\quad \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}), \end{aligned} \quad (22.2)$$

which implies that (22.1) is chart-independent. The usual properties of the exterior derivative then follow by standard calculations. A “natural definition” in our sense should not rely on chart formulas, and it should be built only on the natural operators T (tangent functor) and alt (alternation operator). For the exterior derivative of one-forms this has been carried out in Chapter 13. Here is the pattern for the general case:

- (1) We define the notion of an (*intrinsic*) k -multilinear map $f : T^k M \rightarrow \mathbb{K}$, and we define also *homogeneous* such maps.
- (2) There is a canonical bijection between tensor fields $\underline{\omega} : \times_M^k TM \rightarrow \mathbb{K}$ and homogeneous k -multilinear maps $\overline{\omega} : T^k M \rightarrow \mathbb{K}$.
- (3) If $f : T^k M \rightarrow \mathbb{K}$ is k -multilinear, then $\operatorname{pr}_2 \circ T f : T^{k+1} M \rightarrow \varepsilon \mathbb{K}$ is $k+1$ -multilinear.
- (4) The alternation operator alt maps intrinsic k -multilinear maps onto homogeneous ones.
- (5) Finally, we let $d\underline{\omega} := \operatorname{alt}(\operatorname{pr}_2 T(\overline{\omega}))$. In other words, the natural operator d is defined by the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{T}^{k,0}(M) & \cong & \mathcal{M}_h(T^k M) & \subset & \mathcal{M}(T^k M) \\ d \downarrow & & & & \downarrow T \\ \mathcal{T}^{k+1,0}(M) & \cong & \mathcal{M}_h(T^{k+1} M) & \xleftarrow{\operatorname{alt}} & \mathcal{M}(T^{k+1} M) \end{array}$$

where \mathcal{T} denotes tensor fields and \mathcal{M} (resp. \mathcal{M}_h) intrinsic multilinear (homogeneous) maps.

- (6) The property $d \circ d = 0$ follows from the fact that $\operatorname{alt} \circ T \circ T = 0$ due to symmetry of second differentials.

In the following, we carry out this program.

22.2. Multilinear maps on $T^k M$. We say that a smooth map $\omega : T^k M \rightarrow \mathbb{K}$ is (*intrinsically*) *multilinear* if, for all $x \in M$, the restriction $\omega_x : (T^k M)_x \rightarrow \mathbb{K}$ to the fiber of $T^k M$ over x

¹ Cf., e.g., [Sh97, p. 54]: “As usual in modern differential geometry, we shall be concerned only with the *skew-symmetric* part of the higher derivatives.”

is intrinsically multilinear in the sense of Section MA.15. This means that, for all $u \in T^{k-1}M$ and all $j = 1, \dots, k$, the restriction of ω to the tangent spaces $H_j := T_u(T^{k-1}M)^j \subset T^kM$ is \mathbb{K} -linear, where the index j indicates that the last scalar extension is taken with respect to ε_j . Put differently, the fiber $E := (T^kM)_x$ has a fibration by hyperplanes (fibers of the various projections $T^kM \rightarrow T^{k-1}M$); ω is intrinsically multilinear if all these restrictions are linear. We say that ω is *homogeneous* if ω_x is homogeneous in the sense of MA.15, i.e. the restriction of ω_x to intersections $H_i \cap H_j$ ($i \neq j$) is constant. As has been shown in Prop. MA.16, homogeneous multilinear maps f are in bijection with multilinear maps (in the usual sense) $\underline{f} : \varepsilon_1 T_x M \times \dots \times \varepsilon_k T_x M \rightarrow \mathbb{K}$. Applying this fiberwise, we see that

$$\mathcal{T}^{k,0}(M, \mathbb{K}) \rightarrow \mathcal{M}_h(T^k M, \mathbb{K}), \quad \omega \mapsto \bar{\omega} = (v \mapsto \omega(\text{pr}_1(v), \dots, \text{pr}_k(v)))$$

is a bijection. The inverse map is defined locally in a chart, or globally by a connection L (if a connection exists), by

$$\mathcal{M}(T^k M, \mathbb{K}) \rightarrow \mathcal{T}^{k,0}(M, \mathbb{K}), \quad \bar{\omega} \mapsto \underline{\omega} := \bar{\omega} \circ \iota^L, \quad (22.3)$$

where $\iota^L : \times^k T M \rightarrow T^k M$ is obtained from the Splitting Map $\Phi_k : \oplus_\alpha \varepsilon^\alpha T M \rightarrow T^k M$. The map (22.3) is a surjection and a bijection when restricted to $\mathcal{M}_h(T^k M, \mathbb{K})$, the space of homogeneous multilinear maps. The situation is summarized by the following diagram (cf. (MA.36)):

$$\begin{array}{ccc} T^k M & & \\ \text{pr} \downarrow \uparrow \iota^L & \searrow \bar{\omega} & \\ \times_M^k T M & \xrightarrow{\underline{\omega}} & \mathbb{K} \end{array}$$

Later on, we will consider $\bar{\omega}$ and $\underline{\omega}$ as the same object ω .

Lemma 22.3. *If $\bar{\omega} : T^k M \rightarrow \mathbb{K}$ is k -multilinear, then $\text{pr}_2 \circ T\bar{\omega} : T^{k+1} M \rightarrow \varepsilon \mathbb{K}$ is $k+1$ -multilinear.*

Proof. Let the last tangent functor correspond to scalar extension with ε_{k+1} . Then we have two kinds of tangent spaces in $T^{k+1}M$: the first kind is given by spaces that are stable under ε_{k+1} ; they correspond to projections p_i , $i = 1, \dots, k$, and are tangent bundles of tangent spaces in $T^k M$. The restriction of Tf to such tangent spaces is the tangent map of f restricted to tangent spaces in $T^k M$ and hence is linear. The second kind of tangent spaces corresponds to the last projection p_{k+1} ; but since Tf is defined with respect to ε_{k+1} , the restriction of Tf to such tangent spaces is linear as it is a tangent map. ■

Lemma 22.4. *If $\bar{\omega} : T^k M \rightarrow \mathbb{K}$ is k -multilinear, then*

$$\text{alt } \bar{\omega} : T^k M \rightarrow \mathbb{K}, \quad z \mapsto \sum_{\sigma \in \Sigma_k} \text{sgn}(\sigma) \bar{\omega}(\sigma(z))$$

is homogeneous k -multilinear.

Proof. It is clear that $\text{alt}(\bar{\omega})$ is k -multilinear since it is a sum of multilinear functions. It remains to prove homogeneity: notation being as in MA.15, we have to show that $\text{alt } \bar{\omega}(v) = 0$ for $v \in H_{ij}$ with $i \neq j$. Let $\tau = (ij)$ be transposition between i and j ; then τ acts trivially on H_{ij} since $v = \sum_\alpha \varepsilon^\alpha v_\alpha$ where $\alpha_i = 1 = \alpha_j$ for all α contributing effectively to the sum. It follows that $\tau v = v$ and hence

$$\text{alt } \bar{\omega}(v) = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \bar{\omega}(\sigma v) = \sum_{\sigma \in A_n} (\bar{\omega}(\sigma v) - \bar{\omega}(\sigma \tau v)) = 0. \quad \blacksquare$$

22.5. Exterior derivative. The exterior derivative of a k -form $\underline{\omega} : \times^k T M \rightarrow \mathbb{K}$ is now defined by

$$d\underline{\omega} := \underline{\text{alt}}_{k+1} \text{pr}_2 T\bar{\omega} = \underline{\text{pr}}_2 \underline{\text{alt}}_{k+1} T\bar{\omega}.$$

In fact, the same definition can be given for any tensor field $\underline{\omega} : T^k M \rightarrow \mathbb{K}$ (not necessarily skew-symmetric), but then $d\underline{\omega} = d(\text{alt}\underline{\omega})$, so this gives nothing really new. Similarly, we define the exterior derivative of $\overline{\omega} \in \mathcal{M}_h(T^k M)$ by

$$d\overline{\omega} := \text{alt}_{k+1}(\text{pr}_2 T\overline{\omega}) = \text{pr}_2(\text{alt}_{k+1} T\overline{\omega}) \in \mathcal{M}_h(T^{k+1} M).$$

Theorem 22.6. *For all k -forms $\underline{\omega}$, we have $dd\underline{\omega} = 0$.*

Proof. We show, equivalently, that $dd\overline{\omega} = 0$, i.e.

$$0 = \text{alt}_{k+2} \text{pr}_2 T \text{alt}_{k+1} \text{pr}_2 T\overline{\omega} = \text{pr}_4 \text{alt}_{k+2} T \text{alt}_{k+1} T\overline{\omega}.$$

First of all, for all $\sigma \in \Sigma_k$, acting on $T^k M$, we have $T(\overline{\omega} \circ \sigma) = T\overline{\omega} \circ T\sigma$, where $T\sigma$ is the action of σ on $T^{k+1} M$ if we inject Σ_k into Σ_{k+1} in the natural way. Hence $T \circ \text{alt}_k = \text{alt}_k \circ T$ for all k , and from this follows $\text{pr}_2 T \circ \text{alt}_k = \text{alt}_k \circ \text{pr}_2 T$, and our claim is equivalent to

$$0 = \text{pr}_4 \text{alt}_{k+2} T \text{alt}_{k+1} T \text{pr}_2 T\overline{\omega} = \text{pr}_4 \text{alt}_{k+2} \text{alt}_{k+1} T T\overline{\omega}.$$

But, by symmetry of the second differential, $T T\overline{\omega}$ is invariant under the permutation $(k+2, k+1) \in \Sigma_{k+2}$ which corresponds to exchanging ε_{k+2} and ε_{k+1} , and hence applying the alternation operator alt_{k+2} annihilates the whole expression. ■

The *de Rham cohomology* is now defined as usual:

$$H^k(M) := \ker(d^{k+1}) / \text{im}(d^k).$$

22.7. Exterior derivative and covariant derivative. If L is a linear connection of TM , we define the covariant derivative of a tensor field $\underline{\omega} : \times^k TM \rightarrow \mathbb{K}$ by

$$\nabla \underline{\omega} := \text{pr}_2 T\overline{\omega} \in \mathcal{T}^{k+1,0}(M),$$

where $f \mapsto \underline{f} = f \circ \iota^L$ is given by (22.3) and depends on the connection L .

Proposition 22.8. *For all k -forms $\underline{\omega} : \times^k M \rightarrow \mathbb{K}$,*

$$d\underline{\omega} = \text{alt}_{k+1}(\nabla \underline{\omega}).$$

Proof.

$$\text{alt}_{k+1}(\nabla \underline{\omega}) = \text{alt}_{k+1} \text{pr}_2 T\overline{\omega} = \underline{\text{alt}_{k+1} \text{pr}_2 T\overline{\omega}} = d\underline{\omega}. \quad \blacksquare$$

Corollary 22.9. *In a bundle chart, $d\omega$ is given by (22.1), and evaluation of $d\omega$ on vector fields X_1, \dots, X_k is given by (22.2).*

Proof. Let ∇ be the covariant derivative induced by the chart, i.e. $\nabla = d$ is ordinary derivative. Then

$$(\nabla \omega)(x; v_1, \dots, v_{k+1}) = \partial_{v_{k+1}} \omega(x; v_1, v_2, \dots, v_k)$$

Applying alt_{k+1} gives the first line of (22.1), and it implies the second line since already ω was supposed to be skew-symmetric.

In order to prove the formula for evaluation on vector fields, we first have to check that the result of the right-hand side at a point $p \in M$ depends only on the values of the X_i at p ; then take $X_i = \tilde{v}_i$, the extension to a constant vector field in a chart, and the formula reduces to the one established before. ■

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